GEOMETRIC ALGEBRA AND M–THEORY COMPACTIFICATIONS*

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Received February 20, 2013

We show how supersymmetry conditions for flux compactifications of supergravity and string theory can be described in terms of a flat subalgebra of the Kähler-Atiyah algebra of the compactification space, a description which has wide-ranging applications. As a motivating example, we consider the most general M-theory compactifications on eight-manifolds down to AdS_3 spaces which preserve $N = 2$ supersymmetry in 3 dimensions. We also give a brief sketch of the lift of such equations to the cone over the compactification space and of the geometric algebra approach to 'constrained generalized Killing pinors', which forms the technical and conceptual core of our investigation.

Key words: string theory compactifications, M-theory, supergravity, supersymmetry, differential geometry.

PACS: 11.25.Mj, 11.25.Yb, 04.65.+e, 11.30.Pb, 02.40.-k.

1. INTRODUCTION

Despite their physical relevance, the most general $N = 2$ warped product compactifications of 11-dimensional supergravity down to a three-dimensional Anti-de-Sitter space have not previously been studied in detail. In such compactifications, the internal eight-manifold M carries a Riemannian metric g as well as a one-form f and a four-form F, the latter two of which encode the 4-form field strength of the eleven-dimensional theory. The internal Majorana pinor is a section of the real pin bundle S of M, which is a real vector bundle of rank 16. The condition that such a background preserves exactly $N = 2$ supersymmetry in 3 dimensions amounts to the requirement that the real vector space of solutions to the following algebro-differential system (the so-called *constrained generalized Killing (CGK) pinor equations*):

$$D\xi = Q\xi = 0 \quad (1)$$

*Paper presented at "The 8^{th} Workshop on Quantum Field Theory and Hamiltonian Systems", September 19-22, 2012, Craiova, Romania.

has dimension two. Here, Q is an endomorphism of S given by:

$$Q = \frac{1}{2}\gamma^m \partial_m \Delta - \frac{1}{288} F_{mpqr}\gamma^{mpqr} - \frac{1}{6} f_p \gamma^p \gamma^{(9)} - \kappa \gamma^{(9)}$$

while $D = \nabla^S + A$ is a linear connection on S, where ∇^S is the connection induced on S by the Levi-Civita connection of (M,g) and $A = \mathrm{d}x^m \otimes A_m \in \Omega^1(M, \mathrm{End}(S))$ is an $\mathrm{End}(S)$-valued one form on M, with:

$$A_m = \frac{1}{4} f_p \gamma_m{}^p \gamma^{(9)} + \frac{1}{24} F_{mpqr}\gamma^{pqr} + \kappa \gamma_m \gamma^{(9)} \in \Gamma(M, \mathrm{End}(S)) \ .$$

The quantity κ is a positive parameter related to the cosmological constant Λ of the AdS_3 space through $\Lambda = -8\kappa^2$. The requirement of having $N = 2$ supersymmetry in three-dimensions does not impose any chirality condition on ξ. If one imposes such a condition as an *extraneous* technical assumption (for example, if one adds the condition $\gamma^{(9)}\xi = +\xi$ to the system (1)), then one obtains a drastic simplification leading to the well-known results of [1]. For a number of reasons (having to do, in particular, with efforts to generalize F-theory) we are interested in studying the problem without imposing any such chirality constraint. This leads to unexpected complications, which can be resolved upon re-formulating the problem by using geometric algebra techniques.

2. THE GEOMETRIC ALGEBRA APPROACH TO PINORS

The standard construction of the pin bundle S of a (pseudo)-Riemannian manifold (M,g) of signature (p,q) and dimension $d = p+q$ can be described most briefly by saying that S is a bundle of modules over the Clifford bundle $\mathrm{Cl}(T^*M)$ of the cotangent bundle of M — where T^*M is, of course, endowed with the metric $\hat{g}$ induced by g *. One problem with this approach (which manifests itself in many subtle aspects of spin geometry as constructed in [3]) is that the Clifford bundle is determined by (M,g) only up to isomorphism and hence the association of $\mathrm{Cl}(T^*M)$ to (M,g) is *not* functorial. The issue can be resolved by using a particular realization of the Clifford bundle (going back to Chevalley [4] and Riesz [5]) which is known as the Kähler-Atiyah bundle of (M,g). This removes the ambiguities of the standard approach to spin geometry, since the Kähler-Atiyah bundle of (M,g) is functorially determined by (M,g). The Chevalley-Riesz realization identifies the underlying vector bundle of $\mathrm{Cl}(T^*M)$ with the exterior bundle $\wedge T^*M$ of M, transporting the Clifford product of the former to a non-commutative but unital and associative fiberwise multiplication on the latter which we denote by $\diamond$ and call the

*This is equivalent [2] with giving S as the vector bundle associated with a Clifford^c-structure of M via a representation of the Clifford^c group.

geometric product of (M,g). The geometric product makes $\wedge T^*M$ into the *Kähler-Atiyah bundle* $(\wedge T^*M,\diamond)$, a bundle of associative algebras which is *naturally* (*i.e.* functorially) determined by (M,g). The geometric product (which depends on g) is not homogeneous with respect to the natural $\mathbb{Z}$-grading (given by rank) of the exterior bundle. However, it admits an expansion into a finite sum of binary operations $\triangle_k$ ($k=0\dots d$) which are homogeneous of degree $-2k$ with respect to that grading:

$$\diamond = \sum_{k=0}^{\left[\frac{d}{2}\right]}(-1)^k\,\triangle_{2k} + \sum_{k=0}^{\left[\frac{d-1}{2}\right]}(-1)^{k+1}\,\triangle_{2k+1}\circ(\pi\otimes \mathrm{id}_{\wedge T^*M}) \ , \qquad (2)$$

where π is the parity automorphism, which is defined through:

$$\pi \overset{\mathrm{def.}}{=} \oplus_{k=0}^{d}(-1)^k \mathrm{id}_{\wedge^k T^*M} \ .$$

The binary products $\triangle_k : \wedge T^*M \times_M \wedge T^*M \to T^*M$ are known as *generalized products*. Expansion (2) can be viewed as the semiclassical expansion of the geometric product when the latter is identified with the star product arising in a certain 'vertical' geometric quantization procedure in which the role of the Planck constant is played by the inverse of the overall scale of the metric g. In particular, the classical limit corresponds to $g\to\infty$ and $\diamond$ reduces to the wedge product $\wedge = \triangle_0$ in that limit. The other generalized products $\triangle_k$ ($k>0$) depend on g, being determined on sections of $\wedge T^*M$ by the recursion formula:

$$\omega\,\triangle_{k+1}\,\eta = \frac{1}{k+1} g^{ab}(e_a \lrcorner \omega)\,\triangle_k\,(e_b \lrcorner \eta) = g_{ab}(\iota_{e^a}\omega)\,\triangle_k\,(\iota_{e^b}\eta) \ ,$$

where ι denotes the so-called *interior product* [6].

The pin bundle S can now be viewed as a bundle of modules over the Kähler-Atiyah bundle of (M,g), where the module structure is defined by a morphism of bundle of algebras which we denote by $\gamma : (\wedge T^*M,\diamond) \to (\mathrm{End}(S),\circ)$. Since we are interested in pinors of spin $1/2$, we assume that γ is fiberwise irreducible.

Notations and conventions. We let $(e_m)_{m=1\dots 8}$ denote a local frame of TM, defined on some open subset $U\subset M$ and e^m be the dual local coframe (local frame of T^*M), which satisfies $e^m(e_n)=\delta^m_n$ and $\hat{g}(e^m,e^n)=g^{mn}$, where (g^{mn}) is the inverse of the matrix (g_{mn}). The space of smooth inhomogeneous globally-defined differential forms on M is denoted by $\Omega(M) \overset{\mathrm{def.}}{=} \Gamma(M,\wedge T^*M)$. The fixed rank components of the graded module $\Omega(M)$ are denoted by $\Omega^k(M)=\Gamma(M,\wedge^k T^*M)$, with $k=0,\dots,\dim M$. A general inhomogeneous form $\omega\in\Omega(M)$ expands as:

$$\omega = \sum_{k=0}^{d}\omega^{(k)} =_U \sum_{k=0}^{d}\frac{1}{k!}\omega^{(k)}_{a_1\dots a_k}e^{a_1\dots a_k} \ \text{with} \ \omega^{(k)}\in\Omega^k(M) \ , \qquad (3)$$

where $e^{a_1\ldots a_k} \stackrel{\text{def.}}{=} e^{a_1}\wedge\ldots\wedge e^{a^k}$ and the symbol $=_U$ means that equality holds only after restriction of ω to U . A bundle of real *pinors* over M is an $\mathbb{R}$-vector bundle S over M which is (compatibly) a bundle of modules over the Clifford bundle $\mathrm{Cl}(T^*M)$. Similarly, a bundle of real *spinors* is a bundle of modules over the even Clifford bundle $\mathrm{Cl}^{\mathrm{ev}}(T^*M)$. Of course, a bundle of real pinors is automatically a bundle of real spinors. Hence any pinor is naturally a spinor but the converse need not hold. In this paper, we focus on the case of *pinors* and in particular on the case when S is a bundle of *simple* modules over $\mathrm{Cl}(T^*M)$. We let $\gamma^m \stackrel{\text{def.}}{=} \gamma(e^m)\in\Gamma(U,\mathrm{End}(S))$ and $\gamma_m \stackrel{\text{def.}}{=} g_{mn}\gamma^n\in\Gamma(U,\mathrm{End}(S))$ be the contravariant and covariant 'gamma matrices' associated with the local orthonormal coframe e^m of M and let $\gamma_{m_1\ldots m_k}$ denote the complete antisymmetrization of the composition $\gamma_{m_1}\circ\ldots\circ\gamma_{m_k}$.

Twisted (anti-)selfdual forms. Assuming that M is oriented, we let ν denote the volume form of (M,g). We concentrate on the case when $\nu\diamond\nu=+1$ (this happens, in particular, when M is an eight- or nine-manifold endowed with a Riemannian metric — the two cases which will be relevant for our application). With this assumption, we have the bundle decomposition:

$$\wedge T^*M = (\wedge T^*M)^+\oplus(\wedge T^*M)^- \ \ ,$$

where the spaces of sections of the bundles $(\wedge T^*M)^\pm$ of *twisted (anti-)selfdual forms* are the following $\mathcal{C}^\infty(M,\mathbb{R})$-submodules of $\Omega(M)$:

$$\Omega^\pm(M) \stackrel{\text{def.}}{=} \Gamma(M,(\wedge T^*M)^\pm) = \{\omega\in\Omega(M)|\omega\diamond\nu=\pm\omega\} \ \ .$$

When g is a Riemannian metric, the bundle morphism $\gamma:\wedge T^*M\rightarrow\mathrm{End}(S)$ is injective iff. $d\not\equiv_8 1,5$. When $d\equiv_8 1,5$, we have $\gamma(\nu)=\epsilon_\gamma\mathrm{id}_S$, where $\epsilon_\gamma\in\{-1,1\}$ is a sign factor known [6] as the *signature of* γ . In those cases, we have $\gamma|_{(\wedge T^*M)^{-\epsilon_\gamma}}=0$ while the restriction $\gamma|_{(\wedge T^*M)^{+\epsilon_\gamma}}$ is injective. To uniformly treat all cases, we set:

$$(\wedge T^*M)^\gamma \stackrel{\text{def.}}{=} \begin{cases} \wedge T^*M \ , & \text{if } d\not\equiv_8 1,5 \\ (\wedge T^*M)^{+\epsilon_\gamma} \ , & \text{if } d\equiv_8 1,5 \end{cases}$$

and $\Omega^\gamma(M) \stackrel{\text{def.}}{=} \Gamma(M,(T^*M)^\gamma)$, which is a subalgebra of the Kähler-Atiyah algebra.

Dequantization. In our applications — when M is a Riemannian eight-manifold (the compactification space of M-theory down to 3 dimensions) or a nine-manifold (the metric cone over an eight-dimensional compactification space) — the fiberwise representation given by γ is equivalent with an irreducible representation of the real Clifford algebra $\mathrm{Cl}(8,0)$ or $\mathrm{Cl}(9,0)$ in a 16-dimensional $\mathbb{R}$-vector space, which happens to be surjective. Due to this fact, the map $\gamma^{-1} \stackrel{\text{def.}}{=} (\gamma|_{(\wedge T^*M)^\gamma})^{-1}:\mathrm{End}(S)\rightarrow$

$(\wedge T^*M)^\gamma$ can be used to identify the bundle of endomorphisms of S with the bundle of algebras $((\wedge T^*M)^\gamma, \diamond)$. In particular, every globally-defined endomorphism $T \in \Gamma(M, \mathrm{End}(S))$ admits a *dequantization* $\check{T} \stackrel{\text{def.}}{=} \gamma^{-1}(T) \in \Omega^\gamma(M)$, which is a (generally inhomogeneous) differential form defined on M. Furthermore, the dequantization of a composition $T_1 \circ T_2$ equals the geometric product $\check{T}_1 \diamond \check{T}_2$ of the dequantizations of $T_1, T_2 \in \Gamma(M, \mathrm{End}(S))$.

The Fierz isomorphism. When the Schur algebra [6] of $\mathrm{Cl}(p,q)$ is isomorphic with $\mathbb{R}$ (*i.e.* when γ is surjective), one can define an isomorphism of bundles of algebras $\check{E} : (S \otimes S, \bullet) \xrightarrow{\sim} ((\wedge T^*M)^\gamma, \diamond)$ called *the Fierz isomorphism*, where $(S \otimes S, \bullet)$ is a bundle of algebras known as the *bipinor bundle*. On sections, this induces an isomorphism of $\mathcal{C}^\infty(M,\mathbb{R})$-algebras $\check{E} : (\Gamma(M, S \otimes S), \bullet) \xrightarrow{\sim} (\Omega^\gamma(M), \diamond)$ which identifies the *bipinor algebra* $(\Gamma(M, S \otimes S), \bullet)$ with the subalgebra $(\Omega^\gamma(M), \diamond)$ of the Kähler-Atiyah algebra. Both $\check{E}$ and the multiplication $\bullet$ of the bipinor algebra depend on the choice of an *admissible* [7, 8] pairing $\mathscr{B}$ on S. In our application (when M is an eight- or nine-dimensional Riemannian manifold), $\mathscr{B}$ is a certain admissible bilinear pairing on S which is positive-definite and symmetric. We define $\check{E}_{\xi,\xi'} \stackrel{\text{def.}}{=} \check{E}(\xi \otimes \xi') \in \Omega^\gamma(M)$, where $\xi, \xi' \in \Gamma(M, S)$.

Constrained generalized Killing forms. Using properties of the Fierz isomorphism, the algebraic constraint $Q\xi = 0$ and the generalized Killing pinor equations $D\xi = 0$ translate [6] into the following conditions on the inhomogeneous differential forms $\check{E}_{\xi,\xi'}$, which hold for any global sections $\xi, \xi' \in \Gamma(M, S)$ satisfying (1):

$$\check{D}^{\mathrm{ad}} \check{E}_{\xi,\xi'} = \check{Q} \diamond \check{E}_{\xi,\xi'} = 0 \ . \tag{4}$$

Here, $\check{Q} \stackrel{\text{def.}}{=} \gamma^{-1}(Q) \in \Omega(M)$ is the 'dequantization' of the globally-defined endomorphism $Q \in \Gamma(M, \mathrm{End}(S))$ and $\check{D}^{\mathrm{ad}} = e^m \otimes \check{D}^{\mathrm{ad}}_m$ is the 'adjoint dequantization' of D (see [6]). The operators $\check{D}^{\mathrm{ad}}_m$ are even derivations of the Kähler-Atiyah algebra which are defined through:

$$\check{D}^{\mathrm{ad}}_m \stackrel{\text{def.}}{=} \nabla_m + [\check{A}_m, \ ]_{-,\diamond} \ ,$$

where $\check{A}_m \stackrel{\text{def.}}{=} \gamma^{-1}(A_m)$ and ∇ is the connection induced on $\wedge T^*M$ by the Levi-Civita connection of (M, g). The Fierz identities between the form-valued pinor bilinears $\check{E}_{\xi,\xi'}$ take the concise form [6]:

$$\check{E}_{\xi_1,\xi_2} \diamond \check{E}_{\xi_3,\xi_4} = \mathscr{B}(\xi_3, \xi_2) \check{E}_{\xi_1,\xi_4} \ , \ \ \forall \xi_1, \xi_2, \xi_3, \xi_4 \in \Gamma(M, S) \ ,$$

defining a certain subalgebra of the Kähler-Atiyah algebra of (M, g).

Equations (4) generalize the usual theory of Killing forms in a number of different directions and can be taken as a starting point for a mathematical theory which is of interest in its own right. When expanding the geometric product into generalized products as in (2), these seemingly innocuous equations become a highly non-trivial system whose analysis would be extremely difficult without recourse to the synthetic formulation given above in terms of Kähler-Atiyah algebras. In particular, the geometric algebra formulation given here allows one to easily extract structural properties of such equations and to study them using techniques familiar from the theory of non-commutative algebras and modules over such – thereby providing an interesting connection between spin geometry and non-commutative algebraic geometry. We stress that equations (4) apply in much more general situations than those considered in this brief summary.

3. THE CGK EQUATIONS FOR METRIC CONES

As explained in [9], it is convenient to lift ξ to the metric cone over M, which can be viewed as the warped product $(\hat{M}, g_{\mathrm{cone}}) \approx ((0,\infty), \mathrm{d}r^2) \times_r (M,g)$ (of warp factor equal to r):

$$\mathrm{d}s^2_{\mathrm{cone}} = \mathrm{d}r^2 + r^2 \mathrm{d}s^2 \quad .$$

The one-form

$$\theta \stackrel{\mathrm{def.}}{=} \mathrm{d}r = \partial_r \lrcorner g_{\mathrm{cone}}$$

has unit norm with respect to the cone metric. The pin bundle $\hat{S}$ of the cone can be identified with the pull-back of S through the natural projection $\Pi : \hat{M} \to M$. We define the *lift* $\hat{D}$ of D to be the connection on $\hat{S}$ obtained from D by pull-back to the cone. Then $\hat{D}$ can be expressed as:

$$\hat{D} = \nabla^{\hat{S},\mathrm{cone}} + A^{\mathrm{cone}} \quad , \tag{5}$$

where $\nabla^{\hat{S},\mathrm{cone}}$ is the connection induced on $\hat{S}$ by the Levi-Civita connection of g_{cone}. Since the metric cone $(\hat{M}, g_{\mathrm{cone}})$ over (M,g) has signature $(9,0)$ and since $9-0 \equiv_8 1$, the Clifford algebra $\mathrm{Cl}(9,0)$ corresponds to the normal non-simple case discussed in [6]. In particular, its Schur algebra equals the base field $\mathbb{R}$ and the corresponding pin representation $\gamma_{\mathrm{cone}} : (\wedge T^*\hat{M}, \diamond^{\mathrm{cone}}) \to \mathrm{End}(\hat{S})$ is surjective. We have two inequivalent choices for γ_{cone}, which are distinguished by the signature $\epsilon \in \{-1, 1\}$. The morphism $\gamma_{\mathrm{cone}} : (\wedge T^*\hat{M}, \diamond^{\mathrm{cone}}) \to (\mathrm{End}(\hat{S}), \circ)$ is completely determined by the morphism $\gamma : (\wedge T^*M, \diamond) \to (\mathrm{End}(S), \circ)$ once the signature ϵ has been chosen. In the following, we shall work with the choice $\epsilon = +1$. Setting $\epsilon = +1$ and rescaling the metric on M as $g \to (2\kappa)^2 g$, we find:

$$\nabla^{\hat{S},\mathrm{cone}}_m = \nabla^S_m + \kappa \gamma_{m9} \ , \ A^{\mathrm{cone}}_9 = 0 \ , \ A^{\mathrm{cone}}_m = \frac{1}{4} f^p \gamma_{mp9} + \frac{1}{24} F_{mpqr} \gamma^{pqr} \ .$$

The generalized Killing pinor equations $D_m\xi=0$ $(m=1\ldots 8)$ for pinors $\xi\in\Gamma(M,S)$ defined on M amount to the flatness conditions:

$$\hat{D}_a\hat{\xi}=0 \ \ , \ \ \forall a=1\ldots 9 \ \ ,$$

for pinors $\hat{\xi}\in\Gamma(\hat{M},\hat{S})$ defined on $\hat{M}$. Indeed, the last of the cone flatness equations $\hat{D}_9\hat{\xi}=0$ is equivalent with the requirement that the section $\hat{\xi}$ of $\hat{S}$ is the pull-back of some section ξ of S through the natural projection Π from $\hat{M}$ to M, while the remaining equations amount to the generalized Killing conditions $D_m\xi=0$ on M. Furthermore, the algebraic constraint for ξ is equivalent with the following equation for $\hat{\xi}$:

$$\hat{Q}\hat{\xi}=0 \ \ ,$$

where $\hat{Q}\in\Gamma(\hat{M},\mathrm{End}(\hat{S}))$ is the pullback of $Q\in\Gamma(M,\mathrm{End}(S))$. We refer the reader to [9] for much more detail about the geometric algebra realization of the cone formalism of [10] and for the applications of this realization to the theory of constrained generalized Killing pinors and forms.

4. APPLICATION TO $N=2$ COMPACTIFICATIONS OF M-THEORY DOWN TO THREE DIMENSIONS

In this example, one obtains useful simplifications of the problem by using the geometric algebra reformulation (see [9] and the previous Section) of the cone formalism, which is particularly relevant when seeking a geometric interpretation in terms of reductions of structure group. Using this variant of the cone formalism as well as a software implementation of our approach using `Ricci` [11] and `Cadabra` [12], one can extract and analyse the cone reformulation of (1). Since the detailed theory of the Kähler-Atiyah algebra of cones is somewhat involved and since the equations obtained in this manner for the application at hand are quite complex, we cannot reproduce them here given the space limitations. Instead, we refer the interested reader to [9] and [13].

5. CONCLUSIONS

We summarized an approach to the theory of constrained generalized Killing (s)pinors which is inspired by geometric algebra, a formulation of spin geometry which resolves the lack of naturalism affecting certain traditional constructions. Using this approach, we showed how generalized Killing pinor equations translate succinctly into conditions for differential forms constructed as bilinears in such pinors. We also touched upon the applications of this approach to the study of $N=2$ compactifications of M-theory down to three dimensions, which are discussed in more detail in [9] as well as in [13].

Acknowledgments. This work was supported by the CNCS projects PN-II-RU-TE (contract number 77/2010) and PN-II-ID-PCE (contract numbers 50/2011 and 121/2011). The authors thank the organizers of the 8-th QFTHS Conference for hospitality and interest in their work. C.I.L thanks the Center for Geometry and Physics, Institute for Basic Science and Pohang University of Science and Technology (POSTECH), Pohang, Korea for providing excellent conditions at various stages during the preparation of this work, through the research visitor program affiliated with Grant No. CA1205-1. The Center for Geometry and Physics is supported by the Government of Korea through the Research Center Program of IBS (Institute for Basic Science). He also thanks Perimeter Institute for hospitality for providing an excellent and stimulating research environment during the last stages of this project. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation.

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