ON SINGULAR DISTRIBUTION ON VECTOR BUNDLES*

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The aim of the paper is to study various orthogonals of a singular distribution of vector bundles, using dual vector bundles as well as a riemannian metric. We prove that for a Dirac type distribution in the Pontryagin bundle, an almost Dirac structures on a dense open subset is induced.

Key words: Vector bundle, singular distribution, orthogonal, natural anchor.

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1. INTRODUCTION

The vector subbundles have most of properties of vector subspaces. For example, considering a Riemannian metric on the fibers of a vector bundle, the orthogonal of a vector subbundle is also a vector subbundle and the Whitney sum of the two mutual orthogonal subbundles gives a reduction of the given vector bundle. We see in this paper that the singular distributions (or singular vector subbundles) have different properties and behaviour. In order to see this, we revise the notion of orthogonal.

A singular distribution of a vector bundle is a subset of a given vector bundle such that the intersection with every fiber is a vector subspace of the fiber. A singular distribution is: smooth if it is locally generated by a local smooth section of the distribution, and co-smooth if its annihilator distribution (a distribution on the dual vector bundle) is smooth. Any distribution contains a maximal smooth distribution; the maximal smooth distribution included in the annihilator is the orthogonal smooth singular distribution of the given one. We study some properties of this orthogonal smooth distribution, related to the annihilator (or polar) of a module of sections that generates the distribution.

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2. BASIC CONSTRUCTIONS

Let M be a smooth manifold, *i.e.* of class C^∞, and $\pi : E \to M$ a vector bundle bundle. We denote by $\mathcal{F}(M)$ and $\Gamma(E)$ the algebra of real functions and the corresponding module of sections. We denote also by $\mathcal{F}_{loc}(M)$ and $\Gamma_{loc}(E)$ the local algebras and the local corresponding modules respectively; they are the union of $\mathcal{F}_U(M)$ and $\Gamma_U(E)$ respectively, where $U \subset M$ is open and the index U denotes the restriction on U. For sake of simplicity, we work throughout the paper with global defined objects.

We say that a singular distribution $\mathcal{E} \subset E$ is *co-smooth* if its polar distribution $\mathcal{E}^\circ \subset E^*$ is smooth. Obviously $\mathcal{D} \subset E$ is smooth iff $\mathcal{D}^\circ \subset E^*$ is co-smooth.

If $\mathcal{D} \subset E$ is a distribution, we denote by $\mathcal{D}^\vdash = \mathcal{S}(\mathcal{D}^\circ) \subset E^*$ and we say that it is the *orthogonal* of $\mathcal{D}$. According to the definition of orthogonal $\mathcal{D}^\vdash$, it is always smooth, even when $\mathcal{D}$ is not. In spite of the regular case, when $\mathcal{D}^{\perp\perp} = \mathcal{D}$, in the singular case this property for $\vdash$ it is not always true. For example, for $E = \tau I\!R^2 = I\!R^2 \times I\!R^2$, consider $\mathcal{D}_{(x,y)} = I\!R^2$ for $(x,y) \neq (0,0) = \bar{0}$ and $\mathcal{D}_{\bar{0}} = \{\bar{0}\}$. Then $\mathcal{D}^{\vdash\vdash} = \tau I\!R^2 \neq \mathcal{D}$. But if we consider the singular distribution $\mathcal{D}_0$ generated by the position vector of every point in $I\!R^2$, then $\mathcal{D}_0^{\vdash\vdash} = \mathcal{D}_0^\vdash$. Obviously $\mathcal{D}_0^\vdash$ enjoys the same property.

For a submodule $\Gamma \subset \Gamma(E)$, we denote by $\Gamma^\circ \subset \Gamma(E^*)$ its *polar module*, *i.e.* $\Gamma^\circ = \{\omega \in \Gamma(E^*) : \omega(X) = 0, (\forall) X \in \Gamma\}$. Analogously, for a submodule $\Lambda \subset \Gamma(E^*)$, we denote by $\Lambda^\circ \subset \Gamma(E)$ its *polar module*, *i.e.* $\Lambda^\circ = \{X \in \Gamma(E) : \omega(X) = 0, (\forall)\omega \in \Lambda\}$.

Proposition 1 *If* $\Gamma_0 \subset \Gamma(E)$ *and* $\Lambda_0 \subset \Gamma(E)$ *are submodules, then*

#1 $\Gamma_0^\circ = \Gamma(\mathcal{D}(\Gamma_0))^\circ$, $\Lambda_0^\circ = \Gamma(\mathcal{D}(\Lambda_0))^\circ$,

#2 $\Gamma_0^\circ = \Gamma(\mathcal{D}(\Gamma_0)^\vdash)$, $\Lambda_0^\circ = \Gamma(\mathcal{D}(\Lambda_0)^\vdash)$ *and*

#3 $\Gamma(\mathcal{D}(\Gamma_0))^\circ = \Gamma(\mathcal{D}(\Gamma_0)^\vdash)$, $\Gamma(\mathcal{D}(\Lambda_0))^\circ = \Gamma(\mathcal{D}(\Lambda_0)^\vdash)$.

Proof. We prove the assertions only for Γ_0, since those for Λ_0 follow by duality.

#1 Since $\Gamma_0 \subset \Gamma(\mathcal{D}(\Gamma_0))$, it follows that $\Gamma(\mathcal{D}(\Gamma_0))^\circ \subset \Gamma_0^\circ$. Let us prove the converse inclusion. Let $\omega \in \Gamma_0^\circ$, $Y \in \Gamma(\mathcal{D}(\Gamma_0))$ and $x \in M$. Since $Y_x = Z_x$ for an $Z \in \Gamma_0$, we have $(\omega(Y))_x = \omega_x(Y_x) = \omega_x(Z_x) = (\omega(Z))_x = \bar{0}_x$, thus $\omega(Y) = 0$ and the inclusion $\Gamma_0^\circ \subset \Gamma(\mathcal{D}(\Gamma_0))^\circ$ follows.

#2 We have $\omega \in \Gamma(\mathcal{D}(\Gamma_0)^\vdash)$ iff $\omega \in \Gamma(E^*)$ and $\omega(X) = 0$, $(\forall) X \in \Gamma(\mathcal{D}(\Gamma_0))$ (or like above, $(\forall) X \in \Gamma_0$), *i.e.* iff $\omega \in \Gamma(\mathcal{D}(\Gamma_0))^\circ$.

#3 It follows from #1 and #2 □

Proposition 2 *If* $\mathcal{D} \subset E$ *is a smooth distribution, then* $\Gamma(\mathcal{D})^\circ = \Gamma(\mathcal{D}^\vdash)$.

Proof. If $\mathcal{D} \subset E$ is smooth, then using #3 of Proposition 1 for $\Gamma_0 = \Gamma(\mathcal{D})$, we obtain the conclusion. □

Using the Proposition for a smooth distribution $\mathcal{E} \subset E^*$, we obtain that $\Gamma(\mathcal{E})^\circ = \Gamma(\mathcal{E}^\vdash)$.

Proposition 3 *If a submodule of sections* $\Gamma_0 \subset \Gamma(E)$ *has the property* $(\Gamma_0^\circ)^\circ = \Gamma_0$*, then there is a smooth distribution* $\mathcal{D} \subset E$ *such that* $\Gamma_0 = \Gamma(\mathcal{D})$.

Proof. Let us assume that $(\Gamma_0^\circ)^\circ = \Gamma_0$ and denote $\mathcal{D}_0 = \mathcal{D}(\Gamma_0)$. Then using Propositions 1 and 2, we have $(\Gamma_0^\circ)^\circ = \Gamma(\mathcal{D}_0^\vdash)^\circ = \Gamma((\mathcal{D}_0^\vdash)^\vdash)$, thus $\Gamma_0 = \Gamma(\mathcal{D})$, where $\mathcal{D} = (\mathcal{D}_0^\vdash)^\vdash$. □

If a smooth distribution $\mathcal{D} \subset E$ has the property that $\mathcal{D}^{\vdash\vdash} = \mathcal{D}$, then $\Gamma(\mathcal{D})^{\circ\circ} = \Gamma(\mathcal{D})^\circ$.

3. SUBMODULES OF VECTOR AND COVECTOR FIELDS

We look now at the smooth singular distributions of the tangent and the cotangent bundles; for sake of simplicity, we call them tangent distributions and cotangent distributions respectively.

We say that a submodule $\Gamma \subset \mathcal{X}(M)$ is *involutive* if it is a Lie subalgebra of $\mathcal{X}(M)$ (*i.e.* $[X,Y] \in \Gamma_0$, $(\forall) X,Y \in \Gamma_0$).

Proposition 4 *If* $\Gamma \subset \mathcal{X}(M)$ *is a submodule, there is an involutive one* Γ^∞ *that contains* Γ *and is minimal, in the sense that if* Γ *is a submodule of an involutive submodule* $\Gamma_1 \subset \mathcal{X}(M)$*, then* $\Gamma^\infty \subset \Gamma_1$.

Proof. Let us consider $\Gamma^{(0)} = \Gamma$ and, for $k \geq 1$, $\Gamma^{(k)}$ is the $\mathcal{F}(M)$-module generated by $\Gamma^{(k-1)} + [\Gamma^{(k-1)}, \Gamma^{(k-1)}]$. Then $\Gamma^\infty = \bigcup_{k\geq 0} \Gamma^{(k)}$ is a submodule and a Lie subalgebra of $\mathcal{X}(M)$ as well, thus it is involutive and is minimal, in the sense specified in the statement. □

For a submodule $\Gamma \subset \Gamma(\mathcal{D})$ that generates $\mathcal{D}$, we use the notation in the proof of the above Proposition.

It is easy to see that Γ is involutive iff $\Gamma = \Gamma^{(1)}$; if it is the case, then $\Gamma = \Gamma^{(1)} = \cdots = \Gamma^{(k)} = \cdots = \Gamma^\infty$. Also if there is a k_0 such that $\Gamma^{(k_0)} = \Gamma^{(k_0+1)}$, then $\Gamma^{(k)} = \Gamma^{(k_0)} = \Gamma^\infty$ for $k \geq k_0$.

We say that a tangent distribution $\mathcal{D} \subset TM$ is *involutive* if there is an involutive submodule $\Gamma_0 \subset \Gamma(\mathcal{D})$ that generates $\mathcal{D}$ (*i.e.* $\mathcal{D} = \mathcal{D}(\Gamma_0)$); in order to stress Γ_0, we can say that $\mathcal{D}$ is Γ_0–*involutive*.

Proposition 5 *Any tangent distribution* $\mathcal{D}$ *generated by a submodule* $\Gamma \subset \Gamma(\mathcal{D})$ *is included in the* Γ^∞*–involutive tangent distribution* $\mathcal{D}^\infty = \mathcal{D}(\Gamma^\infty)$ *that is minimal in the sense that if* Γ *is a submodule of an involutive* $\Gamma_1 \subset \mathcal{X}(M)$*, then* $\mathcal{D} \subset \mathcal{D}(\Gamma_1)$.

We denote $\mathcal{D}^{(k)} = \mathcal{D}(\Gamma^{(k)})$.

It is well known that a tangent singular distribution that is involutive is not always integrable, but it is true in the following two cases:

– when $\mathcal{D}$ is a regular distribution (according to Frobenius theorem; see, for example, [1]);

– when $\mathcal{D}$ is generated by an involutive and finitely generated submodule Γ (see, for example, [2], or [4]).

We say that a tangent distribution $\mathcal{D}$ is *involutive* if $\Gamma(\mathcal{D})$ is involutive.

Let $\Gamma \subset \mathcal{F}(M)$ be a submodule, not necessary involutive. We say that $\mathcal{D}(\Gamma)$ is *weak* Γ*–involutive* if $\Gamma^{(1)} \subset \Gamma(\mathcal{D}(\Gamma))$. It is easy to see that if $\mathcal{D}$ is involutive, then $\mathcal{D}$ is weak Γ–involutive for every submodule Γ that generates $\mathcal{D}$ (*i.e.* $\mathcal{D} = \mathcal{D}(\Gamma)$).

It is easy to see that $\mathcal{D}$ is involutive iff $\Gamma(\mathcal{D}) = \Gamma(\mathcal{D})^{(1)}$; if it is the case, then $\mathcal{D} = \mathcal{D}^{(1)} = \cdots = \mathcal{D}^{(k)} = \cdots = \mathcal{D}^{\infty}$. Also if there is a k_0 such that $\Gamma(\mathcal{D})^{(k_0)} = \Gamma(\mathcal{D})^{(k_0+1)}$, then $\Gamma(\mathcal{D})^{(k)} = \Gamma(\mathcal{D})^{(k_0)} = \Gamma(\mathcal{D})^{\infty}$ and $\mathcal{D}^{(k)} = \mathcal{D}^{(k_0)} = \mathcal{D}^{\infty}$ for $k \geq k_0$.

We say that two submodules $\Gamma_1, \Gamma_2 \subset \Gamma(E)$ are *equivalent* if $\mathcal{D}(\Gamma_1) = \mathcal{D}(\Gamma_2)$. According to Proposition 1, if $\Gamma_1, \Gamma_2 \subset \Gamma(E)$ are equivalent, then $\Gamma_1^{\circ} = \Gamma_2^{\circ}$. We say that two submodules $\Gamma \subset \Gamma(E)$ and $\Lambda \subset \Gamma(E^*)$ are

– *orthogonal* if $\Gamma \subset \Lambda^{\circ}$ and $\Lambda \subset \Gamma^{\circ}$;

– *maximal orthogonal* if Γ and Λ are equivalent to Λ° and Γ° respectively.

Let $\mathcal{E} \subset T^*M$ be a cotangent distribution and let $\mathcal{D} = \mathcal{E}^{\vdash}$ be its orthogonal tangent distribution. It is easy to see that a submodule $\Gamma \subset \mathcal{X}(M)$ generates $\mathcal{D}$ iff the polar module $\Gamma^{\circ} \subset \mathcal{X}^*(M)$ or a submodule $\Lambda = \Gamma^{\circ}$, equivalent to Γ°, generates $\mathcal{E}$. Thus $\mathcal{E}$ and $\mathcal{E}^{\vdash}$ are always generated by two maximal orthogonal modules.

If $\Lambda \subset \mathcal{X}^*(M)$ is a submodule, we denote by $\Lambda^{(1)} \subset \mathcal{X}^*(M)$ the submodule spanned by the differential forms ω and $L_{\xi}\omega$, where $\omega \in \Lambda$, $\xi \in \Lambda^{\circ}$ and $L_{\xi} = i_{\xi}d + di_{\xi}$ is the Lie derivative. We say that Λ is *co-involutive* if $\Lambda^{(1)} \subset \Lambda$. We say that the cotangent distribution $\mathcal{E} = \mathcal{D}(\Lambda)$ is:

– Λ*–co-involutive* if Λ is co-involutive;

– *co-involutive* if $\mathcal{D}(\mathcal{E})$ is $\Gamma(\mathcal{E})$–co-involutive.

Proposition 6 *a) Let* $\Gamma \subset \mathcal{X}(M)$ *be involutive. Then* $\mathcal{D}(\Gamma)^{\vdash}$ *is a co-involutive cotangent distribution.*

b) Let $\Lambda \subset \mathcal{X}^*(M)$ *be co-involutive. Then* $\mathcal{D}(\Lambda)^{\vdash}$ *is an involutive tangent distribution.*

Proof. a) For every ξ_1, $\xi_2 \in \Gamma$ and $\omega \in \Gamma^{\circ} = \Gamma(\mathcal{D}(-)^{\vdash})$, we have $L_{\xi_1}\omega(\xi_2) = i_{\xi_1}d\omega(\xi_2) = \omega([\xi_1, \xi_2])$. Let us suppose that $\mathcal{D}(\Gamma)$ is Γ–involutive. Thus $[\xi_1, \xi_2] \in \Gamma$, $(\forall)\xi_1, \xi_2 \in \Gamma$ (*i.e.* $\omega([\xi_1, \xi_2]) = 0$, $(\forall)\omega \in \Gamma(\mathcal{D}^{\vdash})$) iff $L_{\xi_1}\omega(\xi_2) = i_{\xi_1}d\omega(\xi_2) = 0$, $(\forall)\xi_1, \xi_2 \in \Gamma, \omega \in \Gamma(\mathcal{D}^{\vdash})$, thus $\Gamma(\mathcal{D}^{\vdash})^{(1)} \subset \Gamma(\mathcal{D}^{\vdash})$, where $\mathcal{D} = \mathcal{D}(-)$. The statement b) follows analogously. □

Proposition 7 *A smooth tangent distribution* $\mathcal{D}$ *is involutive iff its co-tangent distribution* $\mathcal{D}^{\vdash}$ *is co-involutive.*

Proof. It is a consequence of Proposition 6 for $\Gamma = \Gamma(\mathcal{D})$ and $\Lambda = \Gamma(\mathcal{D}^{\vdash})$. □

4. A SEPARABILITY PROPERTY OF SMOOTH DISTRIBUTIONS

If $\mathcal{D} \subset E$ is a smooth singular distribution, then $\mathcal{D} + \mathcal{D}^{\vdash} \subset E \oplus E^*$ is also a smooth singular distribution. If g is a Riemannian metric in the fibers of E, then the inverse of the musical isomorphism $\# : E \to E^*$ induced by g gives the smooth singular distributions $\#^{-1}(\mathcal{D}^{\vdash}) \overset{not.}{=} \mathcal{D}^{\vdash_g} \subset E$, $\mathcal{D}^{\vdash_g} = \mathcal{S}(\mathcal{D}^{\perp_g})$ and $\mathcal{D} + \mathcal{D}^{\vdash_g} \overset{not.}{=} \mathcal{D}^{\models_g} \subset E$.

We say that $\mathcal{D}$ is *separable* if there are some regular and non-vanishing smooth distributions $\mathcal{D}' \subset \mathcal{D}$ and $\mathcal{D}'' \subset \mathcal{D}^{\perp_g}$. Let us denote $\mathcal{D}_1 = \mathcal{D}' + \mathcal{D}''$ and $\mathcal{D}_0 = \mathcal{D}_1^{\perp_g}$, and also $\mathcal{D}_0' = \mathcal{D} \cap \mathcal{D}_0$ and $\mathcal{D}_0'' = \mathcal{D}^{\vdash_g} \cap \mathcal{D}_0$.

Proposition 8 *Let us suppose that $\mathcal{D}$ is separable and the regular smooth distributions $\mathcal{D}' \subset \mathcal{D}$ and $\mathcal{D}'' \subset \mathcal{D}^{\perp_g}$ are maximal with this property of inclusion. Considering the induced metric g' on the vector bundle $\mathcal{D}_0$, then $\mathcal{D}_0' \subset \mathcal{D}_0$ is a smooth singular distribution that is not separable in $\mathcal{D}_0$ and $\mathcal{D}_0'' = (\mathcal{D}_0')^{\vdash_{g'}}$*

We say that $\mathcal{D}$ is *locally separable* if its restriction to any open subset of the base M is separable.

We say that $\mathcal{D}$ is *locally non-separable* if its restriction to any open subset of the base M is not separable.

A smooth endomorphism Φ in the fibers of E is called a *natural anchor* for $\mathcal{D}$ if its image generates $\mathcal{D}$. According to [3], natural anchors exist for any smooth singular distribution $\mathcal{D}$. Using the existence of a test function proved also in [3], one can prove the following statement.

Proposition 9 *A smooth singular distribution $\mathcal{D}$ is locally non-separable iff it has only two level dimensions, where the minimal one vanishes and the the other is maximal. If it is the case, then there is a positive real function $f \in \mathcal{F}(M)$ such that its zero set is exactly the zero set of $\mathcal{D}$ and $\Phi = f \cdot I_E$ is a natural anchor for $\mathcal{D}$.*

5. ORTHOGONAL PROJECTIONS AND NATURAL ANCHORS

Let $\mathcal{D} \subset E$ be a smooth singular distribution (a s.s.d. for short) of E and consider a Riemannian metric g on the fibers of E. Then $\mathcal{D}^{\perp_g} \subset E$ is just the orthogonal distribution., *i.e.* $\mathcal{D}_x^{\perp_g} = (\mathcal{D}_x)^{\perp_g}$, $(\forall)x \in M$, since $(\forall)x \in M$, the vector space $(\mathcal{D}_x)^{\perp_g}$ is canonically isomorphic with the annihilator $\mathcal{D}_x^{\circ} = \{\omega_x \in E_x^*, \omega_x(X_x) = 0, (\forall)X_x \in \mathcal{D}_x\}$, via the musical isomorphism $\# : E \to E^*$ given by the metric g.

Let us observe that two orthogonals, corresponding to two different Riemannian metrics, are both isomorphic to the annihilator, thus they are isomorphic.

If $\mathcal{D} \subset E$ is a s.s.d. and g is a Riemannian metric on the fibers of E, then we say that

– $\mathcal{D}^{\perp_g}$ is a *cosmooth orthogonal* of $\mathcal{D}$,

– $\mathcal{D}^{\vdash_g} = \mathcal{S}(\mathcal{D}^{\perp_g}(\mathcal{D})) \subset E$ is a *smooth orthogonal* of $\mathcal{D}$ and
– $\mathcal{D}^{\models_g} = \mathcal{D} + \mathcal{D}^{\vdash_g}$ is a *smooth orthogonal completion* of $\mathcal{D}$.

Proposition 10 *For a smooth $\mathcal{D}$, the following properties hold:*

#1 – the smooth orthogonal of $\mathcal{D}^{\models_g}$ is null (i.e. $(\mathcal{D}^{\models_g})^{\vdash_g} = \bar{0}$) and consequently

#2 – the smooth orthogonal completion of $\mathcal{D}^{\models_g}$ is $\mathcal{D}^{\models_g}$ itself (i.e. $(\mathcal{D}^{\models_g})^{\models_g} = \mathcal{D}^{\models_g}$);

#3 – the smooth orthogonal completion $\mathcal{D}^{\models_g} = \mathcal{D} + \mathcal{D}^{\vdash_g}$ has the property that its maximal dimension of the fibers is $m = \dim M$ and is taken on an open dense subset of M;

#4 – in the case when $\mathcal{D}$ has a regular dimension r, then $\mathcal{D}^{\perp_g} = \mathcal{D}^{\vdash_g}$ and $\mathcal{D}^{\models_g} = E$.

Proof. #1 Consider an $x_0 \in M$, an $\bar{v}_0 \in E_{x_0}$ and a local section $s : U \to E$, $x_0 \in U$, $U \subset M$ open, such that $s_{x_0} = \bar{v}_0$ and $s_x \in (\mathcal{D}^{\models_g})^{\vdash_g}$. But it follows by one hand that s is a smooth section of $\mathcal{D}^{\perp_g}$, thus of $\mathcal{D}^{\vdash_g} = \mathcal{S}(\mathcal{D}^{\perp_g}(\mathcal{D}))$, but it must be also orthogonal to this singular distribution. It follows that $s = 0$ on U, thus $\bar{v}_0 = \bar{0}_x$. The assertion #2 follows from #1, since $(\mathcal{D}^{\models_g})^{\models_g} = \mathcal{D}^{\models_g} + (\mathcal{D}^{\models_g})^{\vdash_g} = \mathcal{D}^{\models_g}$. #3 Let's suppose that there is an open set $U \subset M$ where the maximum of dimensions $\dim(\mathcal{D}^{\models_g})_x$, for $x \in U$, is $d_0 < m$. Consider an $x_0 \in U$ where $\dim(\mathcal{D}^{\models_g})_x = d_0$. Then there is an open set $V \subset U$, $x_0 \in V$, such that $\mathcal{D}^{\models_g}_{|V}$ is regular having the dimension of fibers d_0. Thus $\left(\mathcal{D}^{\models_g}_{|V}\right)^{\perp_g} = (\mathcal{D}^{\models_g}_{|V})^{\vdash_g}$ is regular and non-null. But it contradicts #1. For #4 the proof is classical. □

Notice that a simple consequence of #3 of Proposition 10 above is that given the smooth singular distributions $\{\mathcal{D}_i\}_{i=\overline{1,n}}$ of some vector bundles over M, then the intersection $\Sigma = \bigcap_{i=1}^{n} \Sigma^i_{\max}$ of maximal dimensions $\left\{\Sigma^i_{\max}\right\}$ of orthogonal completions $\left\{\mathcal{D}_i^{\models_g}\right\}$ is an open dense subset of M.

As we have already remarked in #3 of Proposition 10, it follows that the sets of maximal dimensions $\Sigma'_{\max}$ and $\Sigma''_{\max}$ of $(\mathcal{D}')^{\models_g}$ and $(\mathcal{D}'')^{\models_g}$ respectively are open dense subsets of M, thus their intersection is also an open dense subset of M.

Considering natural anchors of $\mathcal{D}$ and $\mathcal{D}^{\vdash_g}$, their sum gives a natural anchor of the smooth orthogonal completion $\mathcal{D}^{\models_g}$. Using also Proposition 10, some properties can be summarized in the following statement.

Proposition 11 *Let g be a scalar product on the fibers of E and $\mathcal{D}^{\vdash_g} \subset E$ be the smooth orthogonal of a smooth singular distribution $\mathcal{D}$. If Φ and $\Phi^{\vdash_g} \in End(E)$ are natural anchors of $\mathcal{D}$ and $\mathcal{D}^{\vdash_g}$ respectively, then*

#1 $\Phi \circ \Phi^{\vdash_g} = \Phi^{\vdash_g} \circ \Phi = 0$, i.e. Φ and $\Phi^{\vdash_g}$ are transverse,

#2 $\Phi^{\models_g} = \Phi + \Phi^{\vdash_g}$ is a natural anchor of the smooth orthogonal completion $\mathcal{D}^{\models_g}$,

#3 $\Phi^{\models_g}(\mathcal{D}) = \mathcal{D}$ *and* $\Phi^{\models_g}\left(\mathcal{D}^{\vdash_g}\right) = \mathcal{D}^{\vdash_g}$ *and*

#4 $\Phi^{\models_g}$ *induces on the dense open subset* $\Sigma^{\models_g}_{\max} \subset M$ *an automorphism of the fibres of* E *and a natural splitting* $E_x = \mathcal{D}_x \oplus \mathcal{D}_x^{\vdash_g}$, $(\forall) x \in \Sigma^{\models_g}_{\max}$.

These can be related to some s.s.d.'s in the Pontryagin bundle $P(E) = E \oplus E^*$, where the non-degenerate quadratic form $\varepsilon : P(E) \to \mathcal{F}(M)$, $\varepsilon(X,\omega) = \omega(X)$ can be considered (see [5] for more details). The quadratic form ε has the signature (k,k), where k is the dimension of the fibers of E. Considering a s.s.d. $\mathcal{D} \subset E$ and a Riemannian metric g in the fibers of E, there are canonical isomorphisms:

– $\mathcal{D}^{\perp_g} \cong \mathcal{D}^{\perp} \subset E^*$, $\mathcal{D}^{\vdash_g} \cong \mathcal{S}(\mathcal{D}^{\perp}) \overset{not.}{=} \mathcal{D}^{\vdash} \subset E^*$ and

– $\mathcal{D}^{\models_g} \cong \mathcal{D} + \mathcal{D}^{\vdash} \overset{not.}{=} \mathcal{D}^{\models} \subset P(E)$.

We stress that the above orthogonals are related to the Riemannian metric g and they are different from the orthogonal considered according to the canonical pseudo-Riemannian metric ε (as for example in [5]).

We say that a s.s.d. $\mathcal{D} \subset P(E)$ is *adapted* if the restriction of ε to $\mathcal{D}$ is null. For example, every smooth s.s.d. of E or E^* can be considered as smooth s.s.d. of $P(E)$ of Dirac type; we say that they are *pure*. Notice that there is a decomposition $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$ of $\mathcal{D}$ in its pure Dirac components, $\mathcal{D}_1 \subset E$ and $\mathcal{D}_2 \subset E^*$, call here as the *pure decomposition*. We say also that an adapted s.s.d. $\mathcal{D} \subset P(E)$ is of *Dirac type* if $\mathcal{D}_2 = \mathcal{D}_1^{\vdash}$ or, equivalently, $\mathcal{D} = \mathcal{D}_1^{\models}$, where $\mathcal{D}_1 \subset E$ and $\mathcal{D}_2 \subset E^*$ are its pure components. According to [5], an *almost Dirac structure* $\mathcal{D} \subset P(E)$ is just a regular distribution of Dirac type and of maximal dimensions of fibers, *i.e.* the dimension of the fibers of E.

Using #3 of Proposition 11 and the above observations, the following assertion holds true.

Proposition 12 *If* $\mathcal{D} \subset P(E)$ *is of Dirac type, then there is an open dense subset of* M *where the restriction of* $\mathcal{D}$ *is an almost Dirac structure.*

Some constructions and results presented in the paper for smooth distributions can be translated, by duality, to co-smooth distributions.

6. EXAMPLES

We give now some non-trivial examples of smooth singular distributions $\mathcal{D}$ that are locally non-separable. They are obtained as smooth orthogonal completions of some s.s.d.'s.

Consider a real function $f \in \mathcal{F}(M)$ such that its support is the whole M, *i.e.* its non-zero set is dense in M; for example f has an at most countable set of zeros. It is easy to see that the smooth orthogonal completion of a distribution $\mathcal{D}$ defined as the image of $P_1 = fI$, with f as above, is $\mathcal{D}$ itself, the smooth orthogonal $\mathcal{D}^{\vdash_g}$ is null and its orthogonal completion is $\mathcal{D}$. Examples of singular distributions of this type, even integrable, are given below.

On $I\!R^2$, consider the singular distribution $\mathcal{D}$ spanned by the vector field $\bar{X} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = (x,y,y,-x)$. The g-orthogonal distribution $\mathcal{D}^{\vdash_g}$, also singular, is spanned by the vector field $\bar{Y} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = (x,y,x,y)$. The endomorphisms of $TI\!R^2$, given by $P_1(x,y,X,Y) = (x,y,-(-yX+xY)y,(-yX+xY)x)$ and $P_2(x,y,X,Y) = (x,y,(xX+yY)x,(xX+yY)y)$ are orthogonal projectors on $\mathcal{D}$ and $\mathcal{D}^{\vdash_g}$ respectively. Let us consider the sum $P = P_1 + P_2 = (x^2+y^2)I_2$. It is easy to see that $P_1(x,y,1,0)$ and $P_1(x,y,0,1)$ generate $\mathcal{D}$, while $P_2(x,y,1,0)$ and $P_2(x,y,0,1)$ generate $\mathcal{D}^{\vdash_g}$. This example can be easy extended to a couple of endomorphisms on $I\!R^n$ having as images the singular distributions $\mathcal{D}$, tangent to the spheres centred in the origin and the origin itself as a singular point, and $\mathcal{D}^{\vdash_g}$, generated by the position vector field. For $n=3$, $\mathcal{D}$ is generated by the vector fields $\bar{X}_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} = (x,y,z,0,z,-y)$, $\bar{X}_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} = (x,y,z,-z,0,x)$ and $\bar{X}_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = (x,y,z,y,-x,0)$, while $\mathcal{D}^{\vdash_g}$ is generated by $X_0 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$.

Consider

$$P_1(\bar{X}) = (\bar{X}\cdot\bar{X}_1)\bar{X}_1 + (\bar{X}\cdot\bar{X}_2)\bar{X}_2 + (\bar{X}\cdot\bar{X}_3)\bar{X}_3,$$

or

$$\begin{aligned}P_1(x,y,z,X,Y,Z) = \Big(&x,y,z,(zY-yZ)(0,z,-y) + (-zX+xZ)(-z,0,x)\\ &+ (yX-xY)(y,x,0)) = (x,y,z,(y^2+z^2)X - xyY - xzZ,\\ &-xyX + \left(x^2+z^2\right)Y - yzZ, -xzX - yzY + \left(x^2+y^2\right)Z\Big)\end{aligned}$$

and
$P_2(\bar{X}) = (\bar{X}\cdot\bar{X}_0)\bar{X}_0 = (x,y,z,x^2X + xyY + xzZ, xyX + y^2Y + yzZ + xzX + yzY + z^2Z)$.

Then $P = P_1 + P_2$ has the form $P(\bar{X}) = (x^2+y^2+z^2)\bar{X}$. This example can be extended to $I\!R^n$, considering the singular distribution $\mathcal{D}$ spanned by the vector fields $\bar{X}_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}$, $1 \le i < j \le n+1$. Then $\mathcal{D}^{\vdash_g}$ is generated by $X_0 = x^i\frac{\partial}{\partial x^i}$.

An other example is on an open subset $U \subset I\!R^4$, with coordinates (x,y,z,t). Consider the distribution $\mathcal{D}$ generated by the vectors $X_1 = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial t}$ and $X_2 = u\frac{\partial}{\partial y} + v\frac{\partial}{\partial z}$, where $u,v : U \to I\!R$ are two real functions such that the set of their common zeros is an open dense subset of U; for example, $u = xy$ and $v = zt$. If the set of common zeros of u and v is void, then $\mathcal{D}$ is regular, otherwise $\mathcal{D}$ is a singular distribution. The distribution $\mathcal{D}^{\vdash_g}$ contains the distribution $\mathcal{D}'$ generated by the vector fields $Y_1 = v\frac{\partial}{\partial x} - u\frac{\partial}{\partial t}$ and $X_2 = v\frac{\partial}{\partial y} - u\frac{\partial}{\partial z}$. In fact, it can be proved that $\mathcal{D}^{\vdash_g} = \mathcal{D}'$.

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