

NON-TRIVIAL 2+1-DIMENSIONAL GRAVITY*

D.R. GRIGORE¹, G. SCHARF²

¹Department of Theoretical Physics,
National Institute for Physics and Nuclear Engineering “Horia Hulubei”,
Reactorului 30, RO-077125, POB-MG6, Măgurele-Bucharest, Romania, EU

E-mail: grigore@theory.nipne.ro

²Institut für Theoretische Physik, Universität Zürich,
Winterthurerstr. 190, CH-8057 Zürich, Switzerland

E-mail: scharf@physik.unizh.ch

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We analyze 2+1-dimensional gravity in the framework of quantum gauge theory. We find that Einstein gravity has a trivial physical subspace which reflects the fact that the classical solution in empty space is flat. Therefore we study massive gravity which is not trivial. In the limit of vanishing graviton mass we obtain a non-trivial massless theory different from Einstein gravity. We derive the interaction from descent equations and obtain the cosmological topologically massive gravity. However, in addition to Einstein and Chern-Simons coupling we need coupling to fermionic ghost and anti-ghost fields and to a vector-graviton field with the same mass as the graviton.

Key words: perturbation theory, standard model, quantum gravity.

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1. INTRODUCTION

It is well known that 3-dimensional Einstein gravity is dynamically trivial in the sense that any classical solution in empty space is flat [1]- [3] (and references given there). For this reason one has to consider modifications of Einstein’s theory, in particular massive gravity [4]- [8] in order to have propagating gravitational waves. Most of these studies have been done in the framework of classical Lagrangian field theory.

We advocate a different approach to gravity which has been successfully applied in the 3+1-dimensional case [9]- [11]. We do not start from a classical Lagrangian. Instead we choose a collection of free quantum fields on Minkowski space which are the asymptotic fields of a S-matrix. Some of these fields are gauge fields which have a non-trivial gauge variation involving fermionic ghost fields. The gauge

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variation $d_Q = [Q, \cdot]$ defines the physical subspace of the theory as

$$\mathcal{H}_{\text{phys}} = \text{Ker}Q/\text{Ran}Q. \quad (1)$$

The coupling $T(x)$ is a Wick polynomial of the free fields which is a solution of the gauge invariance condition

$$d_Q T = \partial_\alpha T^\alpha(x), \quad (2)$$

and some generalization of it where $T^\alpha(x)$ is another Wick polynomial. This condition is necessary in order to have unitarity of the S-matrix on the physical subspace. In 3+1 dimensions the method works equally well in the massless and massive case. The difference is only that the gauge structure in the massive case is more rich. It requires the so-called vector graviton field v_μ with the same mass m as the graviton, so that the resulting theory is a vector-tensor theory. The interesting point is that in the limit $m \rightarrow 0$ one does not get Einstein's theory. The now massless vector-graviton field does not decouple from the symmetric tensor field $h_{\mu\nu}$ so that we get an alternative (massless) gravity theory.

It is the purpose of this paper to analyse 2+1-dimensional gravity in exactly the same way. In the next section we introduce the various free quantum fields and their gauge structure. Then we determine the physical Hilbert space. It turns out that in the massless case corresponding to Einstein's theory $\mathcal{H}_{\text{phys}}$ is trivial. This reflects the well-known fact mentioned above that Einstein's theory has no graviton states in 2+1 dimensions. In the massive gauge theory $\mathcal{H}_{\text{phys}}$ is non-trivial, for fixed momentum there exist three physical modes. This remains true in the limit $m \rightarrow 0$ so that we have a non-trivial theory in both cases. In section 3 we construct a concrete Hilbert space representation in order to show that all operators are well defined. This representation is chosen in such a way that the massless limit $m \rightarrow 0$ is smooth. In section 4 we derive the coupling from the so-called descent equations which are a generalization of causal gauge invariance (1.2). The even-parity sector can be treated in exactly the same way as in 3 + 1 dimensions. We carry through the descent procedure in the odd-parity case in all details. We recover the gravitational Chern-Simons coupling. Together with the even-parity coupling we obtain the so-called "cosmological topologically massive gravity"[11]. From the point of view of quantum gauge theory the treatment of this theory in the literature is incomplete because the vector-graviton field is lacking. In the conclusions we point out further differences to the classical theory.

2. MASSLESS AND MASSIVE GRAVITY IN 2+1 DIMENSIONS

As we have said in the Introduction, we use the framework from [9]- [17] which works fine for the four-dimensional case. We must first see if the same framework works in three dimensions. We use the same convention as in 3+1 dimensions as far

as possible. Then many results of the 4-dimensional theory can be taken over without change. The basic free quantum field in massive gravity is a symmetric tensor field $h^{\mu\nu}(x)$ satisfying the Klein-Gordon equation

$$(\square + m^2)h^{\mu\nu} = 0. \quad (3)$$

It is quantized according to

$$[h^{\alpha\beta}(x), h^{\mu\nu}(y)] = -\frac{i}{2}(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu})D_m(x-y), \quad (4)$$

where $\eta^{\mu\nu}$ is the Minkowski tensor with diagonal elements $(1, -1, -1)$. D_m is the 2+1-dimensional Jordan-Pauli distribution with mass m . To make $h^{\mu\nu}$ a gauge field we must introduce ghost and antighost fields with the same mass

$$(\square + m^2)u^\mu = 0 = (\square + m^2)\tilde{u}^\mu.$$

These fields are quantized with anti-commutators

$$\{u^\mu(x), \tilde{u}^\nu(y)\} = i\eta^{\mu\nu}D_m(x-y) \quad (5)$$

and all other anti-commutators vanishing.

Then we can define the gauge variations

$$d_Q h^{\mu\nu} = [Q, h^{\mu\nu}] = -\frac{i}{2}(\partial^\nu u^\mu + \partial^\mu u^\nu - \eta^{\mu\nu}\partial_\alpha u^\alpha) \quad (6)$$

$$d_Q u^\mu = \{Q, u^\mu\} = 0. \quad (7)$$

The gauge variation of \tilde{u}^μ is non-trivial. Since d_Q is nilpotent, $d_Q^2 = 0$, we must introduce a vector field $v^\mu(x)$ with the same mass

$$(\square + m^2)v^\mu = 0$$

which we call vector-graviton field or v -field for short. It is quantized according to

$$[v^\mu(x), v^\nu(y)] = \frac{i}{2}\eta^{\mu\nu}D_m(x-y). \quad (8)$$

This field appears in the gauge variation of \tilde{u}^μ

$$d_Q \tilde{u}^\mu = \{Q, \tilde{u}^\mu\} = i(\partial_\nu h^{\mu\nu} + mv^\mu). \quad (9)$$

Finally

$$d_Q v^\mu = [Q, v^\mu] = -\frac{i}{2}mv^\mu \quad (10)$$

It is not hard to verify nilpotency $d_Q^2 = 0$. Using the commutation rules above one can show that Q is expressed in x -space as follows

$$Q = \int_{x^0=t} d^3x [\partial_\nu h^{\mu\nu}(x) + mv^\mu(x)] \overset{\leftrightarrow}{\partial}_0 u_\mu(x). \quad (11)$$

These relations remain true in the massless case $m = 0$, but the v -field is then completely skipped in ordinary gravity theory.

We now describe the one-particle Hilbert space as in ref. [12] and [14]. First we study the massless case. The generic form of a state $\Psi \in \mathcal{H}^{(1)} \subset \mathcal{H}$ from the one-particle Hilbert subspace is

$$\Psi = \left[\int f_{\mu\nu}(x) h^{\mu\nu}(x) + \int g_{\mu}^{(1)}(x) u^{\mu}(x) + \int g_{\mu}^{(2)}(x) \tilde{u}^{\mu}(x) \right] \Omega \quad (12)$$

with test functions $f_{\mu\nu}, g_{\mu}^{(1)}, g_{\mu}^{(2)}$ verifying the wave equation; we can also suppose that $f_{\mu\nu}$ is symmetric; we denote $f \equiv \eta^{\mu\nu} f_{\mu\nu}$.

The kernel of the gauge charge operator Q (restricted to one-particle states) is given by states of the form

$$\Psi = \left[\int f_{\mu\nu}(x) h^{\mu\nu}(x) + \int g_{\mu}(x) u^{\mu}(x) \right] \Omega \quad (13)$$

with g_{μ} arbitrary and $f_{\mu\nu}$ constrained by the condition $\partial^{\nu} f_{\mu\nu} = \frac{1}{2} \partial_{\mu} f$; so the elements of $\mathcal{H}^{(1)} \cap \text{Ker}(Q)$ are in one-one correspondence with couples of test functions $[f_{\mu\nu}, g_{\rho}]$ with the transversality condition on the first entry.

Now, a generic element $\Psi' \in \mathcal{H}^{(1)} \cap \text{Ran}(Q)$ has the form

$$\Psi' = Q\Phi = \left[-\frac{1}{2} \int (\partial_{\mu} g'_{\nu} + \partial_{\nu} g'_{\mu})(x) h^{\mu\nu}(x) + \int \left(\partial^{\nu} g'_{\mu\nu} - \frac{1}{2} \partial_{\mu} g' \right) (x) u(x) \right] \Omega \quad (14)$$

with $g' = \eta^{\mu\nu} g'_{\mu\nu}$, so if $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is indexed by the couple $[f_{\mu\nu}, g_{\rho}]$ then $\Psi + \Psi'$ is indexed by the couple $[f_{\mu\nu} - \frac{1}{2} (\partial_{\mu} g'_{\nu} + \partial_{\nu} g'_{\mu}), g_{\mu} + (\partial^{\nu} g'_{\mu\nu} - \frac{1}{2} \partial_{\mu} g')]$. If we take $g'_{\mu\nu}$ conveniently we can make $g_{\mu} = 0$ and if we take g'_{μ} convenient we can make $f = 0$; in this case we have the transversality condition $\partial^{\nu} f_{\mu\nu} = 0$. It follows that the equivalence classes from $(\mathcal{H}^{(1)} \cap \text{Ker}(Q)) / (\mathcal{H}^{(1)} \cap \text{Ran}(Q))$ are indexed by wave functions $f_{\mu\nu}$ verifying the conditions of transversality and tracelessness $\partial^{\nu} f_{\mu\nu} = 0, f = 0$.

We go in the momentum space and choose a Lorentz frame such that $P = (1, 0, 1)$. Then the Fourier transform $\tilde{f}_{\mu\nu}(P)$ is restricted by the two conditions above (transversality and tracelessness) and we have the non-null elements of the tensor given by two free parameters:

$$\begin{aligned} \tilde{f}_{00}(P) &= \tilde{f}_{22}(P) = -\tilde{f}_{02}(P) = \alpha \\ \tilde{f}_{01}(P) &= \tilde{f}_{12}(P) = \beta \end{aligned} \quad (15)$$

with α and β two arbitrary complex numbers. Now if we compute the value of the "scalar product"

$$\langle \tilde{f}, \tilde{f} \rangle = \tilde{f}_{\mu\nu}^* \tilde{f}^{\mu\nu} \quad (16)$$

for the previous values we get 0. So there is no way to construct the one-particle Hilbert space as in the four-dimensional case.

The situation changes drastically in the massive case. The generic form of a state $\Psi \in \mathcal{H}^{(1)} \subset \mathcal{H}$ from the one-particle Hilbert subspace is

$$\Psi = \left[\int f_{\mu\nu}(x)h^{\mu\nu}(x) + \int g_{\mu}^{(1)}(x)u^{\mu}(x) + \int g_{\mu}^{(2)}(x)\tilde{u}^{\mu}(x) + \int h_{\mu}(x)v^{\mu}(x) \right] \Omega \quad (17)$$

with test functions $f_{\mu\nu}, g_{\mu}^{(1)}, g_{\mu}^{(2)}, h_{\mu}$ verifying the Klein-Gordon equation; we can also suppose that $f_{\mu\nu}$ is symmetric. Now the elements $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ from the kernel of the gauge charge operator are of the form

$$\Psi = \left[\int f_{\mu\nu}(x)h^{\mu\nu}(x) + \int g_{\mu}(x)u^{\mu}(x) + \frac{2}{m} \int \left(\partial^{\nu} f_{\mu\nu} - \frac{1}{2} \partial_{\mu} f \right) (x)v^{\mu}(x) \right] \Omega \quad (18)$$

with g_{μ} and $f_{\mu\nu}$ arbitrary so $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is indexed by couples of test functions $[f_{\mu\nu}, g_{\mu}]$. Now, a generic element $\Psi' \in \mathcal{H}^{(1)} \cap \text{Ran}(Q)$ has the form

$$\Psi' = Q\Phi = \left[-\frac{1}{2} \int (\partial_{\mu} g'_{\nu} + \partial_{\nu} g'_{\mu})(x)h^{\mu\nu}(x) + \int \left(\partial^{\nu} g'_{\mu\nu} - \frac{1}{2} \partial_{\mu} g' - \frac{m}{2} h'_{\mu} \right) (x)u^{\mu}(x) \right] \Omega \quad (19)$$

with $g' = \eta^{\mu\nu} g'_{\mu\nu}$ so if $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is indexed by the couple $[f_{\mu\nu}, g_{\mu}]$ then $\Psi + \Psi'$ is indexed by the couple $[f_{\mu\nu} - \frac{1}{2} (\partial_{\mu} g'_{\nu} + \partial_{\nu} g'_{\mu}), g_{\mu} + (\partial^{\nu} g'_{\mu\nu} - \frac{1}{2} \partial_{\mu} g' - \frac{m}{2} h'_{\mu})]$. If we take h'_{μ} conveniently we can make $g_{\mu} = 0$ and if we take g'_{μ} convenient we can make

$$\partial^{\nu} f_{\mu\nu} - \frac{1}{2} \partial_{\mu} f = 0. \quad (20)$$

As above we consider a Lorentz reference frame where $P = (m, 0, 0)$ and we get from the condition above that the non-null elements of the expression $\tilde{f}_{\mu\nu}(P)$ depend on three free parameters:

$$\tilde{f}_{11}(P) = \alpha, \quad \tilde{f}_{22}(P) = \beta, \quad \tilde{f}_{12}(P) = \gamma, \quad \tilde{f}_{00}(P) = \alpha + \beta \quad (21)$$

If we compute the value of the ‘‘scalar product’’

$$\langle \tilde{f}, \tilde{f} \rangle = \tilde{f}_{\mu\nu}^* \tilde{f}^{\mu\nu} \quad (22)$$

for the previous values we get in this case

$$\langle f, f \rangle = |\alpha + \beta|^2 + |\alpha|^2 + |\beta|^2 + 2|\gamma|^2 \quad (23)$$

which is positively defined and induces a well-defined scalar product on the physical Hilbert space $\mathcal{H}_{\text{phys}} = \text{Ker}Q/\text{Ran}Q$.

Moreover, if we apply a rotation of angle ϕ (which is an element of the stability group of the momentum P) to the expression $\tilde{f}_{\mu\nu}(P)$ we immediately obtain that the expression $\alpha + \beta$ is invariant (so it describes a spin 0 particle) and the expressions

$\alpha - \beta \pm 2i\gamma$ are transformed by a phase factor $e^{\pm 2i\phi}$ (so they describe two particles of spin ± 2 respectively).

It follows that we have a good description for the massive spin 2 particles in three dimension which is similar to the four-dimensional case. Another construction of the physical Hilbert space is given in the next Section.

3. REPRESENTATION IN MOMENTUM SPACE

To understand the gauge structure better we construct a Hilbert space representation of the massive 2+1-dimensional theory. For this purpose we express the various fields by means of emission and absorption operators. In doing so we have to introduce a positive definite scalar product which breaks Lorentz invariance but defines the topology of the big Fock space of physical and unphysical states and the adjoint operators. It is well known that this Hilbert structure is not unique [18], we shall choose it in such a way that we get a smooth massless limit $m \rightarrow 0$. We follow the discussion of the 4-dimensional case as close as possible [10]. We decompose $h^{\alpha\beta}$ into its traceless part and the trace h

$$h^{\alpha\beta}(x) = H^{\alpha\beta}(x) + \frac{1}{3}\eta^{\alpha\beta}h(x). \quad (24)$$

From (4) we obtain the following commutation relations

$$[h(x), h(y)] = \frac{3i}{2}D_m(x-y) \quad (25)$$

$$[H^{\alpha\beta}(x), H^{\mu\nu}(y)] = -\frac{i}{2} \left(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \frac{2}{3}\eta^{\alpha\beta}\eta^{\mu\nu} \right) D_m(x-y), \quad (26)$$

and

$$[H^{\alpha\beta}(x), h(y)] = 0. \quad (27)$$

It is easy to verify that the fields in (25) (26) can be represented as follows

$$H^{\alpha\beta}(x) = (2\pi)^{-1} \int \frac{d^2k}{\sqrt{2E_k}} \left(a_{\alpha\beta}(\vec{k})e^{-ikx} + \eta^{\alpha\alpha}\eta^{\beta\beta}a_{\alpha\beta}^+(\vec{k})e^{ikx} \right). \quad (28)$$

Here $E_k = \sqrt{\vec{k}^2 + m^2}$, $a_{\alpha\beta} = a_{\beta\alpha}$ is symmetric and satisfies the commutation relation

$$[a_{\alpha\beta}(\vec{k}), a_{\mu\nu}^+(\vec{k}')] = \eta^{\alpha\alpha}\eta^{\beta\beta} \left(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \frac{2}{3}\eta^{\alpha\beta}\eta^{\mu\nu} \right) \delta(\vec{k} - \vec{k}'). \quad (29)$$

The trace part is given by

$$h(x) = (2\pi)^{-1} \int \frac{d^2k}{\sqrt{2E_k}} \left(a(\vec{k})e^{-ikx} - a^+(\vec{k})e^{ikx} \right) \quad (30)$$

with

$$[a(\vec{k}), a^+(\vec{k}')] = \frac{3}{2}\delta(\vec{k} - \vec{k}'). \quad (31)$$

Since the right-hand side is positive, the h -sector of Fock space can be constructed in the usual way by applying products of a^+ 's to the vacuum.

The situation is not so simple in the H -sector because the righthand side of (3.6) is not a diagonal matrix. We perform a linear transformation of the diagonal operators $a_{\alpha\alpha}$ and $a_{\alpha\alpha}^+$ in such a way that the new operators are usual annihilation and creation operators satisfying

$$[\tilde{a}_{\alpha\alpha}(\vec{k}), \tilde{a}_{\beta\beta}^+(\vec{k}')] = \delta_{\alpha\beta}\delta(\vec{k} - \vec{k}'). \quad (32)$$

This is achieved by the following transformation:

$$\begin{aligned} a_{00} &= \sqrt{\frac{2}{3}}(\tilde{a}_{11} + \tilde{a}_{22} + \tilde{a}_{33}), \\ a_{11} &= \alpha_1\tilde{a}_{11} + \alpha_2\tilde{a}_{22}, \\ a_{22} &= \alpha_2\tilde{a}_{11} + \alpha_2\tilde{a}_{22}, \end{aligned} \quad (33)$$

with

$$\alpha_1 = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{2}}, \quad \alpha_2 = \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}}. \quad (34)$$

We note that \tilde{a}_{00} does not appear because one pair of absorption and emission operators is superfluous due to the trace condition $H^\alpha{}_\alpha = 0$. In fact, from (3.10) we see

$$\sum_{j=1}^2 a_{jj} = a_{00}.$$

The Fock representation can now be constructed as usual by means of $\tilde{a}_{11}^+, \tilde{a}_{22}^+$ and $a_{\alpha\beta}^+$ with $\alpha \neq \beta$.

The other fields have the following representation in terms of emission and absorption operators:

$$\begin{aligned} u^\mu(x) &= (2\pi)^{-1} \int \frac{d^2k}{\sqrt{2E_k}} \left(c_2^\mu(\vec{k})e^{-ikx} - \eta^{\mu\mu} c_1^\mu(\vec{k})^+ e^{ikx} \right) \\ \tilde{u}^\mu(x) &= (2\pi)^{-1} \int \frac{d^2k}{\sqrt{2E_k}} \left(-c_1^\mu(\vec{k})e^{-ikx} - \eta^{\mu\mu} c_2^\mu(\vec{k})^+ e^{ikx} \right) \end{aligned} \quad (35)$$

$$v^\mu(x) = (2\pi)^{-1} \int \frac{d^2k}{2\sqrt{E_k}} \left(b^\mu(\vec{k})e^{-ikx} - \eta^{\mu\mu} b^\mu(\vec{k})^+ e^{ikx} \right) \quad (36)$$

with the following (anti)commutation relations

$$\{c_j^\mu(\vec{k}), c_l^\nu(\vec{k}')^+\} = \delta_{jl}\delta_\nu^\mu\delta^3(\vec{k} - \vec{k}'), \quad (37)$$

$$[b^\mu(\vec{k}), b^\nu(\vec{k}')^+] = \delta_\nu^\mu\delta^3(\vec{k} - \vec{k}'). \quad (38)$$

Then the gauge charge Q (11) can be written in momentum space as follows

$$Q = \int d^3k \left(A^\alpha(\vec{k})^+ c_2^\gamma(\vec{k}) - B^\alpha(\vec{k}) c_1^\gamma(\vec{k})^+ \right) \eta_{\alpha\gamma}, \quad (39)$$

where

$$A^\alpha = \eta^{\alpha\alpha} \eta^{\beta\beta} a^{\alpha\beta}(\vec{k}) k^\beta - \frac{k^\alpha}{4} d(\vec{k}) - im_1 \eta^{\alpha\alpha} b^\alpha \quad (40)$$

$$B^\alpha = (a^{\alpha\beta}(\vec{k}) k_\beta + \frac{k^\alpha}{4} d(\vec{k}) + im_1 b^\alpha) \eta^{\alpha\alpha}, \quad (41)$$

$$m_1 = \frac{m}{\sqrt{2}}. \quad (42)$$

The adjoint is given by

$$Q^+ = \int d^2k \left(c_2^\beta(\vec{k})^+ A^\alpha(\vec{k}) - c_1^\beta(\vec{k}) B^\alpha(\vec{k})^+ \right) \eta_{\alpha\beta}. \quad (43)$$

The physical Hilbert space can be expressed by means of the gauge charge Q in the following equivalent form

$$\mathcal{H}_{\text{phys}} = \text{Ker}(QQ^+ + Q^+Q). \quad (44)$$

We must study the selfadjoint operator

$$\begin{aligned} \{Q, Q^+\} = & \int d^3k d^3k' \left(A^\alpha(\vec{k})^+ A^\beta(\vec{k}') \{c_2^\gamma(\vec{k}), c_2^\delta(\vec{k}')^+\} \right. \\ & + B^\beta(\vec{k}')^+ B^\alpha(\vec{k}) \{c_1^\delta(\vec{k}'), c_1^\gamma(\vec{k})^+\} + c_2^\delta(\vec{k}')^+ c_2^\gamma(\vec{k}) [A^\beta(\vec{k}'), A^\alpha(\vec{k})^+] \\ & \left. + c_1^\gamma(\vec{k})^+ c_1^\delta(\vec{k}') [B^\alpha(\vec{k}), B^\beta(\vec{k}')^+] \right) \eta_{\alpha\gamma} \eta_{\beta\delta}. \quad (45) \end{aligned}$$

We restrict to the graviton sector because the ghost sector is totally unphysical:

$$\{Q, Q^+\}|_{\text{graviton}} = \int d^3k \sum_{\alpha=0}^3 \left(A^{\alpha+} A^\alpha + B^{\alpha+} B^\alpha \right). \quad (46)$$

It is convenient to introduce time-like and space-like components:

$$\begin{aligned} A^0 &= k_0(a^{00} - a_{\parallel}^0 - \frac{a}{3} - \frac{im_1}{k_0} b^0), \\ A^j &= k_0(-a^{0j} + a_{\parallel}^j - \frac{k^j}{k_0} \frac{a}{3} + \frac{im_1}{k_0} b^j), \\ B^0 &= k_0(a^{00} + a_{\parallel}^0 + \frac{a}{3} + \frac{im_1}{k_0} b^0), \\ B^j &= k_0(-a^{0j} - a_{\parallel}^j - \frac{k^j}{k_0} \frac{a}{3} - \frac{im_1}{k_0} b^j), \end{aligned} \quad (47)$$

where

$$a_{\parallel}^\mu = \frac{k_j}{k_0} a^{\mu j}. \quad (48)$$

We choose a Lorentz frame where $k^\mu = (k_0, 0, k_2)$ and substitute the diagonal operators $a^{\mu\mu}$ by \tilde{a}_{jj} (33). Then we get for the integrand in (46)

$$\begin{aligned} \sum_{\alpha=0}^3 \left(A^{\alpha+} A^\alpha + B^{\alpha+} B^\alpha \right) = & 2k_0^2 \left\{ \frac{2}{3} (\tilde{a}_{11+} + \tilde{a}_{22+}) (\tilde{a}_{11} + \tilde{a}_{22}) + \frac{k_2^2}{k_0^2} a^{02+} a^{02+} \right. \\ & + \frac{im_1 k_2}{k_0^2} a^{02+} b_0 + \frac{1}{3} \frac{k_2}{k_0} a^{02+} a + \frac{1}{9} a^+ a + \frac{im_1}{3k_0} a^+ b_0 - \frac{imk_2}{k_0^2} b_0^+ a^{02} - \frac{im}{3k_0} b_0^+ a^+ \\ & + \frac{m_1^2}{k_0^2} b_0^+ b_0 + a^{01+} a^{01} + a^{02+} a^{02} + \frac{k_2}{3k_0} a^{02+} a + \frac{k_2^2}{k_0^2} a^{12+} a^{12+} \\ & + \frac{k_2^2}{k_0^2} (\alpha_2 \tilde{a}_{11}^+ + \alpha_1 \tilde{a}_{22}^+) (\alpha_2 \tilde{a}_{11} + \alpha_1 \tilde{a}_{22}) + \frac{im_1 k_2}{k_0^2} a^{12+} b^1 + \\ & + \frac{im_1 k_2}{k_0^2} (\alpha_2 \tilde{a}_{11}^+ + \alpha_1 \tilde{a}_{22}^+) b^2 + \frac{k_2}{3k_0} a^+ a^{02} + \frac{k_2^2}{9k_0^2} a^+ a - \frac{im_1 k_2}{k_0^2} b^{1+} a^{12-} \\ & \left. - \frac{im_1 k_2}{k_0^2} b^{2+} (\alpha_2 \tilde{a}_{11}^+ + \alpha_1 \tilde{a}_{22}^+) + \frac{m_1^2}{k_0^2} (b^{1+} b^1 + b^{2+} b^2) \right\}. \quad (49) \end{aligned}$$

Since a^{11+} does not appear inhere, the states $a^{11+}\Omega$ where Ω is the Fock vacuum certainly belong to the kernel of (46) and, hence, are in the physical subspace.

The quadratic form (49) can be represented in matrix notation $A^+ X A$ where A^+ stands for the emission operators

$$A^+ = (\tilde{a}_{11}^+, \tilde{a}_{22}^+, b_2^+, b_1^+, a_{12}^+, a_{02}^+, a^+, b_0^+, a_{01}^+). \quad (50)$$

The matrix X has block diagonal form with the following three submatrices:

$$\begin{aligned} X_0 &= \begin{pmatrix} 1 + \frac{k_2^2}{k_0^2} & \frac{k_2}{3k_0} & \frac{im_1 k_2}{k_0^2} \\ \frac{k_2}{3k_0} & \frac{1}{9} + \frac{k_2^2}{9k_0^2} & \frac{im_1}{3k_0} \\ -\frac{imk_2}{k_0^2} & 0 & \frac{m_1^2}{k_0^2} \end{pmatrix} \\ X_1 &= \begin{pmatrix} \frac{m^2}{k_0^2} & -\frac{im_1 k_2}{k_0^2} \\ \frac{im_1 k_2}{k_0^2} & \frac{k_2^2}{k_0^2} \end{pmatrix} \\ X_2 &= \begin{pmatrix} \frac{2}{3} + \frac{\alpha_2^2 k_2^2}{k_0^2} & \frac{2}{3} + \frac{\alpha_1 \alpha_2 k_2^2}{k_0^2} & \frac{im_1 \alpha_2 k_2}{k_0^2} \\ \frac{2}{3} + \frac{\alpha_1 \alpha_2 k_2^2}{k_0^2} & \frac{2}{3} + \frac{\alpha_1^2 k_2^2}{k_0^2} & \frac{im_1 \alpha_1 k_2}{k_0^2} \\ -\frac{im_1 \alpha_2 k_2}{k_0^2} & 0 & \frac{m_1}{k_0^2} \end{pmatrix}. \end{aligned} \quad (51)$$

The kernel (44) now consists of the null-vectors of these matrices. Only X_1 and X_2

have eigenvalue 0, the corresponding eigenvectors are

$$\begin{aligned}\psi_1 &= \left(b_1^+ - \frac{im_1}{k_2} a^{12+}\right)\Omega \\ \psi_2 &= \left(b_2^+ + \frac{im_1}{\sqrt{2}k_2} (\tilde{a}_2^+ - \tilde{a}_1^+)\right)\Omega.\end{aligned}\tag{52}$$

In the limit $m \rightarrow 0$ these two physical states go over into the free vector-graviton states. Consequently, the physical modes of the massless theory are one transversal graviton state $a^{11+}\Omega$ plus these two vector-graviton states. In the massive case there is some admixture of other graviton states (3.29).

4. INTERACTION FROM DESCENT EQUATIONS

There are a number of ideas which must be used to determine in an unique way the expression of the interaction Lagrangian respecting the gauge invariance condition (2). First, because we are in three dimensions, power counting allows us to consider Wick polynomials of canonical dimension $\omega(T), \omega(T^\alpha) \leq 6$ and tri-linear in the fields and their derivatives.

Next, we can obtain from (2) by a standard procedure the chain of relations

$$\begin{aligned}d_Q T^\alpha &= i \partial_\beta T^{[\alpha\beta]}, \\ d_Q T^{[\alpha\beta]} &= i \partial_\gamma T^{[\alpha\beta\gamma]} \\ d_Q T^{[\alpha\beta\gamma]} &= 0,\end{aligned}\tag{53}$$

where the carets emphasize complete antisymmetry. These relations can be solved starting from the last one (in top ghost number equal to 3). Going backwards in this chain of relations we are always reduced to solve co-cycle conditions of the type $d_Q C = 0$. This is the descent procedure.

The cohomology of the operator d_Q has been investigated in [14] in the four-dimensional case. Because we have preserved the algebraic structure of the gauge charge operator the analysis from this reference remains unchanged: the space dimension plays no rôle. One must determine the invariants with respect to the gauge charge *i.e.* solutions of the equation $d_Q C = 0$ which cannot be expressed as co-boundaries *i.e.* in the form $C = d_Q B$ and of canonical dimension bounded by some integer n . We denote by $\mathcal{P}^n, Z_Q^n, B_Q^n$ the space of cochains, cocycles and coboundaries respectively and we require $\omega(B_Q^n) \leq n - 1$.

In the massless case these invariants are u_μ , the antisymmetric first-order derivative

$$u_{[\mu\nu]} \equiv \frac{1}{2} (\partial_\mu u_\nu - \partial_\nu u_\mu)\tag{54}$$

and the (linear) Riemann tensor and its derivatives. One defines it as follows: first

we introduce the *Christoffel symbols* according to:

$$\Gamma_{\mu;\nu\rho} \equiv \partial_\rho \hat{h}_{\mu\nu} + \partial_\nu \hat{h}_{\mu\rho} - \partial_\mu \hat{h}_{\nu\rho} \quad (55)$$

where

$$h \equiv \eta^{\mu\nu} h_{\mu\nu} \quad \hat{h}_{\mu\nu} \equiv h_{\mu\nu} - \eta_{\mu\nu} h \quad (56)$$

and then the Riemann tensor is:

$$R_{\mu\nu;\rho\sigma} \equiv \partial_\rho \Gamma_{\mu;\nu\sigma} - (\rho \leftrightarrow \sigma). \quad (57)$$

One must eliminate in the systematic way all traces from the derivatives of the Riemann tensor to obtain true invariants; the traces are coboundaries. Then one can prove that any cocycle is cohomologous to a Wick polynomial in the invariants.

In the massive case some invariants are lost: u_μ and $u_{[\mu\nu]}$ become coboundaries by (10) and a new invariant appears:

$$\phi_{\mu\nu} \equiv -\partial_\mu v_\nu - \partial_\nu v_\mu + \eta_{\mu\nu} \partial_\rho v^\rho + m h_{\mu\nu}, \quad \phi \equiv \eta^{\mu\nu} \phi_{\mu\nu} \quad (58)$$

and their derivatives. (Again one must conveniently eliminate the traces of the various derivative). However, the situation is still more involved in the massive case. Any cocycle is cohomologous to an expression of the form

$$p_1 + d_Q p_2 \quad (59)$$

where p_1 depends only on the invariants and p_2 is a Wick polynomial of canonical dimension equal to n and such that $d_Q p_2$ has also canonical dimension n . This is because the expression $d_Q p_2$ is a cocycle in any canonical dimension bigger than n i.e. in \mathcal{P}^m , $m > n$ but is not a co-boundary in \mathcal{P}^n .

A final idea is related to the fact that we are working in 3 dimensions so we have the Lorentz invariant and completely antisymmetric tensor $\epsilon_{\mu\nu\rho}$ which allows us to trade a couple of antisymmetric indices for only one index. For instance, instead of $u_{[\mu\nu]}$ we prefer to work with the variables

$$\lambda_\mu \equiv \frac{1}{2} \epsilon_{\mu\rho\sigma} u^{[\rho\sigma]} \quad \Leftrightarrow \quad u_{[\mu\nu]} = \epsilon_{\mu\nu\rho} \lambda^\rho. \quad (60)$$

We also define the expressions

$$\Gamma_{\mu;\nu} \equiv \epsilon_\mu^{\rho\sigma} \Gamma_{\rho;\sigma\nu} \quad (61)$$

and observe that

$$\partial_\nu \lambda_\mu = -\frac{i}{2} d_Q \Gamma_{\mu;\nu}. \quad (62)$$

Finally we define

$$V_\mu \equiv \epsilon_{\mu\rho\sigma} \partial^\rho v^\sigma \quad \Leftrightarrow \quad \partial_\mu v_\nu - \partial_\nu v_\mu = \epsilon_{\mu\nu\rho} V^\rho \quad (63)$$

so that we have

$$d_Q V_\mu = im \lambda_\mu. \quad (64)$$

We are ready to start the descent procedure. We must start with the expression $T^{[\alpha\beta\gamma]}$ of the descent system (53). One must use the limitations on the canonical dimension ($\omega(T^{\alpha\beta\gamma}) \leq 6$), ghost number ($gh(T^{\alpha\beta\gamma}) = 3$) and complete antisymmetry. In canonical dimension 5 we have the same expression as in four dimensions. The descent procedure goes though in exactly the same way as in [14] and [16] so we obtain in the end the same interaction terms related to the Einstein-Hilbert Lagrangian (see the Conclusions). However, in canonical dimension 6 a new expression appears:

$$T^{[\alpha\beta\gamma]} = c_0 \epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} \lambda_\mu \lambda_\nu \lambda_\rho + i d_Q B^{[\alpha\beta\gamma]} \quad (65)$$

The first term can be rewritten (up to a constant) as $d_Q(\epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} V_\mu \lambda_\nu \lambda_\rho)$; the expression in the bracket is of canonical dimension 6 so it is of the form $d_Q p_2$ from (59). For simplicity we take $c_0 = 1$ because anyway we can rescale the final solution by a constant. We substitute this in the second equation (53), use (62) and obtain

$$d_Q \left(T^{[\alpha\beta]} - \frac{3}{2} \epsilon^{\alpha\beta\gamma} \lambda_\mu \lambda_\nu \Gamma_{\rho;\gamma} + \partial_\gamma B^{[\alpha\beta\gamma]} \right) = 0. \quad (66)$$

Using the description of the cocycles for d_Q we get from here:

$$T^{[\alpha\beta]} = \frac{3}{2} \epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} \lambda_\mu \lambda_\nu \Gamma_{\rho;\gamma} + i d_Q B^{[\alpha\beta]} - \partial_\gamma B^{[\alpha\beta\gamma]} + T_0^{[\alpha\beta]} \quad (67)$$

where the last term $T_0^{[\alpha\beta]}$ depends only on the invariants. Now we substitute the preceding expression in the first equation (53) and after some computations we arrive at

$$\begin{aligned} d_Q \left[T^\alpha + \frac{3}{4} \epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} \lambda_\mu \Gamma_{\nu;\beta} \Gamma_{\rho;\gamma} \right. \\ \left. + 3 \epsilon^{\mu\nu\rho} \lambda_\mu \lambda_\nu (\partial_\rho \tilde{u}^\alpha + \partial^\alpha \tilde{u}_\rho - \delta_\rho^\alpha \partial_\sigma \tilde{u}^\sigma) + 3 \epsilon_{\mu\nu\rho} \lambda^\mu V^\nu \phi^{\rho\alpha} + \partial_\beta B^{[\alpha\beta]} \right] \\ = \partial_\beta T_0^{[\alpha\beta]}. \quad (68) \end{aligned}$$

It is easy to verify this by computing d_Q of the terms in the bracket and comparing with $\partial_\beta T^{\alpha\beta}$ from (67).

It follows that the divergence in the right hand side must be a coboundary. It is rather straightforward to write a general ansatz for the expression $T_0^{[\alpha\beta]}$ and use this condition. The result is that $T_0^{[\alpha\beta]}$ is a relative cocycle *i.e.* an expression of the form

$$T_0^{[\alpha\beta]} = i d_Q B_0^{[\alpha\beta]} - \partial_\gamma B_0^{[\alpha\beta\gamma]} \quad (69)$$

so we can get rid of the last term in (67) if we redefine $B^{[\alpha\beta]}$ and $B^{[\alpha\beta\gamma]}$ properly. We are left with

$$T^{[\alpha\beta]} = \frac{3}{2} \epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} \lambda_\mu \lambda_\nu \Gamma_{\rho;\gamma} + i d_Q B^{[\alpha\beta]} - \partial_\gamma B^{[\alpha\beta\gamma]} \quad (70)$$

and (68) becomes

$$d_Q \left[T^\alpha + \frac{3}{4} \epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} \lambda_\mu \Gamma_{\nu;\beta} \Gamma_{\rho;\gamma} + 3 \epsilon^{\mu\nu\rho} \lambda_\mu \lambda_\nu (\partial_\rho \tilde{u}^\alpha + \partial^\alpha \tilde{u}_\rho - \delta_\rho^\alpha \partial_\sigma \tilde{u}^\sigma) \right. \\ \left. + 3 \epsilon_{\mu\nu\rho} \lambda^\mu V^\nu \phi^{\rho\alpha} + \partial_\beta B^{[\alpha\beta]} \right] = 0. \quad (71)$$

Using again the cohomology of the operator d_Q we obtain that

$$T^\alpha = -\frac{3}{4} \epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} \lambda_\mu \Gamma_{\nu;\beta} \Gamma_{\rho;\gamma} - 3 \epsilon^{\mu\nu\rho} \lambda_\mu \lambda_\nu (\partial_\rho \tilde{u}^\alpha + \partial^\alpha \tilde{u}_\rho - \delta_\rho^\alpha \partial_\sigma \tilde{u}^\sigma) \\ - 3 \epsilon_{\mu\nu\rho} \lambda^\mu V^\nu \phi^{\rho\alpha} + T_0^\alpha + i d_Q B^\alpha - \partial_\beta B^{[\alpha\beta]} \quad (72)$$

where T_0^α depends only on the invariants. A generic expression is

$$T_0^\alpha = a_1 \lambda^\alpha \phi^2 + a_2 \lambda^\alpha \phi^{\rho\sigma} \phi_{\rho\sigma} + a_3 \phi^{\alpha\beta} \phi_{\beta\mu} \lambda^\mu + a_4 \phi^{\alpha\beta} \phi \lambda_\beta. \quad (73)$$

We now substitute the expression (72) in the basic equation (2). For this purpose we calculate $\partial_\alpha T^\alpha$ and write it as a coboundary $d_Q(\dots)$ plus a rest. Then we obtain

$$d_Q \left[T + \frac{1}{8} \epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} \Gamma_{\mu;\alpha} \Gamma_{\nu;\beta} \Gamma_{\rho;\gamma} - 3 \epsilon^{\mu\nu\rho} \lambda_\mu \Gamma_{\nu;\alpha} (\partial_\rho \tilde{u}^\alpha + \partial^\alpha \tilde{u}_\rho - \delta_\rho^\alpha \partial_\sigma \tilde{u}^\sigma) \right. \\ \left. + \frac{3}{2} \epsilon_{\mu\nu\rho} \Gamma^{\mu;\alpha} V^\nu \phi_\alpha{}^\rho - 6 m \epsilon_{\mu\nu\rho} \lambda^\mu V^\nu \tilde{u}^\rho - t + \partial_\alpha B^\alpha \right] \\ = \left(a_3 + \frac{3}{2} \right) \lambda_\mu \partial^\rho \phi^{\mu\sigma} \phi_{\rho\sigma} + \left(2a_2 - \frac{3}{2} \right) \lambda_\mu \partial^\mu \phi^{\rho\sigma} \phi_{\rho\sigma} \\ + \left(a_4 - \frac{3}{2} \right) \lambda_\mu \partial_\mu \phi \phi^{\mu\nu} + \left(2a_1 + \frac{3}{2} \right) \lambda_\mu \partial^\mu \phi \phi \quad (74)$$

where

$$t \equiv a_3 \left(m \tilde{u}^\beta \phi_{\beta\mu} \lambda^\mu + \frac{1}{2} \phi^{\alpha\beta} \phi_{\beta\mu} \Gamma^{\mu;\alpha} \right) + a_4 \left(m \tilde{u}^\beta \phi \lambda^\beta + \frac{1}{2} \phi^{\alpha\beta} \phi \Gamma_{\beta;\alpha} \right) \quad (75)$$

It follows that we must choose

$$a_1 = -\frac{3}{4}, \quad a_2 = \frac{3}{4}, \quad a_3 = -\frac{3}{2}, \quad a_4 = \frac{3}{2}. \quad (76)$$

and the preceding relation is

$$d_Q \left[T + \frac{1}{8} \epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} \Gamma_{\mu;\alpha} \Gamma_{\nu;\beta} \Gamma_{\rho;\gamma} - 3 \epsilon^{\mu\nu\rho} \lambda_\mu \Gamma_{\nu;\alpha} (\partial_\rho \tilde{u}^\alpha + \partial^\alpha \tilde{u}_\rho - \delta_\rho^\alpha \partial_\sigma \tilde{u}^\sigma) \right. \\ \left. + \frac{3}{2} \epsilon_{\mu\nu\rho} \Gamma^{\mu;\alpha} V^\nu \phi_\alpha{}^\rho - 6 m \epsilon_{\mu\nu\rho} \lambda^\mu V^\nu \tilde{u}^\rho - t + \partial_\alpha B^\alpha \right] = 0, \quad (77)$$

where now

$$t \equiv \frac{3}{2} \left(-m \tilde{u}^\beta \phi_{\beta\mu} \lambda^\mu - \frac{1}{2} \phi^{\alpha\beta} \phi_{\beta\mu} \Gamma^{\mu;\alpha} + m \tilde{u}^\beta \phi \lambda^\beta + \frac{1}{2} \phi^{\alpha\beta} \phi \Gamma_{\beta;\alpha} \right) \quad (78)$$

Finally, we apply once more the description of the cohomology of the operator d_Q and get

$$T = -\frac{1}{8} \epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} \Gamma_{\mu;\alpha} \Gamma_{\nu;\beta} \Gamma_{\rho;\gamma} + 3 \epsilon^{\mu\nu\rho} \lambda_\mu \Gamma_{\nu;\alpha} (\partial_\rho \tilde{u}^\alpha + \partial^\alpha \tilde{u}_\rho - \delta_\rho^\alpha \partial_\sigma \tilde{u}^\sigma) \\ - \frac{3}{2} \epsilon_{\mu\nu\rho} \Gamma^{\mu;\alpha} V^\nu \phi_\alpha{}^\rho + 6 m \epsilon_{\mu\nu\rho} \lambda^\mu V^\nu \tilde{u}^\rho + t + T_0 \\ + i d_Q B - \partial_\alpha B^\alpha, \quad (79)$$

where the expression T_0 depends only on invariants. The generic form is

$$T_0 = c_1 \phi^{\mu\nu} \phi_{\nu\rho} \phi_\mu{}^\rho + c_2 \phi^{\mu\nu} \phi_{\mu\nu} \phi + c_3 \phi^3. \quad (80)$$

The final result is given by the formula (79) together with (78) and (80). One can prove that

$$\epsilon^{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} \Gamma_{\mu;\alpha} \Gamma_{\nu;\beta} \Gamma_{\rho;\gamma} = 8 \epsilon^{\alpha\beta\gamma} \Gamma_{\rho\alpha}^\nu \Gamma_{\mu\beta}^\rho \Gamma_{\nu\gamma}^\mu \quad (81)$$

which is the well-known tri-linear part of the Chern-Simons coupling for gravity in three dimensions. The second line in (79) is a ghost - antighost - graviton coupling, and the third line gives the coupling to the vector-graviton field. Without these couplings quantum gauge invariance is violated.

5. CONCLUSIONS

We have derived the coupling of 2+1-dimensional massive quantum gauge theory on Minkowski space from descent equations without using any classical Lagrangian. The complete coupling T including the parity conserving part is equal to

$$T = \kappa \left\{ -h^{\alpha\beta} \partial_\alpha h \partial_\beta h + 2h^{\alpha\beta} \partial_\alpha h_{\mu\nu} \partial_\beta h^{\mu\nu} + 4h_{\alpha\beta} \partial_\nu h^{\beta\mu} \partial_\mu h^{\alpha\nu} \right. \\ \left. + 2h_{\alpha\beta} \partial_\mu h^{\alpha\beta} \partial^\mu h - 4h_{\alpha\beta} \partial_\nu h^{\alpha\mu} \partial^\nu h_\mu^\beta - 4h^{\mu\alpha} \partial_\alpha v^\nu \partial_\mu v_\nu \right. \\ \left. - 4u^\mu \partial_\beta \tilde{u}_\nu \partial_\mu h^{\nu\beta} + 4\partial_\nu u^\beta \partial_\mu \tilde{u}_\beta h^{\mu\nu} - 4\partial_\nu u^\nu \partial_\mu \tilde{u}^\beta h^{\beta\mu} + 4\partial_\nu u^\mu \partial_\mu \tilde{u}_\beta h^{\nu\beta} \right. \\ \left. + m^2 \left(\frac{4}{3} h^{\mu\nu} h_{\mu\beta} h^{\nu\beta} - h^{\mu\beta} h_{\mu\beta} h + \frac{1}{6} h^3 \right) + 4m u^\mu \tilde{u}^\nu \partial_\mu v_\nu \right\} \\ + \frac{8}{\mu} \left\{ \epsilon^{\alpha\beta\gamma} \Gamma_{\rho\alpha}^\nu \Gamma_{\mu\beta}^\rho \Gamma_{\nu\gamma}^\mu + 3 \epsilon^{\mu\nu\rho} \lambda_\mu \Gamma_{\nu;\alpha} (\partial_\rho \tilde{u}^\alpha + \partial^\alpha \tilde{u}_\rho - \delta_\rho^\alpha \partial_\sigma \tilde{u}^\sigma) \right. \\ \left. - \frac{3}{2} \epsilon_{\mu\nu\rho} \Gamma^{\mu;\alpha} V^\nu \phi_\alpha{}^\rho + 6 m \epsilon_{\mu\nu\rho} \lambda^\mu V^\nu \tilde{u}^\rho + \right. \\ \left. + \frac{3}{2} \left(-m \tilde{u}^\beta \phi_{\beta\mu} \lambda^\mu - \frac{1}{2} \phi^{\alpha\beta} \phi_{\beta\mu}{}^\cdot \Gamma_{\mu;\alpha} + m \tilde{u}^\beta \phi \lambda^\beta + \frac{1}{2} \phi^{\alpha\beta} \phi \Gamma_{\beta;\alpha} \right) \right. \\ \left. + c_1 \phi^{\mu\nu} \phi_{\nu\rho} \phi_\mu{}^\rho + c_2 \phi^{\mu\nu} \phi_{\mu\nu} \phi + c_3 \phi^3 \right\}. \quad (82)$$

This is the lowest order trilinear coupling, higher orders can be computed from higher orders causal gauge invariance as in four dimensions [11], [18]. Note that in the limit $m \rightarrow 0$ the vector-graviton field v^μ which is also contained in V^μ and $\phi_{\mu\nu}$ does not decouple from the graviton field $h^{\mu\nu}$ in both parity-even and odd sectors. That means the massless limit of the massive theory does not agree with the $m = 0$ theory constructed without the v -field. The latter is trivial as far as the physical Hilbert space is concerned whereas the former is non-trivial.

In order to make contact with classical field theory one certainly asks: what is the classical Lagrangian which after expansion around flat background leads to the coupling (82). The answer is simple as far as the m -independent pure h -terms are concerned [11]: one takes the Einstein-Hilbert Lagrangian

$$L_{EH} = -\frac{2}{\kappa^2} \sqrt{-g} R \quad (83)$$

and expands the metric in the form

$$\sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu} \quad (84)$$

using the so-called Goldberg variables. Then the terms $O(\kappa)$ agree with the m -independent pure graviton terms in (82) (if an overall factor 4 is multiplied in (83)). The m -dependent terms in the fourth line of (82) are obtained if we add to (83) a cosmological term

$$-\frac{2}{\kappa^2} \sqrt{-g} 2\Lambda \quad (85)$$

and put

$$m^2 = -2\Lambda. \quad (86)$$

The same mass value comes out from $O(\kappa^0)$ and $O(\kappa^2)$ of the expansion. Consequently, we arrive at the cosmological topologically massive gravity with a negative cosmological constant which has been intensively studied in the literature.

The classical theory is usually expanded around anti-de-Sitter background which brings out the behaviour in the large. Our quantum theory describes the local aspects, *i.e.* the quantum fluctuations around the local Minkowski space. As a consequence the two pictures are rather different, for example, the mass of the classical AdS_3 graviton depends on the Chern-Simons coupling constant μ [8] whereas we have the simple relation (86). Another difference is that the classical Chern-Simons coupling contains a quadratic term $\Gamma\partial\Gamma$ which does not appear in our quantum theory. Our theory is only consistent if the vector-graviton field $v^\mu(x)$ with the same mass m as the graviton is included. This is required for quantum gauge invariance which is necessary for unitarity of the S-matrix on the physical Hilbert space. In section 3 we have shown that the v -field carries physical degrees of freedom. Therefore, it is hard

to believe that the classical theory can be consistent without the v -field.

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