KILLING FORMS ON 5D SASAKI-EINSTEIN SPACES*

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We start with a short review of Killing forms on Sasaki spaces. We then summarize recent results on the construction of Killing forms on Sasaki-Einstein manifolds. We present an explicit construction of the Killing forms on 5-dimensional Sasaki-Einstein spaces $L^{a,b,c}$ on $S^2 \times S^3$.

Key words: Killing forms, Sasaki-Einstein spaces, Calabi-Yau metric cone.

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1. INTRODUCTION

In the last time Sasaki-Einstein geometry has been the focus of much attention. The interest in this subject has arisen due to its importance in the AdS/CFT correspondence [1]. A Sasaki-Einstein space M_{2n+1} in $(2n+1)$-dimensions may be defined as a complete Riemannian manifold such that the metric cone $C(M_{2n+1})$ with the complex dimension $(n+1)$ is Ricci flat and Kähler, *i.e.* Calabi-Yau. In complex dimensions three and four, a Calabi-Yau cone is AdS/CFT dual to a symmetric field theory in dimensions four and three respectively. The purpose of this paper is to investigate the symmetries of the 5-dimensional Sasaki-Einstein spaces $L^{a,b,c}$ which are relevant in IIB supergravity theories.

The explicit symmetries of the configuration space are represented by isometries of the metric generated by Killing vector fields. An extension of the Killing vector fields is given by conformal Killing vector fields with flows preserving a given conformal class of metrics. The symmetries of the full phase space of a system are described by (conformal) Killing tensors which could be symmetric or antisymmetric. The antisymmetric generalization is represented by (conformal) Killing-Yano tensors. Killing-Yano tensors are also called Yano tensors or Killing forms, and conformal Killing-Yano tensors are sometimes referred as conformal Yano tensors, conformal Killing forms or twistor forms.

The symmetric generalization of the conformal Killing vectors is represented by Stäckel-Killing tensors. These tensors give rise to conserved quantities of higher order in particle momenta.

The organization of the paper is as follows: In the next Section we review some basic facts about Killing forms and Sasaki-Einstein spaces. In Section 3 we present the explicit construction of all Killing forms on 5-dimensional Sasaki-Einstein spaces $L^{a,b,c}$. The paper ends with conclusions in Section 4.

2. MATHEMATICAL PRELIMINARIES

For convenience we summarize the definitions and properties of Killing forms and Sasaki-Einstein spaces which are needed in the study of the hidden symmetries on $L^{a,b,c}$ spaces.

2.1. KILLING FORMS

Definition 1 *A vector field X on a Riemannian manifold (M,g) is said to be a Killing vector field if the Lie derivative of the metric g to X vanishes*

$$\mathcal{L}_X g = 0 \,. \tag{1}$$

A natural generalization of Killing vector fields is given by the conformal Killing vector fields, *i.e.* vector fields with flows preserving a given conformal class of metrics.

Definition 2 *A conformal Killing-Yano tensor of rank p on a Riemannian manifold (M,g) is a p-form ω which satisfies:*

$$\nabla_X \omega = \frac{1}{p+1} X \lrcorner \, d\omega - \frac{1}{n-p+1} X^* \wedge d^* \omega \,, \tag{2}$$

for any vector field X on M.

Here ∇ is the Levi-Civita connection of g, n is the dimension of M, X^* is the 1-form dual to the vector field X with respect to the metric g, $\lrcorner$ is the operator dual to the wedge product and d^* is the adjoint of the exterior derivative d. If ω is co-closed in (2), then we obtain the definition of a Killing-Yano tensor [2]. A particular class of Killing forms is represented by the special Killing forms:

Definition 3 *A Killing form ω is said to be a special Killing form if it satisfies for some constant c the additional equation*

$$\nabla_X (d\omega) = c X^* \wedge \omega \,, \tag{3}$$

for any vector field X on M.

Let us note that most known Killing forms are special.

Definition 4 *A symmetric tensor $K_{(i_1...i_r)}$ obeying the equation*

$$K_{(i_1...i_r;j)} = 0 \,, \tag{4}$$

is called a Stäckel-Killing tensor of rank r.

For any geodesic with a tangent vector u^i the following object

$$P_K = K_{i_1...i_r} u^{i_1} \cdots u^{i_r} \,, \tag{5}$$

is conserved.

These two generalizations of the Killing vectors could be related. Given two Killing-Yano tensors $\omega^{i_1,...,i_r}$ and $\sigma^{i_1,...,i_r}$ it is possible to associate with them a Stäckel-Killing tensor of rank 2:

$$K_{ij}^{(\omega,\sigma)} = \omega_{i i_2...i_r} \sigma_j^{\;i_2...i_r} + \sigma_{i i_2...i_r} \omega_j^{\;i_2...i_r} \,. \tag{6}$$

Having a Killing form, one can always construct the corresponding Stäckel-Killing tensor. On the other hand, not every Stäckel-Killing tensor can be decomposed in terms of Killing forms.

The existence of enough integrals of motion leads to complete integrability or even superintegrability of the mechanical system when the number of functionally independent constants of motion is larger than its number of degrees of freedom.

2.2. SASAKI-EINSTEIN SPACES

Traditionally the Sasakian structure on a manifold M_{2n+1} of $2n+1$ dimensions was defined as a metric contact structure (g,η,ξ,φ) satisfying an additional condition called *normality* [3]. Lately it was emphasized the relation between Sasakian structure and Kähler one. In fact Sasakian structures are sandwiched between two Kähler structures: the Kähler cone of complex dimension $n+1$, and the transverse Kähler structure of complex dimension n .

Definition 5 *A Riemannian manifold (M_{2n+1}, g) is Sasakian if the metric cone $(C(M_{2n+1}), \bar{g})$, $C(M_{2n+1}) := \mathbb{R}_+ \times M_{2n+1}$ and $\bar{g} = dr^2 + r^2 g$ is Kähler, that is $\bar{g}$ admits a compatible almost complex structure J so that $(C(M_{2n+1}), \bar{g}, J)$ is a Kähler structure.*

Here $r \in (0, \infty)$ may be regarded as a coordinate on the positive real line $\mathbb{R}_+$.

The metric cone has holonomy $Hol(C(M_{2n+1}), \bar{g}) \subseteq U(n+1)$ and its Kähler form is

$$\Omega_{cone} = \frac{1}{4} d\, d^c r^2 \,, \tag{7}$$

where $d^c = J \circ d = i(\bar{\partial} - \partial)$. The vector field $r\, \partial/\partial r$ is holomorphic and

$$\xi = J(r \frac{\partial}{\partial r}) \tag{8}$$

is holomorphic and also a Killing field (1), $\mathcal{L}_\xi g = 0$. ξ is known as the *Reeb vector field* and the contact form η is

$$\eta = d^c \log r. \tag{9}$$

The Sasakian manifold (M_{2n+1}, g) is naturally isometrically embedded into the metric cone via the inclusion

$$M_{2n+1} = \{r = 1\} = \{1\} \times M_{2n+1} \subset C(M_{2n+1}), \tag{10}$$

and the Kähler structure of the cone $(C(M_{2n+1}), \bar{g})$ induces an almost contact metric (g, η, ξ, φ) on M_{2n+1}. We shall use the same notations for ξ and η on $C(M_{2n+1})$ and their restrictions to M_{2n+1}.

If the orbits of ξ close, ξ generates an isometry $U(1)$ action on (M_{2n+1}, g) and the Sasakian structure is called *quasi-regular*. If the isotropy subgroups for all points are trivial then the $U(1)$ action is free, and the Sasakian structure is called *regular*. In the opposite case, if the orbits of ξ do not all close, the Sasakian structure is said to be *irregular*.

In what follows we shall concentrate on Einstein spaces. A simple calculation of the Ricci tensors shows that [4]

$$Ric_{\bar{g}} = Ric_g - 2n\,g = Ric_{g_T} - 2(n+1)\,g_T, \tag{11}$$

where g_T is the metric of the transverse Kähler-Einstein space M_{2n}. Thus the Kähler cone $(C(M_{2n+1}), \bar{g})$ is Ricci-flat if and only if (M_{2n+1}, g) is a Sasaki-Einstein manifold, $Ric_g = 2n\,g$, and further the base space (M_{2n}, g_T) is Kähler-Einstein, $Ric_{g_T} = 2(n+1)\,g_T$.

The Sasaki-Einstein manifold M_{2n+1} can be written as a fibration over the Kähler-Einstein base manifold (M_{2n}, g_T) twisted by the overall $U(1)$ part of the connection [5]

$$g = (d\psi_n + A)^2 + g_T, \tag{12}$$

where dA is given as the Kähler form $\Omega_{M_{2n}}$ of the Kähler-Einstein base

$$dA = 2\Omega_{M_{2n}}, \tag{13}$$

and the Sasakian 1-form of the Sasaki-Einstein metric is

$$\eta = A + d\psi_n. \tag{14}$$

Consequently the metric $\bar{g}$ on the cone manifold can be written as

$$\bar{g} = dr^2 + r^2 g = dr^2 + r^2 \left(d\psi_n + A \right)^2 + g_T, \tag{15}$$

and Kähler form (7) of the Calabi-Yau cone $C(M_{2n+1})$ is

$$\Omega_{cone} = r\,dr \wedge (d\psi_n + A) + r^2 \Omega_{M_{2n}}. \tag{16}$$

The hidden symmetries of the Sasaki-Einstein manifold M_{2n+1} are described by the special Killing $(2k+1)$-forms [6]

$$\Theta_k = \eta \wedge (d\eta)^k, \quad k = 0, 1, \cdots, n-1. \tag{17}$$

Moreover in the case of the Calabi-Yau cone, the holonomy is $SU(n+1)$ and there are *two additional Killing forms* of degree $(n+1)$. To write explicitly these additional Killing forms we shall note that there is an intimate connection between the Kähler form Ω_{cone} and the volume form of the metric cone

$$d\mathcal{V} = \frac{1}{(n+1)!}\,\Omega_{cone}^{n+1}, \tag{18}$$

where $d\mathcal{V}$ denotes the volume form of $C(M_{2n+1})$, Ω_{cone}^{n+1} is the wedge product of Ω_{cone} with itself $n+1$ times. On the other hand, if the volume of a Kähler manifold is written as

$$d\mathcal{V} = \frac{i^{n+1}}{2^{n+1}}(-1)^{n(n+1)/2}\,dV \wedge \overline{dV}, \tag{19}$$

then dV is the complex volume holomorphic $(n+1,0)$ form of $C(M_{2n+1})$. The real additional Killing forms are given by real respectively imaginary parts of the holomorphic volume form dV [6,7].

In order to extract the corresponding additional Killing forms on the Sasaki-Einstein space we make use of the fact that for any p-form ω on the space M_{2n+1} we can define an associated $(p+1)$-form ω^C on the cone $C(M_{2n+1})$

$$\omega^C := r^p\, dr \wedge \omega + \frac{r^{p+1}}{p+1}\,d\omega. \tag{20}$$

Moreover ω^C is parallel if and only if ω is a special Killing form (3) with constant $c = -(p+1)$ [6]. The 1-1-correspondence between special Killing p-forms on M_{2n+1} and parallel $(p+1)$-forms on the metric cone $C(M_{2n+1})$ allows us to describe the additional Killing forms on Sasaki-Einstein spaces. Therefore in order to find the additional Killing forms on the Sasaki-Einstein manifold M_{2n+1} we must identify the ω form in the complex volume form of the Calabi-Yau cone. An explicit example is presented in the next Section.

3. $L^{a,b,c}$ SPACES

Recently an infinite family $Y^{p,q}$ of explicit Sasaki-Einstein metrics on $S^2 \times S^3$ have been constructed [8,9]. The following theorem states the results:

Theorem 1 *([9]) There exist a countably infinite number of Sasaki-Einstein metrics $Y^{p,q}$ on $S^2 \times S^3$, labeled naturally by $p,q \in \mathbb{N}$ where $gcd(p,q) = 1$, $q < p$. $Y^{p,q}$ is quasi-regular if and only if $4p^2 - 3q^2$ is the square of a natural number, otherwise is irregular.*

The construction of $Y^{p,q}$ spaces was generalized and the following theorem describes a new infinite class of Sasaki-Einstein manifolds in 5-dimensions:

Theorem 2 *([10, 11]) There exist a countably infinite number of Sasaki-Einstein metrics $L^{a,b,c}$ on $S^2 \times S^3$, labeled naturally by $a,b,c \in \mathbb{N}$ where $a \leq b, c \leq b, d = a+b-c, gcd(\{a,b\},\{c,d\}) = 1$ The metrics are generically irregular.*

Here $gcd(\{a,b\},\{c,d\}) = 1$ means that each of the pair $\{a,b\}$ must be coprime to each of $\{c,d\}$. In particular $L^{p-q,p+q,p} = Y^{p,q}$.

The explicit local metric of the 5-dimensional $L^{a,b,c}$ manifold is given by the line element [12]

$$g = g_T + (d\psi' + A)^2,\tag{21}$$

with

$$g_T = \frac{(\zeta-\xi)}{2F(\xi)}d\xi^2 + \frac{2F(\xi)}{(\zeta-\xi)}(d\Phi+\zeta d\Psi)^2 + \frac{(\zeta-\xi)}{2G(\zeta)}d\zeta^2 + \frac{2G(\zeta)}{(\zeta-\xi)}(d\Phi+\xi d\Psi)^2,\tag{22}$$

where

$$F(\xi) = -\kappa(\alpha_1-\xi)(\alpha_2-\xi)(\alpha_3-\xi),\tag{23}$$
$$G(\zeta) = \kappa(\alpha_1-\zeta)(\alpha_2-\zeta)(\alpha_3-\zeta)+\gamma,\tag{24}$$
$$A = -\frac{1}{2}(\xi+\zeta)d\Phi - \frac{1}{2}\xi\zeta d\Psi.\tag{25}$$

The metrics $L^{a,b,c}$ are specified by three integers a,b,c (see [10–12] for details and the ranges of variables).

The holomorphic $(3,0)$ form on the cone is [12]:

$$dV = e^{i\psi'}r^2\Omega_4 \wedge (dr+ir\eta),\tag{26}$$

with

$$\Omega_4 = 2(\zeta-\xi)\sqrt{FG}\hat{\eta}_1 \wedge \hat{\eta}_2,\tag{27}$$

where

$$\hat{\eta}_1 = \frac{1}{2F}d\xi + \frac{i}{\zeta-\xi}(d\Phi+\zeta d\Psi),\tag{28}$$
$$\hat{\eta}_2 = \frac{1}{2G}d\zeta + \frac{i}{\zeta-\xi}(d\Phi+\xi d\Psi).\tag{29}$$

Now it is straightforward to find the Killing forms on $L^{a,b,c}$ spaces. First of all the forms (17) for $k = 0, 1$ are special Killing forms:

$$\Theta_0 = \eta = d\psi' - \frac{1}{2}(\xi + \zeta)d\Phi - -\frac{1}{2}\xi\zeta d\Psi \,, \tag{30}$$

$$\begin{aligned}
\Theta_1 &= \eta \wedge d\eta \\
&= -\frac{1}{2}d\psi' \wedge d\xi \wedge d\Phi - \frac{1}{2}d\psi' \wedge d\zeta \wedge d\Phi - \frac{1}{2}\zeta d\psi' \wedge d\xi \wedge d\Psi - \frac{1}{2}\xi d\psi' \wedge d\zeta \wedge d\Psi \\
&\quad - \frac{1}{4}(\xi + \zeta)(\zeta d\xi \wedge d\Phi \wedge d\Psi + \xi d\zeta \wedge d\Phi \wedge d\Psi) + \frac{1}{4}\xi\zeta d\zeta \wedge d\Phi \wedge d\Psi \,. \tag{31}
\end{aligned}$$

Let us note also that $(d\eta)^k$ for $k = 1, 2$ are closed conformal Killing forms.

Finally, the additional Killing forms are connected with the real and imaginary parts of the holomorphic $(3,0)$ form (26). The evaluation of the holomorphic form Ω_4 (27) gives:

$$\begin{aligned}
\Omega_4 =& 2(\zeta - \xi)\sqrt{FG}\Big[\frac{1}{4FG}d\xi \wedge d\zeta + \frac{i}{2F(\zeta - \xi)}(d\xi \wedge d\Phi + \xi d\xi \wedge d\Psi) \\
&- \frac{i}{2G(\zeta - \xi)}(d\zeta \wedge d\Phi + \zeta d\zeta \wedge d\Psi) + \frac{1}{\zeta - \xi}d\Phi \wedge d\Psi\Big]. \tag{32}
\end{aligned}$$

Extracting from the complex volume form (26) the form ω on the Einstein-Sasaki space according to (20) for $p = 2$ we get the following additional Killing 2-forms of the $L^{a,b,c}$ spaces written as real 2-forms:

$$\begin{aligned}
\Xi = \mathrm{Re}\,\omega =& 2(\zeta - \xi)\sqrt{FG} \\
&\times \left(\cos\psi'\Big[\frac{1}{4FG}d\xi \wedge d\zeta + \frac{1}{\zeta - \xi}d\Phi \wedge d\Psi\Big] - \right. \tag{33} \\
&\left. -\sin\psi'\frac{1}{2(\zeta - \xi)}\Big[\frac{1}{F}(d\xi \wedge d\Phi + \xi d\xi \wedge d\Psi) - \frac{1}{G}(d\zeta \wedge d\Phi + \zeta d\zeta \wedge d\Psi)\Big]\right),
\end{aligned}$$

$$\begin{aligned}
\Upsilon = \mathrm{Im}\,\omega =& 2(\zeta - \xi)\sqrt{FG} \\
&\times \left(\sin\psi'\Big[\frac{1}{4FG}d\xi \wedge d\zeta + \frac{1}{\zeta - \xi}d\Phi \wedge d\Psi\Big] - \right. \tag{34} \\
&\left. +\cos\psi'\frac{1}{2(\zeta - \xi)}\Big[\frac{1}{F}(d\xi \wedge d\Phi + \xi d\xi \wedge d\Psi) - \frac{1}{G}(d\zeta \wedge d\Phi + \zeta d\zeta \wedge d\Psi)\Big]\right).
\end{aligned}$$

The Stäckel-Killing tensors associated with the Killing forms Θ_0 (30), Θ_1 (31), Ξ (33), Υ (34) are constructed as in (6). The list of the non vanishing components of these Stäckel-Killing tensors is quite long and will be given elsewhere. Together with the Killing vectors $P_\Phi, P_\Psi, P_{\psi'}$ these Stäckel-Killing tensors provide the super-integrability of the $L^{a,b,c}$ geometries.

Let us note that one recovers the $Y^{p,q}$ metrics in the limit $a = p - q$, $b = p + q$, $c = p$. The corresponding Killing forms are given in [13].

4. CONCLUSIONS

In general it is a hard task to find solutions of the (conformal) Killing-Yano equations. However in the case of spaces endowed with special geometrical structures, the existence of Killing forms and their explicit construction is granted.

In this paper we presented the complete set of Killing forms on 5-dimensional Einstein-Sasaki spaces $L^{a,b,c}$. The multitude of Killing-Yano and Stäckel-Killing tensors makes possible a complete integrability of geodesic equations.

The explicit construction of the Killing forms on 5-dimensional Sasaki-Einstein manifolds can be extended to other spaces of interest. We mention here the 7-dimensional Sasaki-Einstein [14] and 3-Sasakian spaces [15].

The remarkable properties of the Killing forms offer new perspectives in the investigation of the supersymmetries, separability of Hamilton-Jacobi, Klein-Gordon and Dirac equations on Einstein-Sasaki spaces.

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