DYNAMICAL SYMMETRIES OF GENERALIZED TAUB-NUT AND MULTI-CENTRE METRICS*

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Hidden symmetries of generalized Kaluza-Klein-type metrics are studied using van Holten's systematic analysis [1] based on Killing tensors. Applied to generalized Taub-NUT metrics, Kepler-type symmetries with associated Runge-Lenz-type conserved quantities are constructed. In the multi-centre case, the subclass of two-centre metrics gives rise to a conserved Runge-Lenz-type scalar, while no Kepler-type constant of the motion does exist for non aligned ($N \geq 3$)-centres. We also investigated the diatomic molecule system of Wilczek *et al.*, where "truly" non-Abelian gauge fields mimicking monopole-like fields arose. From the latter system we deduced a new conserved charge.

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1. INTRODUCTION TO KALUZA-KLEIN THEORIES

Kaluza-Klein (KK) theories have been extensively studied as schemes attempting to unify gravitation and gauge theory [2,3], through the physical assumption that the world admits, in addition to 4D space-time, an unobservable extra dimension. Thus, ordinary general relativity in five dimensions is considered to possess a local $U(1)$ gauge symmetry arising from a "vertical" translation along the hidden extra dimension.

A perfect illustration of the KK framework has been given by the Sorkin, then Gross and Perry solution of the vacuum Einstein equation, involving an Abelian monopole potential [4,5] which carries, unexpectedly, Kepler-type dynamical symmetries. Further examples are provided by the multi-centre metrics [6–9] for which similar hidden symmetry properties have been revealed.

Let us consider the 5D KK metric tensor,

$$
g_{AB} = \begin{pmatrix} \gamma_{\mu\nu} + V A_\mu A_\nu & A_\mu V \\ A_\nu V & V \end{pmatrix},
$$

$$
A, B = 0, \cdots, 4; \qquad \mu, \nu = 0, \cdots, 3; \qquad A_0 = 0,
$$

(1)

where the 5D manifold can be viewed as a direct product of a 4D space-time (where x^0 is the time coordinate) with an unobservable space-like loop, $\mathcal{M}^4 \otimes \mathcal{S}^1$, and where all components of g_{AB} are independent of the extra coordinate x^4. Here $\gamma_{\mu\nu}$ is the metric of the 4D manifold $\mathcal{M}^4$. From (1), the dynamics of a classical point-like test particle of unit mass is given by the 5D geodesic motion,

$$\frac{d^2 x^A}{d\tau^2} + \Gamma^A_{BC} \frac{dx^B}{d\tau} \frac{dx^C}{d\tau} = 0, \tag{2}$$

where τ denotes the proper time. Using the effective theory (1) in (2), a routine calculation yields the equations of the motion,

$$\begin{cases} \frac{d}{d\tau}\left(V A_\mu \frac{dx^\mu}{d\tau} + V \frac{dx^4}{d\tau}\right) = \frac{dq}{d\tau} = 0, \\ \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} - q F^\mu_\lambda \frac{dx^\lambda}{d\tau} - \frac{q^2}{2} \frac{\partial^\mu V}{V^2} = 0. \end{cases} \tag{3}$$

Here the x^μ are the coordinates on the 4D Manifold $\mathcal{M}^4$ while x^4 represents the "extra" coordinate. The first equation in (3) tells us that the "charge",

$$q = V\left(A_\mu \frac{dx^\mu}{d\tau} + \frac{dx^4}{d\tau}\right), \tag{4}$$

is conserved along the 5D geodesics. The latter can also be viewed as being associated with translations in the "extra" direction, generated by the Killing vector ∂_{x^4}. The second equation in (3) is a 4D geodesic equation involving an interaction with the scalar field V in addition to the Lorentz force . See [10–12] for further references.

The non-Abelian generalization of the 5D KK approach was given by Kerner [13] through generalizing the previous 5D manifold to a (4+d)-dimensional one, namely
$\mathcal{M} = \mathcal{M}^4 \otimes \mathcal{S}^d$ whose base space $\mathcal{M}^4$ is the usual space-time and where $\mathcal{S}^d$ represents an unobservable d-dimensional extra space. Here the (4+d)D diffeomorphism symmetry is broken to 4D infinitesimal coordinate transformations augmented with translations along the extra dimensions,

$$x^\mu \rightarrow x^\mu + \delta x^\mu, \quad y^a \rightarrow y^a + f^i(x^\nu)\xi^a_i(y), \tag{5}$$

where $f^i(x^\nu)$ are infinitesimal functions and ξ^a_i denoting the isometry generators on the compact manifold $\mathcal{S}^d$. The (4+d)D generalized metric, invariant under (5), then reads

$$\tilde{g}_{CD} = \begin{pmatrix} \gamma_{\mu\nu} + \kappa_{ab} B^a_\mu B^b_\nu & B^b_\mu \kappa_{ba} \\ \kappa_{ab} B^a_\nu & \kappa_{ab} \end{pmatrix},$$

where κ_{ab} represents the $SU(d-1)$ invariant metric on $\mathcal{S}^d$, and $B^a_\mu = A^b_\mu \xi^a_b$ includes the $SU(d-1)$ Lie algebra valued one-form A^b_μ, identified as a *Yang-Mills field*.

We can now deduce the geodesic equations which yield the equations of motion of an isospin-carrying particle in a curved space plus a Yang-Mills field,

$$
\begin{cases}
\mathcal{D}_\tau \mathcal{I}_c = \frac{d\mathcal{I}_c}{d\tau} - \mathcal{I}_a \varepsilon^a_{\ bc} A^b_\mu \frac{dx^\mu}{d\tau} = 0, \\
\frac{d^2 x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \gamma^{\nu\beta} F^b_{\mu\nu} \mathcal{I}_b \frac{dx^\mu}{d\tau} = 0 \,.
\end{cases}
\tag{6}
$$

Analogously to the Abelian case (3), the first equation in the pair (6) identifies the *classical isospin variable*,

$$
\mathcal{I}_a = \kappa_{ab} \left(dy^b/d\tau + A^b_\nu dx^\nu/d\tau \right),
$$

which is parallel transported and describes the motion in non-Abelian internal space. The isospin is analogous to the electric charge in the Abelian KK theory. Indeed, it is also obtained by contracting the Killing vector which generates the "vertical" translations with the direction field of the geodesic. The second equation in (6) still describes the motion in 4D real space. Note here the generalized Lorentz force

$$
\gamma^{\nu\beta} F^b_{\mu\nu} \mathcal{I}_b (dx^\mu/d\tau)
$$

due to the Yang-Mills field and with the electric charge replaced by the isospin.

The equations (6) are known as the *Kerner-Wong equations*, as they were also obtained by Wong [14] by "dequantizing" the Dirac equation. They can be derived from a variational principle [15]; alternatively, they can be studied using a symplectic approach [16, 17].

2. VAN HOLTEN'S METHOD TO DERIVE THE CONSTANTS OF THE MOTION

Now, inquiring about the symmetries of the KK-type metrics, we recall that constants of the motion denoted as Q, which are polynomial in the momenta, can be derived following van Holten's algorithm [1, 18]. The recipe is to expand Q into a power series of the covariant momentum,

$$
Q = C + C^i \Pi_i + \frac{1}{2!} C^{ij} \Pi_i \Pi_j + \frac{1}{3!} C^{ijl} \Pi_i \Pi_j \Pi_l + \cdots
$$

and to require Q to Poisson-commute with the Hamiltonian augmented with an effective potential, $\left\{ \mathcal{Q}, \mathcal{H} = \vec{\Pi}^2/2 + G(\vec{x}) \right\} = 0$. This yields the series of constraints,

$$
\begin{cases}
C^m \partial_m G = 0 & \text{o(0)} \\
\partial_n C = q F_{nm} C^m + C^m_n \partial_m G & \text{o(1)} \\
\mathcal{D}_{(i} C_{l)} = q(F_{im} C^m_l + F_{lm} C^m_i) + C^{\ \ k}_{il} \partial_k G & \text{o(2)} \\
\mathcal{D}_{(i} C_{lj)} = q F_{im} C^m_{lj} + q F_{jm} C^m_{il} \\
\qquad\qquad + q F_{lm} C^{\ \ m}_{ij} + C^{\ \ \ m}_{ijl} \partial_m G & \text{o(3)} \\
\ldots\ldots
\end{cases}
\tag{7}
$$

Here the zeroth-order constraint can be interpreted as a consistency condition for the effective potential. It is worth noting that the expansion can be truncated at a finite order provided some higher-order constraint reduces to a Killing equation,

$$\mathcal{D}_{(i_1} C_{i_2 \cdots i_n)} = 0, \tag{8}$$

where the covariant derivative is constructed with the Levi-Civita connection so that

$$\mathcal{D}_i C^j = \partial_i C^j + \Gamma^j_{ik} C^k. \tag{9}$$

Then $C_{i_1 \cdots i_p} = 0$ for all $p \geqslant n$ and the constant of the motion takes the polynomial form,

$$Q = \sum_{k=0}^{p-1} \frac{1}{k!} C^{i_1 \cdots i_k} \Pi_{i_1} \cdots \Pi_{i_k}. \tag{10}$$

It is worth noting that apart from zeroth-order conserved charges which are independent of the covariant momentum, all order-*n* invariants are deduced by the van Holten method (7) involving rank-*n* Killing tensors of the curved manifold [19]. Generating symmetries using Killing tensors was first advocated by Carter in the context of the Kerr metric [20].

In our set of constraints (7), a given Killing tensor is the highest-order coefficient of the expansion (10), allowing us to solve the truncated series of constraints and thus generating a conserved quantity. The intermediate-order constraints in (7) then determine the other coefficient-terms of the invariant (10).

In what follows, our strategy will be to find conditions for lifting to the "Kaluza-Klein" 4-space those Killing tensors which generate, in flat space, the conserved angular momentum and the Runge-Lenz vector of planetary motion, respectively.

3. HIDDEN SYMMETRIES OF GENERALIZED TAUB-NUT METRICS

Let us first investigate the symmetries of the [Abelian] Kaluza-Klein monopole. The latter, obtained by embedding the Taub-NUT gravitational instanton into Kaluza-Klein theory [4, 5, 21], provides us with an exact solution of four-dimensional Euclidean gravity approaching the vacuum solution at spatial infinity,

$$\begin{cases} ds^2 = g_{ij}(\vec{x}) dx^i dx^j + h(\vec{x}) \big(dx^4 + A_k dx^k \big)^2 \\ g_{ij}(\vec{x}) = f(\vec{x}) \delta_{ij}(\vec{x}) \end{cases} \tag{11}$$

where $f(\vec{x})$, $h(\vec{x})$ and A_k are real functions and the gauge potential of an Abelian magnetic field, respectively. The Lagrangian function associated reads

$$\mathcal{L} = \frac{1}{2} f(\vec{x}) \dot{\vec{x}}^2 + \frac{1}{2} f^{-1}(\vec{x}) \big(\frac{dx^4}{dt} + A_k \frac{dx^k}{dt} \big)^2 - U(r),$$

where $U(r)$ is an external scalar potential.

Inspired by Kaluza's hypothesis, as the fourth dimension x^4 is considered to be cyclic, we use the conservation of the "vertical" component of the momentum interpreted as a conserved electric charge,

$$p_4 = h(\vec{x})(dx^4/dt + A_k dx^k/dt) = q,$$

to reduce the four-dimensional problem to one in three dimensions, where we have strong candidates for the way dynamical symmetries act [22]. Then, the lifting problem can be conveniently solved using the van Holten technique (7) based on Killing tensors. The geodesic motion on the 4-manifold therefore projects onto the curved 3-manifold with the metric $g_{ij}(\vec{x})$, augmented with an effective potential,

$$V(\vec{x}) = q^2/2h(\vec{x}) + U(r),$$

such as

$$\begin{cases} \dot{x}^i = g^{ij}\Pi_j, \quad \Pi_j = p_j - q A_j, \\ \dot{\Pi}_i = q F_{ij}\dot{x}^j - \partial_i V + \Gamma_{ij}^k \Pi_k \dot{x}^j. \end{cases} \tag{12}$$

Note that the Lorentz equation in (12) involves also in addition to the monopole and potential terms, a curvature term which is quadratic in the velocity.

Let us now focus Kaluza-Klein-type metrics (11) whose hidden symmetries have been extensively investigated [6–8, 21–37]. For geodesic motion on hyperbolic space, for instance, Gibbons and Warnick [9] found a large class of systems admitting such hidden symmetries.

We assume that the metric (11) is radial,

$$f(\vec{x}) = f(r), \quad h(\vec{x}) = h(r),$$

and the magnetic field is that of a Dirac monopole of charge g. In that event, the generator of spatial rotations,

$$C_i = g_{ij}(r)\,\epsilon_{kl}^{j}\,n^k\,x^l, \tag{13}$$

applied to (7) provides us with the conserved angular momentum involving the typical monopole term,

$$\vec{J} = \vec{x} \times \vec{\Pi} - qg\frac{\vec{x}}{r}. \tag{14}$$

Turning to second order Runge-Lenz-type conserved quantities, we use the rank-2 Killing tensor,

$$C_{ij} = 2g_{ij}(r)n_k x^k - g_{ik}(r)n_j x^k - g_{jk}(r)n_i x^k, \tag{15}$$

inspired by its form in the Kepler problem. We therefore deduce from (7) that:

1. For the original Taub-NUT case [4, 5] with no external scalar potential,

$$f(r) = \frac{1}{h(r)} = 1 + \frac{4m}{r}, \tag{16}$$

where m is real* [7,8] we obtain, for the energy $\mathcal{E}$ and the charge $g = \pm 4m$, the conserved Runge-Lenz vector,

$$\vec{K} = \vec{\Pi} \times \vec{J} - 4m\left(\mathcal{E} - q^2\right)\frac{\vec{x}}{r}. \tag{17}$$

2. Lee and Lee [38] argued that for monopole scattering with independent Higgs expectation values, the geodesic Lagrangian derived from (11) should be replaced by $\mathcal{L} \to \mathcal{L} - W(r)$, where

$$W(r) = \frac{1}{2}\frac{a_0{}^2}{1 + \frac{4m}{r}}. \tag{18}$$

It is easy to see that this addition merely shifts the value of the energy by a constant $a_0{}^2/2$, so the previously found Runge-Lenz vector (17) is still valid.

3. The metric associated with winding strings [39],

$$f(r) = 1, \qquad h(r) = \frac{1}{\left(1 - \frac{1}{r}\right)^2}, \tag{19}$$

with charge $g = \pm 1$, leads to the conserved Runge-Lenz vector,

$$\vec{K} = \vec{\Pi} \times \vec{J} - q^2\frac{\vec{x}}{r}. \tag{20}$$

4. The extended Taub-NUT metric [25, 26]

$$f(r) = b + \frac{a}{r}, \quad h(r) = \frac{ar + br^2}{1 + dr + cr^2}, \tag{21}$$

with (a, b, c, d) real. With no external scalar potential and charge $g = \pm 1$, we have the conserved Runge-Lenz vector,

$$\vec{K} = \vec{\Pi} \times \vec{J} - (a\mathcal{E} - \frac{1}{2}dq^2)\frac{\vec{x}}{r}. \tag{22}$$

5. Considering the oscillator-type metric discussed by Iwai and Katayama [25], the functions $f(r)$ and $h(r)$ take the particular form

$$f(r) = b + ar^2, \ h(r) = \frac{ar^4 + br^2}{1 + cr^2 + dr^4}. \tag{23}$$

A direct calculation leads to the following Runge-Lenz-type vector [25] ,

$$\vec{K} = \left(b + ar^2\right)\dot{\vec{x}} \times \vec{J} + \beta\frac{\vec{x}}{r}, \tag{24}$$

conserved only for a scalar potential of the form

$$W(r) = (\frac{q^2g^2}{2r^2} + \frac{\beta}{r})\frac{1}{f(r)} - \frac{q^2}{h(r)}. \tag{25}$$

*Monopole scattering corresponds to $m = -1/2$ [21].

4. N-CENTRE METRICS WITH SCALAR CONSTANTS OF THE MOTION

The multi-centre metrics in which we are interested here are Euclidean vacuum solutions of the Einstein equations with self-dual curvature; they can be viewed as generalizations of the previously investigated Taub-NUT metrics.

Let us consider a particle moving in Gibbons-Hawking space [40]. The Lagrangian function associated with this dynamical system, derived from (11), is

$$\mathcal{L} = \frac{1}{2} f(\vec{x}) \dot{\vec{x}}^2 + \frac{1}{2} f^{-1}(\vec{x}) \left(\frac{dx^4}{dt} + A_k \frac{dx^k}{dt} \right)^2 - U(\vec{x}),$$

where the functions $f(\vec{x})$ obey the "self-dual" condition $\vec{\nabla} f(\vec{x}) = \pm \vec{\nabla} \times \vec{A}$. $f(\vec{x})$ satisfies therefore the three-dimensional Laplace equation,

$$\Delta f(\vec{x}) = 0,$$

whose most general solution is given by

$$f(\vec{x}) = f_0 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|}, \ (f_0, m_i) \in \mathbb{R}^{N+1}.$$

The multi-centre metric admits multi-NUT singularities so that the position of the i-th NUT singularity with the charge m_i is $\vec{a}_i$. These singularities can be removed provided to have all NUT charges equal. In this case, the cyclic variable x^4 is periodic with range $0 \leq x^4 \leq 4\pi m$. We are thus interested in the projection of the motion on the curved 3-manifold whose metric is,

$$g_{jk}(\vec{x}) = \left(f_0 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|} \right) \delta_{jk}. \tag{26}$$

For simplicity, we limit ourselves to two-centre metrics,

$$f(\vec{x}) = f_0 + \frac{m_1}{|\vec{x} - \vec{a}|} + \frac{m_2}{|\vec{x} + \vec{a}|}. \tag{27}$$

Turning to rotational symmetry, the rank-1 Killing tensor satisfying the second-order equation in (7),

$$C_i = g_{im} \epsilon^m_{\ lk} \hat{a}^l x^k, \quad \hat{a} = \vec{a}/a, \tag{28}$$

generates rotational symmetry around the axis through the two centres. The corresponding conserved quantity,

$$\mathcal{J}_a = (\vec{x} \times \vec{\Pi}) \cdot \hat{a} - q(f(\vec{x}) - f_0) \vec{x} \cdot \hat{a} + \frac{qam_1}{|\vec{x} - \vec{a}|} - \frac{qam_2}{|\vec{x} + \vec{a}|},$$

is the *projection of the angular momentum* onto the axis of the two centres.

Now we study quadratic conserved quantities by considering the reducible rank-2 Killing tensor,

$$C_{ij} = 2g_{im}g_{jn}\epsilon^m_{\ lk}\epsilon^n_{\ pq}\hat{a}^l\hat{a}^p x^k x^q + 2g_{il}g_{jm}\hat{a}^l\hat{a}^m, \tag{29}$$

which is a symmetrized product of the Killing-Yano tensors, $C_i = g_{im}\epsilon^m_{\ lk}\,\hat{a}^l x^k$ generating rotations around the axis of the two centres and $C_j = g_{jm}\hat{a}^m$ generating spatial translation along the axis of the two centres. Injecting (29) into the system of constraint (7) yields, for vanishing effective potential $U = 0$, the Casimir, which combines the *square of the projected angular momentum with the square of the component of the covariant momentum along the axis*,

$$Q = \mathcal{J}_a^2 + \Pi_a^2. \tag{30}$$

As expected from the construction of the associated Killing tensor [6], the obtained conserved quantity is a combination of two constant of the motion [27].

Now introducing into (7) the rank two Killing tensor generating Kepler-type dynamical symmetry provides us with the conserved *scalar*,

$$K_a = \left(\vec{\Pi} \times \vec{J}\right) \cdot \frac{\vec{a}}{a} + \frac{\beta}{q}\left(\mathcal{L}_a - \mathcal{J}_a\right), \tag{31}$$

where $\beta \in \mathbb{R}$. K_a therefore represents, for the two-centre metrics (26), a conserved Runge-Lenz-type *scalar* for particle motion confined onto the "Appolonius" two-sphere[†] [22] of centre at $\vec{a}\rho$ and with radius $R = a\sqrt{\rho^2 - 1}$, provided the effective potential is of the form,

$$W = \frac{q^2}{2}(f(\vec{x}) - f_0)^2 + \beta(f(\vec{x}) - f_0) + \gamma \tag{32}$$

γ a constant, which satisfies the consistency condition given by the zeroth-order constraint of (7).

It is worth mentioning that a scalar Runge-Lenz-type conserved quantity does exist only for a particle moving along the axis of the two centres, or for motions confined onto the "Apollonius" two-sphere defined above. In the Eguchi-Hanson case $m_1 = m_2$, and the 2-sphere is replaced by the median plane of the two centres [22].

The $(N \geq 3)$-centres metrics of the form

$$g_{jk}(\vec{x}) = f(\vec{x})\delta_{jk} = (f_0 + \sum_{i=1}^{N \geq 3} \frac{m_i}{|\vec{x} - \vec{a}_i|})\delta_{jk} \tag{33}$$

can also be investigated, but they carry no Runge-Lenz-type symmetry for $N \geq 3$ non aligned centres. To obtain this result, let us first generalize the rank-2 Killing

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[†]This sphere was known already by Apollonius of Perga in the 2th century BC.

tensor (15) generating the Runge-Lenz vector as

$$C_{ij} = 2g_{ij}(\vec{x})n_k x^k - g_{ik}(\vec{x})n_j x^k - g_{jk}(\vec{x})n_i x^k \,. \tag{34}$$

Requiring the Killing equation to be satisfied as

$$\mathcal{D}_{(k} C_{ij)} = 0, \tag{35}$$

a tedious calculation then provides us with the condition

$$\vec{n} \times \left(\vec{x} \times \vec{\nabla} f(\vec{x}) \right) = 0\,. \tag{36}$$

For the metric (33) above this requires

$$\sum_{i=1}^{N} \frac{\left(\vec{n}\cdot\vec{x}\right)\vec{a}_i - \left(\vec{n}\cdot\vec{a}_i\right)\vec{x}}{|\vec{x} - \vec{a}_i|^3} = 0, \tag{37}$$

which can not be satisfied for more than two non-aligned centres. The only possibility we get for (37) to be satisfied by more than two centres is to have all of them in the same alignment describing a straight line of centres.

5. THE NON-ABELIAN CASE : THE DIATOMIC MOLECULE

In Ref. [41] Moody, Shapere and Wilczek have shown that nuclear motion in a diatomic molecule can be described, in the Born-Oppenheimer approximation, by an effective non-Abelian gauge field of "hedgehog" form,

$$A_i^a = (1-\kappa)\epsilon_{iaj} \frac{x^j}{r^2}, \quad F_{ij}^a = (1-\kappa^2)\epsilon_{ijk}\frac{x^k x^a}{r^4} \,. \tag{38}$$

This gauge field mimics the structure of that of a non-Abelian monopole [42, 43]. Note here *the unquantized constant real factor* $(1-\kappa^2)$. The potential (38) becomes that of a Wu-Yang [*i.e.*, an embedded Dirac] monopole of unit charge when $\kappa = 0$; for other values of κ, it is a truly non-Abelian configuration – except for $\kappa = \pm 1$, when it is a gauge transform of the vacuum.

Now we investigate the symmetries of an isospin-carrying particle carrying unit charge, evolving in the monopole-like field of the diatom (38) plus a scalar potential. The Hamiltonian describing the dynamics of this particle is finally expressed as

$$\mathcal{H} = \frac{\vec{\pi}^2}{2} + V(\vec{x}, \vec{\pi}, \mathcal{I}^a), \quad \pi_i = p_i - A_i^a \mathcal{I}^a \,.$$

Defining the covariant Poisson-brackets as

$$\left\{ M, N \right\} = D_j M \frac{\partial N}{\partial \pi_j} - \frac{\partial M}{\partial \pi_j} D_j N + \mathcal{I}^a F_{jk}^a \frac{\partial M}{\partial \pi_j} \frac{\partial N}{\partial \pi_k} - \epsilon_{abc} \frac{\partial M}{\partial \mathcal{I}^a} \frac{\partial N}{\partial \mathcal{I}^b} \mathcal{I}^c,$$

the non-vanishing brackets are

$$\{x^i, \pi_j\} = \delta_j^i, \quad \{\pi_i, \pi_j\} = \mathcal{I}^a F_{ij}^a, \quad \{\mathcal{I}^a, \mathcal{I}^b\} = -\epsilon_{abc}\mathcal{I}^c.$$

The equations of motion governing an isospin-carrying particle in the static non-Abelian gauge field (38) read

$$\begin{cases} \ddot{x}_i - \mathcal{I}^a F_{ij}^a \dot{x}^j + D_i V = 0, \\ \dot{\mathcal{I}}^a + \epsilon_{abc} \mathcal{I}^b (A_j^c \dot{x}^j - \frac{\partial V}{\partial \mathcal{I}^c}) = 0. \end{cases} \tag{39}$$

These equations generalize the Kerner-Wong equations (6) to an additional scalar potential.

Turning to the conserved quantities constructed with the van Holten algorithm, the zeroth-order conserved charge which used-to-be interpreted as electric charge for $\kappa \neq 0$,

$$Q = \frac{\vec{x} \cdot \vec{\mathcal{I}}}{r}, \tag{40}$$

is *not more covariantly conserved* in general,

$$\{Q, \mathcal{H}\} = \vec{\pi} \cdot \vec{D} Q, \quad D_j Q = \frac{\kappa}{r}(\mathcal{I}^j - Q\frac{x_j}{r}). \tag{41}$$

Detailed calculation shows that the equation $D_j Q = 0$ can only be solved, for an embedded Abelian monopole field, when $\kappa = 0, \pm 1$.

Nor is Q^2 conserved, $\{Q^2, \mathcal{H}\} = 2\kappa Q(\vec{\pi} \cdot \vec{D} Q)$. but note that, unlike Q^2, the length of the isospin, $\mathcal{I}^2$, *is* conserved,

$$\{\mathcal{H}, \mathcal{I}^2\} = 0. \tag{42}$$

These results are consistent with those in [44].

Turning to linear conserved quantities, we use the Killing vector generating spatial rotations,

$$\vec{C} = \vec{n} \times \vec{x}, \tag{43}$$

to build up the conserved angular momentum,

$$\vec{J} = \vec{x} \times \vec{\pi} - (1 - \kappa) Q \frac{\vec{x}}{r} - \kappa \vec{\mathcal{I}}. \tag{44}$$

Moody, Shapere and Wilczek [41] did find this expression for $\kappa = 0$ but, as they say it, "they are not aware of a canonical derivation when $\kappa \neq 0$". Our construction here is an alternative to that of Jackiw [45], who obtained it using the method of Ref. [46].

We now consider the rank-2 Killing tensor,

$$C_{ij} = 2\delta_{ij} x^2 - 2x_i x_j, \tag{45}$$

which satisfies the third-order constraint in (7). The Killing tensor (45) thus yields the *conserved* Casimir,

$$L^2 = (\vec{x} \times \vec{\pi})^2 = x^2 \vec{\pi}^2 - (\vec{x} \cdot \vec{\pi})^2, \tag{46}$$

which is the *square* of *non-conserved orbital angular momentum*, $\vec{L} = \vec{x} \times \vec{\pi}$. Since J^2 and L^2 are both conserved, the new charge,

$$\Gamma = J^2 - L^2 = \left(1 - \kappa\right)^2 Q^2 - \kappa^2 \vec{\mathcal{I}}^2 - 2\kappa \vec{J} \cdot \vec{\mathcal{I}}, \tag{47}$$

is conserved for motion in the monopole-like field of diatomic molecule. *It is worth noting that the charge Γ becomes, in the Abelian limit $\kappa = 0$, the square of conserved electric charge.*

Just like $\vec{J}$, J^2 and L^2, the charge Γ is conserved for any radially symmetric potential, $V(r)$. Γ can also be obtained by using the Killing vector,

$$\vec{C} = 2\kappa \left(\vec{x} \times \vec{\mathcal{I}}\right), \tag{48}$$

in the van Holten algorithm (7). We note at last, that no Runge-Lenz-type conserved quantity could be found in this case except in the Abelian case, cf. [18].

6. CONCLUSION

We studied the geodesic motion of a particle in Kaluza-Klein-type monopole spaces and in its Gibbons-Hawking generalization. As illustrations, we treated in detail the generalized Taub-NUT metrics, for which we derived Runge-Lenz-type vectors. We considered the subclass of two-centre metrics into which a conserved Runge-Lenz-type scalar has been revealed in the special case of motions confined onto a particular "Apollonius" sphere. For the $(N \geq 3)$-centre metrics, we demonstrated that no symmetry of the Kepler-type occurs for non-aligned centres. We also treated the case of the effective "truly" non-Abelian monopole-like field generated by nuclear motion in a diatomic molecule. This system is due to Wilczek *et al.*, where *despite the non-conservation of the electric charge (41)*, we surprisingly constructed, in addition to the "unusual" angular momentum (44), a new conserved charge (47).

It is worth mentioning that apart from the generic importance of constructing conserved quantities to confine classical trajectories to conic sections as in the case of Kepler-type systems, the existence of quadratic conserved quantity like Runge-Lenz vector yields, in particular, the separability of the Hamilton-Jacobi equation for the generalized Taub-NUT and two-centre metrics.

"Hidden" symmetries also play a rôle in quantum mechanics. The system may be quantized by the usual procedure of replacing Poisson brackets with commutators and in this way the energy levels and degeneracies of Kepler-type systems may be found using dynamical symmetries.

We mention in conclusion that the van Holten's method can also be extended to study supersymmetries, cf. [47].

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