

# DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS WITH $\mathbb{Z}_N$ AND $\mathbb{D}_N$ -REDUCTIONS\*

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*Received February 20, 2013*

Special types of derivative nonlinear Schrödinger equations with  $\mathbb{Z}_N$  and  $\mathbb{D}_N$ -reductions are analysed. They admit Lax pairs whose fundamental analytic solutions can be related to Riemann-Hilbert problem on a set of straight lines closing angles equal to  $\pi/N$ . We briefly analyse the spectral properties of the Lax operator, the hierarchies of conservation laws and Hamiltonian structures.

*Key words:*  $N$ -wave equations, Group of reductions, Lax representation .

*PACS:* 03.75.Lm, 42.65.Tg, 52.35.Sb, 71.15.Ap.

## 1. INTRODUCTION

The general theory of the nonlinear evolution equations (NLEE) allowing Lax representation is well developed, [1, 6, 9, 14–16, 24, 25]. This paper is an extension of the report [11] and the paper [12] and deals with NLEE that allow Lax representations with deep reductions. This means that they can be written as the commutativity condition of two ordinary differential operators of the type:

$$\mathcal{L}\chi(x, t, \lambda) \equiv \left( \frac{d}{dx} + U(x, t, \lambda) \right) \chi(x, t, \lambda) = 0, \quad (1)$$

$$\mathcal{M}\chi(x, t, \lambda) = \left( \frac{d}{dt} + V(x, t, \lambda) \right) \chi(x, t, \lambda) = \lambda^2 \chi(x, t, \lambda) V_2, \quad (2)$$

where the potentials  $U(x, t, \lambda)$  and  $V(x, t, \lambda)$  are polynomials of  $\lambda$ . For simplicity we assume that:

$$U(x, t, \lambda) = U_0(x, t) + \lambda U_1, \quad (3)$$

$$V(x, t, \lambda) = V_0(x, t) + \lambda V_1(x, t) + \lambda^2 V_2. \quad (4)$$

We request also that the Lax pair (1) possesses  $\mathbb{Z}_N$  and  $\mathbb{D}_N$ -reduction groups [22]. For the case of  $\mathbb{Z}_N$ -reduction this means that we impose on (1) and (2) a  $\mathbb{Z}_N$ -

\*Paper presented at “The 8<sup>th</sup> Workshop on Quantum Field Theory and Hamiltonian Systems”, September 19-22, 2012, Craiova, Romania.

reduction by [22]:

$$U(x, t, \lambda) = K^{-1}U(x, t, \lambda\omega)K, \quad V(x, t, \lambda) = K^{-1}V(x, t, \lambda\omega)K, \quad (5)$$

where  $K^N = \mathbb{1}$  and  $\omega = \exp(2\pi i/N)$ . We choose  $K$  to be:

$$K = \sum_{j=1}^N E_{j,j+1}, \quad (E_{jk})_{lm} = \delta_{jl}\delta_{km}; \quad (6)$$

In the case of  $\mathbb{D}_N$ -reduction besides (5) we impose also a  $\mathbb{Z}_2$ -reduction as follows:

$$U^\dagger(x, t, \lambda) = K_0^{-1}U(x, t, -\lambda^*)K_0, \quad V^\dagger(x, t, \lambda) = K_0^{-1}V(x, t, -\lambda^*)K_0, \quad (7)$$

where  $K_0^2 = \mathbb{1}$ . Below we consider only the simplest possible case, when the underlying algebra is  $sl(N, \mathbb{C})$  and the group of reduction is  $\mathbb{Z}_N$ ,  $\mathbb{D}_N$  and  $\mathbb{Z}_2 \times \mathbb{Z}_N$ . The class of relevant NLEE may be considered as generalizations of the derivative NLS equations [11], see also [19, 22]:

$$i \frac{\partial \psi_k}{\partial t} + \gamma \frac{\partial}{\partial x} \left( \cotan \frac{\pi k}{N} \cdot \psi_{k,x} + i \sum_{p=1}^{N-1} \psi_p \psi_{k-p} \right) = 0, \quad k = 1, 2, \dots, N-1, \quad (8)$$

where  $\gamma$  is a constant and the index  $k-p$  should be understood modulus  $N$ ,  $\psi_0 = \psi_N = 0$ . The system (8) allows also the involutions:

$$\begin{aligned} \text{a)} \quad & \psi_k = -\psi_k^*, & \gamma &= -\gamma^*, \\ \text{b)} \quad & \psi_k = \psi_{N-k}^*, & \gamma &= \gamma^*. \end{aligned} \quad (9)$$

These DNLS are members of the hierarchy of 2-dimensional integrable equations which contains also the two-dimensional Toda field theories [8, 19, 22] and the references therein.

Section 2 contains preliminaries necessary to derive the DNLS. In particular we provide a convenient basis of  $sl(N)$  which is compatible with the  $\mathbb{Z}_N$ -reduction. In Section 3 we derive the constraints on the Lax operator  $L$  which lead to the reductions (9) and their consequences for the scattering matrix and scattering data of  $L$ . We also give several particular cases of the DNLS eqs. and their reductions. In the next Section 4 we outline the construction of the fundamental analytic solutions (FAS) of  $L$ , their asymptotics for  $x \rightarrow \pm\infty$  and their symmetry properties. Section 5 shows that the FAS are directly related to solutions of a Riemann-Hilbert problem (RHP) on a set of lines intersecting at the origin and closing angles  $\pi/N$ . Thus instead of solving the inverse scattering problem for the operator  $L$  we can treat the RHP and use the dressing Zakharov-Shabat method for constructing the soliton solutions [22, 24]. We also derive the simplest integrals of motion of the DNLS eq. Section 6 treats the Hamiltonian structures of the DNLS.

## 2. PRELIMINARIES

Let us consider the Lax operator (1). To this end we will use a convenient basis in the algebra  $sl(N)$  which is compatible with the  $\mathbb{Z}_N$ -reduction. Here and below all indices are understood modulus  $N$ , so that the last term with  $j = N$  in (6) equals  $E_{N,1}$ . The automorphism  $\text{Ad}_K$  defines a grading in the Lie algebra

$$sl(N, \mathbb{C}) = \bigoplus_{k=0}^{N-1} \mathfrak{g}^{(k)}, \quad (10)$$

where  $\mathfrak{g}^{(k)}$  is the eigen-space of  $\text{Ad}_K$  corresponding to its eigenvalue  $\omega^{-k}$ ,  $k = 0, 1, \dots, N-1$ . The calculations are much simpler if we introduce a convenient basis in  $\mathfrak{g}^{(k)}$ , namely:

$$J_s^{(k)} = \sum_{j=1}^N \omega^{kj} E_{j,j+s}, \quad K^{-1} J_s^{(k)} K = \omega^{-k} J_s^{(k)}. \quad (11)$$

Obviously,  $J_s^{(k)}$  satisfies the commutation relations:

$$[J_s^{(k)}, J_l^{(m)}] = (\omega^{ms} - \omega^{kl}) J_{s+l}^{(k+m)} \quad (12)$$

and (5) is identically satisfied if  $U_k, V_k \in \mathfrak{g}^{(k)}$ . Let us put

$$U_0(x) = \sum_{j=1}^{N-1} \psi_j(x, t) J_j^{(0)}, \quad U_1 = -a\omega^{-1/2} J_0^{(1)}, \quad V_2 = -b J_0^{(2)}. \quad (13)$$

Then the requirement that (1) and (2) are compatible for all values of  $\lambda$  allows us to express  $V_0(x, t)$  and  $V_1(x, t)$  in terms of  $\psi_j(x, t)$  as follows:

$$\begin{aligned} V_1(x, t) &= \sum_{k=1}^N v_{1,k}(x, t) J_k^{(1)}, & v_{1,p} &= -\frac{b}{a} \omega^{(p+1)/2} \cos(p\pi/N) \psi_p(x, t), \\ V_0(x, t) &= \sum_{k=1}^{N-1} v_{0,k}(x, t) J_k^{(0)}, \end{aligned} \quad (14)$$

where

$$v_{0,p} = \gamma \left( i \cotan \frac{p\pi}{N} \psi_{p,x} - \sum_{k+s=p}^N \psi_k \psi_s(x, t) \right), \quad \gamma = \frac{b\omega}{a^2}.$$

The  $\lambda$ -independent term in the Lax representation vanishes whenever the functions  $\psi_k$  satisfy the DNLS eq. (8).

### 3. ADDITIONAL INVOLUTIONS AND EXAMPLES

Along with the  $\mathbb{Z}_N$ -reduction (5), we can introduce one of the following involutions ( $\mathbb{Z}_2$ -reductions):

$$\begin{aligned} \text{a)} \quad & K_0^{-1}U^\dagger(x, t, -\lambda^*)K_0 = -U(x, t, \lambda), \\ \text{b)} \quad & K_0^{-1}U^*(x, t, \lambda^*)K_0 = U(x, t, \lambda), \\ \text{c)} \quad & U^T(x, t, -\lambda) = -U(x, t, \lambda), \end{aligned} \quad (15)$$

where

$$K_0 = \sum_{k=1}^N E_{k, N-k+1}.$$

An immediate consequences of eq. (15) are the constraints on the potentials:

$$\begin{aligned} \text{a)} \quad & K_0^{-1}U_0^\dagger(x, t)K_0 = -U_0(x, t), & K_0^{-1}U_1^\dagger K_0 &= U_1, \\ \text{b)} \quad & K_0^{-1}U_0^*(x, t)K_0 = U_0(x, t), & K_0^{-1}U_1^* K_0 &= U_1, \\ \text{c)} \quad & U_0^T(x, t) = -U_0(x, t), & U_1 &= U_1, \end{aligned} \quad (16)$$

and the constraints on the fundamental solutions:

$$\begin{aligned} \text{a)} \quad & K_0^{-1}\chi^\dagger(x, t, -\lambda^*)K_0 = \chi^{-1}(x, t, \lambda), \\ \text{b)} \quad & K_0^{-1}\chi^*(x, t, \lambda)K_0 = \chi(x, t, \lambda), \\ \text{c)} \quad & \chi^T(x, t, \lambda) = \chi^{-1}(x, t, \lambda), \end{aligned} \quad (17)$$

More specifically from eq. (16) there follows:

$$\begin{aligned} \text{a)} \quad & \psi_j^*(x, t) = -\psi_j(x, t), & j &= 1, \dots, N-1. \\ \text{b)} \quad & \psi_j^*(x, t) = \psi_{N-j}(x, t), & j &= 1, \dots, N-1, \\ \text{c)} \quad & \psi_j(x, t) = -\psi_{N-j}(x, t), & j &= 1, \dots, N-1. \end{aligned} \quad (18)$$

The involutions a) and b) lead directly to the constraints in eq. (9). All three involutions are valid reductions for the  $L$  operator. However, the third involution is not applicable to the DNLS eqs. and thus the constraint (18c) is not compatible with eq. (8). The reason for that is that the second operator  $M$  is not compatible with it. In fact the reduction (16c) is compatible only with  $M$ -operators whose highest order term in  $\lambda$  is of odd power. In other words, this involutions is good only for NLEE that have odd dispersion laws.

Let us write down examples of DNLS systems of equations. The involution (9a) reduces eq. (8) to a system of equations for  $N$  real-valued functions  $u_k = i\psi_k$ ,  $\gamma = i\gamma_0$ :

$$\frac{\partial u_k}{\partial t} + \gamma_0 \frac{\partial}{\partial x} \left( \cotan \frac{\pi k}{N} \cdot u_{k,x} - \sum_{p=1}^{N-1} u_p u_{k-p} \right) = 0, \quad k = 1, 2, \dots, N-1. \quad (19)$$

The other two examples are obtained with involution (9b). If  $N = 5$  the involution (9b) leads to:  $\psi_0 = \psi_5 = 0$ ,  $\psi_1 = \psi_4^*$ ,  $\psi_2 = \psi_3^*$ , *i.e.*, we have only two independent complex-valued fields and

$$\begin{aligned} i \frac{\partial \psi_1}{\partial t} + \gamma \cotan \frac{\pi}{5} \frac{\partial^2 \psi_1}{\partial x^2} + i\gamma \frac{\partial}{\partial x} (2\psi_2 \psi_1^* + (\psi_2^*)^2) &= 0, \\ i \frac{\partial \psi_2}{\partial t} + \gamma \cotan \frac{2\pi}{5} \frac{\partial^2 \psi_2}{\partial x^2} + i\gamma \frac{\partial}{\partial x} (2\psi_1^* \psi_2^* + (\psi_1)^2) &= 0, \end{aligned} \quad (20)$$

For  $N = 6$  and  $\psi_1 = \psi_5^*$ ,  $\psi_2 = \psi_4^*$ ,  $\psi_3 = \psi_3^*$ , so we have a system for two complex-valued fields  $\psi_1$  and  $\psi_2$  and the real field  $\psi_3$ :

$$\begin{aligned} i \frac{\partial \psi_1}{\partial t} + \gamma \cotan \frac{\pi}{6} \frac{\partial^2 \psi_1}{\partial x^2} + 2i\gamma \frac{\partial}{\partial x} (\psi_1^* \psi_2 + \psi_2^* \psi_3) &= 0, \\ i \frac{\partial \psi_2}{\partial t} + \gamma \cotan \frac{2\pi}{6} \frac{\partial^2 \psi_2}{\partial x^2} + i\gamma \frac{\partial}{\partial x} (\psi_1^2 + 2\psi_1^* \psi_3 + (\psi_2^*)^2) &= 0, \\ \frac{\partial \psi_3}{\partial t} + 2\gamma \frac{\partial}{\partial x} (\psi_1 \psi_2 + \psi_1^* \psi_2^*) &= 0, \end{aligned} \quad (21)$$

#### 4. THE FAS OF THE LAX OPERATORS WITH $\mathbb{Z}_n$ -REDUCTION

Skipping the details, see [3–5, 18, 22], we just outline the procedure of constructing the FAS. First we have to determine the regions of analyticity. For smooth potentials  $U_0(x)$  that fall off fast enough for  $x \rightarrow \pm\infty$  these regions are the  $2N$  sectors  $\Omega_\nu$  separated by the rays  $l_\nu$  on which  $\operatorname{Re} \lambda(a_j - a_k) = 0$ , where by  $a_j$  here and below we mean  $a_j = -U_{1,jj} = -\omega^{j-1/2}$ . The rays  $l_\nu$  are given by:

$$l_\nu: \arg(\lambda) = \frac{\pi(\nu - 1)}{n}, \quad \nu = 1, \dots, 2N, \quad (22)$$

and close angles equal to  $\pi/N$ . Here without restrictions we have put  $a = 1$ ; indeed, we can always change  $\lambda \rightarrow \lambda' = a\lambda$ .

The next step is to construct the set of integral equations equivalent to (1) whose solution will be analytic in  $\Omega_\nu$ . To this end we associate with each sector  $\Omega_\nu$  the relations (orderings)  $\underset{\nu}{>}$  and  $\underset{\nu}{<}$  by:

$$\begin{aligned} \underset{\nu}{i > j} & \quad \text{if} & \quad \operatorname{Re} \lambda(a_i - a_j) < 0 & \quad \text{for } \lambda \in \Omega_\nu, \\ \underset{\nu}{i < j} & & \quad \operatorname{Re} \lambda(a_i - a_j) > 0 & \quad \text{for } \lambda \in \Omega_\nu. \end{aligned} \quad (23)$$

Then the solution of the system

$$\begin{aligned}\xi_{ij}^\nu(x, \lambda) &= \delta_{ij} + i \int_{-\infty}^x dy e^{-\lambda(a_i - a_j)(x-y)} \sum_{p=1}^h U_{0;ip}(y) \xi_{pj}^\nu(y, \lambda), \quad i \geq j; \\ \xi_{ij}^\nu(x, \lambda) &= i \int_{\infty}^x dy e^{-\lambda(a_i - a_j)(x-y)} \sum_{p=1}^h U_{0;ip}(y) \xi_{pj}^\nu(y, \lambda), \quad i < j;\end{aligned}\quad (24)$$

will be the FAS of  $L$  in the sector  $\Omega_\nu$ . The asymptotics of  $\xi^\nu(x, \lambda)$  and  $\xi^{\nu-1}(x, \lambda)$  along the ray  $l_\nu$  can be written in the form [12, 18]:

$$\begin{aligned}\lim_{x \rightarrow -\infty} e^{-\lambda U_1 x} \xi^\nu(x, \lambda e^{i0}) e^{\lambda U_1 x} &= S_\nu^+(\lambda), \quad \lambda \in l_\nu, \\ \lim_{x \rightarrow -\infty} e^{-\lambda U_1 x} \xi^{\nu-1}(x, \lambda e^{-i0}) e^{\lambda U_1 x} &= S_\nu^-(\lambda), \quad \lambda \in l_\nu, \\ \lim_{x \rightarrow \infty} e^{-\lambda U_1 x} \xi^\nu(x, \lambda e^{i0}) e^{\lambda U_1 x} &= T_\nu^- D_\nu^+(\lambda), \quad \lambda \in l_\nu, \\ \lim_{x \rightarrow \infty} e^{-\lambda U_1 x} \xi^{\nu-1}(x, \lambda e^{-i0}) e^{\lambda U_1 x} &= T_\nu^+ D_\nu^-(\lambda), \quad \lambda \in l_\nu,\end{aligned}\quad (25)$$

where the matrices  $S_\nu^+$ ,  $T_\nu^+$  (resp.  $S_\nu^-$ ,  $T_\nu^-$ ) are upper-triangular (resp. lower-triangular) with respect to the  $\nu$ -ordering. They provide the Gauss decomposition of the scattering matrix with respect to the  $\nu$ -ordering, *i.e.*:

$$T_\nu(\lambda) = T_\nu^-(\lambda) D_\nu^+(\lambda) \hat{S}_\nu^+(\lambda) = T_\nu^+(\lambda) D_\nu^-(\lambda) \hat{S}_\nu^-(\lambda), \quad \lambda \in l_\nu. \quad (26)$$

More careful analysis shows [18] that in fact  $T_\nu(\lambda)$  belongs to a subgroup  $\mathcal{G}_\nu$  of  $SL(N, \mathbb{C})$ . Indeed, with each ray  $l_\nu$  one can relate a subalgebra  $\mathfrak{g}_\nu \subset sl(N, \mathbb{C})$ . Each such  $sl(2)$ -subalgebra can be specified by a pair of indices  $(k, s)$  and is generated by:

$$h^{(k,s)} = E_{kk} - E_{ss}, \quad e^{(k,s)} = E_{ks}, \quad f^{(k,s)} = E_{sk}, \quad k < s. \quad (27)$$

Then the scattering matrix  $T_\nu(\lambda)$  will be a product of mutually commuting matrices  $T_\nu^{(k,s)}$  of the form:

$$T_\nu^{(k,s)} = \mathbb{1} + (a_{\nu;ks}^+(\lambda) - 1) E_{kk} + (a_{\nu;ks}^-(\lambda) - 1) E_{ss} - b_{\nu;ks}^-(\lambda) E_{ks} + b_{\nu;ks}^+(\lambda) E_{sk}, \quad (28)$$

where  $k < s$ , with only 4 non-trivial matrix elements, just like the ZS (or AKNS) system. The  $\mathbb{Z}_n$ -symmetry imposes the following constraints on the FAS and on the scattering matrix and its factors:

$$\begin{aligned}C_0 \xi^\nu(x, \lambda \omega) C_0^{-1} &= \xi^{\nu-2}(x, \lambda), & C_0 T_\nu(\lambda \omega) C_0^{-1} &= T_{\nu-2}(\lambda), \\ C_0 S_\nu^\pm(\lambda \omega) C_0^{-1} &= S_{\nu-2}^\pm(\lambda), & C_0 D_\nu^\pm(\lambda \omega) C_0^{-1} &= D_{\nu-2}^\pm(\lambda),\end{aligned}\quad (29)$$

where the index  $\nu - 2$  should be taken modulo  $2N$ . Consequently we can view as independent only the data on two of the rays, e.g. on  $l_1$  and  $l_{2N} \equiv l_0$ ; all the rest will be recovered using (29).

If in addition we impose the  $\mathbb{Z}_2$ -symmetry (17a), then we will have also:

$$\begin{aligned} \text{a)} \quad & K_0^{-1}(\xi^\nu(x, -\lambda^*))^\dagger K_0 = (\xi^{N+1-\nu}(x, \lambda))^{-1}, \\ & K_0^{-1}(S_\nu^\pm(-\lambda^*))K_0 = (S_{N+1-\nu}^\mp(\lambda))^{-1}, \\ \text{b)} \quad & K_0^{-1}(\xi^\nu(x, \lambda^*))^* K_0 = (\xi^\nu(x, \lambda))^{-1}, \\ & K_0^{-1}(S_\nu^\pm(\lambda^*))K_0 = (S_{N+1-\nu}^\mp(\lambda))^{-1}, \end{aligned} \quad (30)$$

and analogous relations for  $T_\nu^\pm(\lambda)$  and  $D_\nu^\pm(\lambda)$ . One can prove also that  $D_\nu^+(\lambda)$  (resp.  $D_\nu^-(\lambda)$ ) allows analytic extension for  $\lambda \in \Omega_\nu$  (resp. for  $\lambda \in \Omega_{\nu-1}$ ). Another important fact is that  $D_\nu^+(\lambda) = D_{\nu+1}^-(\lambda)$  for all  $\lambda \in \Omega_\nu$  [18].

## 5. THE INVERSE SCATTERING PROBLEM AND THE RIEMANN-HILBERT PROBLEM

The next important step is the possibility to reduce the solution of the ISP for the GZSs to a (local) RHP. More precisely, we have:

$$\begin{aligned} \xi^\nu(x, t, \lambda) &= \xi^{\nu-1}(x, t, \lambda)G_\nu(x, t, \lambda), & \lambda \in l_\nu, \\ G_\nu(x, t, \lambda) &= e^{\lambda U_1 x - \lambda^2 V_2 t} G_{0,\nu}(\lambda) e^{-\lambda U_1 x + \lambda^2 V_2 t}, & G_{0,\nu}(\lambda) = \hat{S}_\nu^- S_\nu^+(\lambda) \Big|_{t=0}. \end{aligned} \quad (31)$$

The collection of all relations (31) for  $\nu = 1, 2, \dots, 2N$  together with

$$\lim_{\lambda \rightarrow \infty} \xi^\nu(x, t, \lambda) = \mathbb{1}, \quad (32)$$

can be viewed as a local RHP posed on the collection of rays  $\Sigma \equiv \{l_\nu\}_{\nu=1}^{2N}$  with canonical normalization. Rather straightforwardly we can prove that if  $\xi^\nu(x, \lambda)$  is a solution of the RHP (31), (32) then  $\chi^\nu(x, \lambda) = \xi^\nu(x, \lambda)e^{-\lambda U_1 x}$  is a FAS of  $L$  with potential

$$U_0(x, t) = \lim_{\lambda \rightarrow \infty} \lambda \left( U_1 - \xi^\nu(x, t, \lambda) U_1 \hat{\xi}^\nu(x, t, \lambda) \right). \quad (33)$$

The analyticity properties of  $D_k^\pm(\lambda)$  allow one to reconstruct them from the sewing function  $G(\lambda)$  (31) and from the locations of their simple zeroes and poles through

$$\ln \mathcal{D}_k(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \ln \mu_{G_0, k} + \sum_{j=1}^N \sum_{s=0}^{N-1} \ln \frac{\lambda - \lambda_{j,s}^+}{\lambda - \lambda_{j,s}^-}, \quad (34)$$

where  $\mu_{G_0, k}$  is the principal upper minor of  $G_0(t, \lambda)$  of order  $k$ . The zeroes and poles  $\lambda_{j,s}^\pm$  of  $\mathcal{D}$  are in fact discrete eigenvalues of  $L$ , which due to the reductions come in multiplets:

$$\begin{aligned} \text{a)} \quad & \lambda_{j,s}^+ = \lambda_j^+ \omega^{s-1}, \quad \lambda_{j,s}^- = -(\lambda_{j,s}^+)^*, \quad \lambda_j^+ \in \Omega_1, \\ \text{b)} \quad & \lambda_{j,s}^+ = \lambda_j^+ \omega^{s-1}, \quad \lambda_{j,s}^- = (\lambda_{j,s}^+)^*, \quad \lambda_j^+ \in \Omega_1. \end{aligned} \quad (35)$$

Consider first case a). For odd  $N$  and  $\lambda_j^+ \in \Omega_1$  it is easy to check that all eigenvalues are inside sectors  $\Omega_{2\nu-1}$  with odd indices; if  $\lambda_j^+ \in \Omega_0$  then all eigenvalues are inside sectors  $\Omega_{2\nu}$  with even indices. Thus for each generic choice of  $\lambda_j^+$  we have a multiplet of  $2N$  discrete eigenvalues. However, if we choose  $\arg \lambda_j^+ = \pi/(2N)$ , then the set of  $\lambda_{j,s}^-$  coincides with the set of  $\lambda_{j,s}^+$  and we have smaller multiplets with  $N$  discrete eigenvalues each. Thus one may conclude that for odd  $N$  the DNLS eqs. have two types of one-soliton solutions corresponding to the two different types of multiplets. For even  $N$  the situation is different. All multiplets containing  $\lambda_j^+$  has exactly  $2N$  discrete eigenvalues, one in each of the  $2N$  sectors  $\Omega_\nu$ . So for odd  $N$  only one type of one-soliton solutions exists.

In the case b) we will have multiplets of  $2N$  eigenvalues, one for each of the sectors  $\Omega_\nu$  both for  $N$  even and  $N$  odd.

More detailed analysis shows that  $\mathcal{D}_k^+(\lambda)$  (resp.  $\mathcal{D}_k^-(\lambda)$ ) are related to the principle upper (resp. lower) minors of order  $k$  of the scattering matrix  $T(t, \lambda)$  by:

$$\mathcal{D}_k(\lambda) = \begin{cases} \ln m_k^+(\lambda), & \lambda \in \mathbb{C}_+ \\ -\ln m_{n-k}^-(\lambda), & \lambda \in \mathbb{C}_-. \end{cases} \quad (36)$$

One can view  $\mathcal{D}_k(\lambda)$  as generating functionals of the conserved quantities for the related NLEE. Using the fact that  $\ln m_{\nu,k}^+(\lambda)$  allow asymptotic expansions

$$\ln m_{\nu,k}^+(\lambda) = \sum_{s=1}^{\infty} M_{\nu,k}^{(s)} \lambda^{-s}. \quad (37)$$

we are able to calculate the local integrals of motion for the DNLS eqs. We illustrate it by the two first integrals of motion of the  $\mathbb{Z}_n$ -NLS equation:

$$M_{1,1}^{(1)} = \frac{1}{2\omega} \int_{-\infty}^{\infty} dx \sum_{p=1}^n \psi_p \psi_{n-p}(x, t), \quad (38)$$

$$M_{1,1}^{(2)} = \frac{1}{2\omega^2} \int_{-\infty}^{\infty} dx \left\{ \sum_{p=1}^n i \cotan\left(\frac{\pi p}{n}\right) \left( \frac{d\psi_p}{dx} \psi_{n-p} - \psi_p \frac{d\psi_{n-p}}{dx} \right) - \frac{2}{3} \sum_{p+k+l=n} \psi_p \psi_k \psi_l(x, t) \right\}. \quad (39)$$

## 6. HAMILTONIAN STRUCTURES

The system of equations (8) allows a hierarchy of Hamiltonian structures. Here we will use the approach of Kulish and Reiman [20], developed for generic (non-restricted) systems. In cases, which involve reductions one should: (i) calculate the



generic Poisson brackets; (ii) impose the reduction as constraints on these brackets; (iii) calculate the corresponding Dirac brackets.

However, if we define  $\psi_j(x)$  as linear functionals of  $U(x, t, \lambda)$  by:

$$\psi_j(x, t) = \frac{1}{N} \text{tr} U(x, t, \lambda) J_{N-j}^{(0)},$$

then the results of [20] surprisingly give us the correct answer [21]:

$$\{\psi_j(x, t), \psi_k(x, t)\} = \delta_{k+j-N} \delta'(x - y). \quad (40)$$

The Poisson brackets (40) and the Hamiltonian  $H = 2\omega^2 \gamma M_{1,1}^{(2)}$  (39) lead to the DNLS eq.(8).

## 7. CONCLUSIONS

The generic integrable NLEE possesses a hierarchy of Hamiltonian structures, generated by a certain recursion, or  $\Lambda$ -operator [1, 10, 13, 14, 16, 17]. Its spectral expansions and applications to the theory of the corresponding NLEE are well known, see e.g. [17]. However, the reduced case has some peculiarities [10] which must be taken care of when studying the properties of the corresponding  $\Lambda$ -operator.

The leading power of  $\lambda$  in  $\mathcal{M}$  (2) should be an exponent of the corresponding algebra  $\mathfrak{g}$ . Since 2 is an exponent only for  $A_n$ , there would be no new examples of  $\mathbb{Z}_N$ -reduced NLS type equations. However, 3 is an exponent for all the algebras from the classical series. Choosing  $\mathcal{M}$  to be cubic in  $\lambda$  one will get series of KdV type equations; the simplest of them are known, see [7, 8, 23].

*Acknowledgments.* I am grateful to professors R. Constantinescu and G. Djordjevich for making my participation in the conference possible and to the SEENET-TMP network programm for supporting my visit to Craiova University, Romania, where this paper was finished. I am grateful also to professors S. V. Manakov, A. V. Mikhailov, J. Gibbons and G. Wilson for useful discussions.

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