A NEW LOOK AT THE MILNE UNIVERSE AND ITS GROUND STATE WAVE FUNCTIONS*

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In this paper we present the Milne model, as a special case of the Friedmann-Lemaitre-Robertson-Walker model in the limit of zero mass density. We consider this model in its quantum form, both on real, and p-adic, *i.e.* ultra-metric spaces, with a special emphasize on evaluation of all vacuum states of its wave function. Although the Milne model predicts matter density and spatial curvature which are in contradictions with observations, nevertheless this approach remains active and interesting for quite a number of researchers. We discuss possibilities for a plausible role of this model within the framework of Quantum Cosmology and in particular in 2+1 dimensions. Finally we comment a possibility to formulate the Milne model in a complete adelic form.

Key words: Milne Universe, Quantum cosmology.

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1. INTRODUCTION

Milne model of the universe (E. A. Milne, 1935) [1] describes the universe as an open one with hyperbolic geometry in the limit of vanishing energy density. Despite an obvious contradiction between Milne model [2] and nowadays observational data and standard ΛCMD model this model shows a strong viability and attracts interest of many researchers from a variety of approaches and motivations.

Milne model of the universe is based on special relativity. Again, despite such a unrealistic base the Milne universe does illustrate some useful points. It shows the similarities between the Big Bang and Hubble expansion because the Milne universe has a center while curved Friedmann universe not. The Milne model is very useful in the illustration of the difference between 3-curvature and 4-curvature. The Milne universe ($k=-1$) has a nonzero spatial curvature, but zero 4-curvature until spatially flat universe ($k=0$) has a nonzero 4-curvature.

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Besides this, mainly pedagogical reasons to study such a weird cosmology, in the last decade we see increasing interest for Milne approach. Let us mention articles in which are investigated existence/absence of tachyons [3], some cosmological string models [4, 5], trans-Planckian modification of QFT [6], motion of the quantum particle across the big-crunch/big-bang type singularity [7], etc.

A new motivation for application of the Milne cosmological model comes from two as yet directly unobserved components of the universe, dark matter and dark energy, that constitute more than 95% of the Universe [8]. According to the inflation-based ΛCDM cosmological model our Universe is composed of only 5% of normal matter (baryons), $\sim$25% of dark matter and $\sim$70% of dark energy. Dark matter is supposed to be responsible for anomaly large rotational velocities in the spiral galaxies, while dark energy behaves in a similar way as a vacuum energy and therefore appear to be responsible for the acceleration of the expansion of the Universe. Although dark matter and dark energy are well adopted among majority of scientists, they are not directly verified in the laboratory and still remain hypothetical forms of matter. For these reasons, there is not yet commonly accepted theoretical explanation of the Universe acceleration.

Faced with this uncomfortable situation where we understand less than 5% of our Universe, in the paper [8] is studied the unconventional cosmology of a symmetric universe *i.e.* containing equal quantities of matter and antimatter. Antimatter is supposed to present a negative active gravitational mass. Motivations for attributing a negative active gravitational mass to antimatter (with gravitational repulsion between matter and antimatter) comes from the work on Kerr-Newmann geometry describing charged rotating black holes [9]. The main consequence of this hypothesis is that on large scales, the expansion factor evolves linearly with time, which is reminiscent of the Milne cosmology. In the symmetric Milne Universe we have a flat space-time (the space is hyperbolic!) with no dark matter and no dark energy.

In this paper is formulated real p-adic version of (2+1)-dimensional Milne cosmology, in both classical and quantum way. A strong impetus to the development of p-adic models was given by the hypothesis about a possible p-adic structure of physical space-time at subPlanck distances ($\leq 10^{-33}$ cm) [10]. p-Adic strings were introduced in 1987 [11] and a adelic formula was obtained. Application of p-adic numbers in construction of various models is presented in [12] and [13].

This paper is organized as follows. Following the Introduction, in Section 2 we recapitulate basic facts about p-adics, p-adic and adelic analysis and nonarchimedean geometry. Section 3 is a brief recapitulation of the Milne model and General Relativity in 2+1 dimensions. In Section 4 we show how quantum cosmology is related to relativistic particle mechanics. The main and new results are presented in Section 5, where we present the Milne cosmological model in 2+1 dimension in both the classical and quantum case, and in both the p-adic and real approach. Here we are focussed on a particular case *i.e.* $p=2$. We complete this paper with concluding remarks and suggestions for further investigation.

2. p-ADIC AND ADELIC MATHEMATICS

Completion of the field of rational numbers $\mathbb{Q}$ with respect to the standard absolute value ($|\cdot|_\infty$) gives $\mathbb{R}$, and an algebraic extension of $\mathbb{R}$ makes $\mathbb{C}$, and any non-trivial norm on $\mathbb{Q}$ is equivalent to the absolute value $|\cdot|_\infty$ or to the p-adic norm $|\cdot|_p$ (Ostrowski) [10], where p is a prime number. p-Adic norm is the nonarchimedean one and for a rational number, $0\neq x\in\mathbb{Q},\ x=p^\nu\frac{m}{n}$, $0\neq n$, $\nu,m\in\mathbb{Z}$, has a value $|x|_p=p^{-\nu}$. Completion of Q with respect to the p-adic norm for a fixed p leads to the corresponding field of p-adic numbers $\mathbb{Q}_p$. Completions of $\mathbb{Q}$ with respect to $|\cdot|_\infty$ and all $|\cdot|_p$ exhaust all possible completions of $\mathbb{Q}$. The norm of p-adic number x is $|x|_p=p^{-\nu}$ and satisfies not only the triangle inequality, but also the stronger one

$$|x+y|_p\leq\max(|x|_p,|y|_p). \tag{1}$$

Real and p-adic numbers are unified in the form of the adeles. An adele is an infinite sequence $a=(a_\infty,a_2,...,a_p,...)$, where $a_\infty\in\mathbb{Q}_\infty$, and $a_p\in\mathbb{Q}_p$, with restriction to $a_p\in\mathbb{Z}_p$ ($\mathbb{Z}_p=\{x\in\mathbb{Q}_p:|x|_p\leq1\}$) for all but a finite set S of primes p. If we introduce $\mathcal{A}(S)=\mathbb{Q}_\infty\times\prod\limits_{p\in S}\mathbb{Q}_p\times\prod\limits_{p\notin S}\mathbb{Z}_p$ then the space of all adeles is $\mathcal{A}=\bigcup\limits_S\mathcal{A}(S)$, which is a topological ring. An important function on $\mathcal{A}$ is the additive character $\chi(x),\ x\in\mathcal{A}$, which is a continuous and complex-valued function:

$$|\chi(x)|_\infty=1,\quad \chi(x+y)=\chi(x)\chi(y). \tag{2}$$

This additive character may be presented as

$$\chi(x)=\prod_\upsilon\chi_\upsilon(x_\upsilon)=\exp(-2\pi ix_\infty)\prod_p\exp(2\pi i\{x_p\}_p), \tag{3}$$

where $\upsilon=\infty,2,\cdots,p,\cdots$, and $\{x\}_p$ is the fractional part of the p-adic number x. Map $\varphi:\mathcal{A}\to\mathbb{C}$, which has the form

$$\varphi(x)=\varphi_\infty(x_\infty)\prod_{p\in S}\varphi_p(x_p)\prod_{p\notin S}\Omega(|\,x_p\,|_p), \tag{4}$$

where $\varphi_\infty(x_\infty)\in D(\mathbb{Q}_\infty)$ is an infinitely differentiable function on $\mathbb{Q}_\infty$ and falls to zero faster than any power of $|\,x_\infty\,|_\infty$ as $|\,x_\infty\,|_\infty\to\infty$, $\varphi_p(x_p)\in D(\mathbb{Q}_p)$ is a locally constant function with compact support, and

$$\Omega(|x|_p)=\begin{cases}1, & |x|_p\leq1,\\ 0, & |x|_p>1,\end{cases} \tag{5}$$

is called an elementary function on $\mathcal{A}$. It is worth noting that Ω-function is a counterpart of the Gaussian in the real case, since it is invariant with respect to the Fourier transform. It is also an important issue for consideration of the ground state(s) of quantum mechanical systems at high energies, where the use of p-adic numbers and nonarchimedean geometry in "modelling" should be fully justified.

The integrals of the Gauss type over the p-adic sphere S_ν, p-adic ball B_ν and over any $\mathbb{Q}_v$ are (for $|4\alpha|_p \geq p^{2-2\nu}$):

$$\int_{S_\nu} \chi_p\left(\alpha x^2 + \beta x\right) dx = \begin{cases} \lambda_p(\alpha)|2\alpha|_p^{-1/2}\chi_p\left(-\frac{\beta^2}{4\alpha}\right), & \left|\frac{\beta}{2\alpha}\right|_p = p^\nu, \\ 0, & \left|\frac{\beta}{2\alpha}\right|_p \neq p^\nu, \end{cases} \tag{6}$$

$$\int_{B_\nu} \chi_p(\alpha x^2 + \beta x) dx = \begin{cases} p^\nu \Omega(p^\nu |\beta|_p), & |\alpha|_p p^{2\nu} \leq 1, \\ \frac{\lambda_p(\alpha)}{|2\alpha|_p^{1/2}}\chi_p\left(-\frac{\beta^2}{4\alpha}\right)\Omega\left(p^{-\nu}\left|\frac{\beta}{2\alpha}\right|_p\right), & |4\alpha|_p p^{2\nu} > 1, \end{cases} \tag{7}$$

$$\int_{\mathbb{Q}_v} \chi_p(\alpha x^2 + \beta x) dx = \lambda_v(\alpha)|2\alpha|_v^{-1/2}\chi_v\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0. \tag{8}$$

For more about the arithmetic functions $\lambda_v(a): \ \mathbb{Q}_v \mapsto \mathbb{C}$, $v = \infty, 2, 3, 5, \cdots$, see [10].

3. MILNE MODEL AND GENERAL RELATIVITY IN $2+1$ DIMENSIONS

As it is well known Milne model describes the universe as open with hyperbolic geometry $(k = -1)$ in the limit of vanishing energy density. With this assumptions the 2nd Friedman equation is reduced to $\dot{a}^2 = 1$, *i.e.* $a = 1$. The corresponding metric

$$ds^2 = dt^2 - t^2(d\chi^2 + \sinh^2\chi d\Omega^2), \tag{9}$$

describes space-time of a Milne universe [2]. Because of introduced assumptions this metric is a solution of the Einstein equations for an isotropic space without matter *i.e.* it must be Minkowskian type. Indeed if we start with the Minkowski metric

$$ds^2 = d\tau^2 - dr^2 - r^2 d\Omega^2, \tag{10}$$

with replacement: $\tau = t\cosh\chi$, $r = t\sinh\chi$, metric (10) reduces to (9). Further analysis shows that Milne coordinates cover only a quarter of Minkowski space-time. There is also relation between Milne space-time and Rindler space-time [14].

In any space-time, the curvature tensor may be decomposed into a curvature scalar R, a Ricci tensor $R_{\mu\nu}$ and a remaining trace-free, conformally invariant, the Weyl tensor $C^\sigma_{mu\nu\rho}$. In $2+1$ dimensions, however, Weyl tensor vanishes identically, and the curvature tensor is determined by the curvature scalar and Ricci tensor [15]

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma})\mathcal{R}. \tag{11}$$

This imply that any solution of vacuum Einstein field equations is flat, and that any solution of the field equations with a cosmological constant $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$ has constant curvature. In $2+1$ dimensions, space-time has no local degrees of freedom: there are no gravitational waves in the classical theory, and no gravitons in the quantum theory. Moreover, in three dimensions a mini super-space reduction might be expected to faithfully represent the full theory [16].

4. COSMOLOGY AS RELATIVISTIC PARTICLE MECHANICS

Cosmology can be viewed as geodesic motion in an appropriate metric on an target space - mini super-space [17, 18]. Motion of this type can be deduced from an effective relativistic point particle Lagrangian, in which the Friedmann constraint arises as a mass-shell constraint.

If we choose for the metric of the three dimensional gravity [16]

$$ds^2 = -N^2(t)dt^2 + h_{ij}(t)dx^i dx^j, \quad i,j = 1,2. \tag{12}$$

the extrinsic curvature is $K_{ij} = -\frac{1}{2N}\dot{h}_{ij}$, and $K = h^{ij}K_{ij} = -\frac{1}{2N}\frac{d}{dt}\log h$, ($h = \det h_{ij}$). The Einstein-Hilbert action

$$\begin{aligned} S &= \int d^3x\sqrt{-g}\left({}^{(3)}\mathcal{R} - 2\Lambda\right) + 2\int d^2x\sqrt{h}K \\ &= \int dt d^2x\sqrt{h}N\left(-K^2 + K_{ij}K^{ij} - 2\Lambda\right), \end{aligned} \tag{13}$$

after introduced coset coordinates U^M $(M = 1,2)$ and invariant metric $\dot{U}^M G_{MN}\dot{U}^N = -\frac{1}{2}\dot{h}_{ij}\dot{h}^{ij}$ as well as the field redefinition $\rho = RN$, $R = \sqrt{h}$, yields

$$S = \frac{1}{2}\int dt\left[\frac{1}{\rho}\left(-\dot{R}^2 + R^2\dot{U}^M G_{MN}\dot{U}^N\right) - 4\Lambda\rho\right]. \tag{14}$$

Usual action for a free relativistic particle

$$S = -m\int_{\tau_1}^{\tau_2} d\tau\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = -m\int_{\tau_1}^{\tau_2} d\tau\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{15}$$

is nonlinear and unsuitable for quantization. However, a free relativistic particle can be considered as a system with the (mass-shell or Hamiltonian) constraint $\eta_{\mu\nu}k^\mu k^\nu + m^2 = 0$, which lead to the canonical Hamiltonian [19] $H_c = N(\eta_{\mu\nu}k^\mu k^\nu + m^2)$, ($N$ is Lagrange multiplier) and to the Lagrangian $L = \dot{x}_\mu k^\mu - H_c = \frac{\dot{x}^2}{4N} - m^2 N$. Instead of (15), the corresponding action, suitable for quantum-mechanical investigations of a free relativistic particle is

$$S = \int_{\tau_1}^{\tau_2} d\tau\left(\frac{\dot{x}^2}{4N} - m^2 N\right). \tag{16}$$

Let us note, this relativistic particle is tachyonic in AdS. If we compare (14) and (16) we will see that the action (14) is the action of a relativistic particle in three dimensions with $(\text{mass})^2 = 4\Lambda$ and in the Lorentzian spacetime with metric [16]

$$ds^2 = -dR^2 + R^2 dU^M G_{MN} dU^N. \tag{17}$$

The metric (17) can be presented in the form

$$ds^2 = -dR^2 + R^2 \frac{d\tau_1^2 + d\tau_2^2}{\tau_2^2} \tag{18}$$

$(-\infty < \tau_1 < \infty, 0 < \tau_2$ and $0 < R)$. This space is three dimensional Milne Universe. By the transformations $R = \sqrt{Z^2 - X^2 - Y^2}$, $\tau_1 = \frac{Y}{Z-X}$, $\tau_2 = \frac{R}{Z-X}$, the metric becomes the flat three dimensional Minkowski one $ds^2 = -dZ^2 + dX^2 + dY^2$, with corresponding action

$$S = \frac{1}{2}\int dt \left(\frac{1}{\rho}(-\dot{Z}^2 + \dot{X}^2 + \dot{Y}^2) - 4\Lambda\rho\right). \tag{19}$$

Lagrangian

$$L = \frac{1}{2\rho}(-\dot{Z}^2 + \dot{X}^2 + \dot{Y}^2) - 2\Lambda\rho. \tag{20}$$

leads to the canonical Hamiltonian

$$H_c = \rho\left(-\frac{1}{2}(k^Z)^2 + \frac{1}{2}(k^X)^2 + \frac{1}{2}(k^Y)^2 + 2\Lambda\right), \tag{21}$$

(with the Lagrange multiplier ρ). Classical action is

$$\bar{S} = \frac{(t''-t')}{2}\left\{\frac{1}{\rho}\left(-\left(\frac{Z''-Z'}{t''-t'}\right)^2 + \left(\frac{X''-X'}{t''-t'}\right)^2 + \left(\frac{Y''-Y'}{t''-t'}\right)^2\right) - 4\Lambda\rho\right\}, \tag{22}$$

or

$$\bar{S} = -\frac{(Z''-Z')^2}{2T} + \frac{(X''-X')^2}{2T} + \frac{(Y''-Y')^2}{2T} - 2\Lambda T, \tag{23}$$

where $T = \rho(t''-t')$. Note that the classical action (23) can be presented in the form

$$\bar{S} = \left(-\frac{(Z''-Z')^2}{2T} + 2\Lambda T\right) + \left(\frac{(X''-X')^2}{2T} - 2\Lambda T\right) + \left(\frac{(Y''-Y')^2}{2T} - 2\Lambda T\right)$$
$$= \bar{S}^Z + \bar{S}^X + \bar{S}^Y, \tag{24}$$

which is quadratic in all space-time coordinates.

5. CLASSICAL AND QUANTUM DYNAMICS

5.1. CLASSICAL AND QUANTUM DYNAMICS IN STANDARD CASE

From the (20), in the gauge $\rho = 1$, straight line geodesic are

$$Z = Z' + k^Z t, \quad X = X' + k^X t, \quad Y = Y' + k^Y t, \tag{25}$$

subject to a mass-shell constraint $-(k^Z)^2+(k^X)^2+(k^Y)^2=4\Lambda$.

Details of the classical dynamics are presented in the ref. [16]. It is very interesting that you can find the solution which correspond to the Kasner, de Sitter and, as well as, anti de Sitter case.

If you want to define quantum dynamics in the case considered three-dimensional gravity, we have to solve Wheeler-de Witt equation

$$\left\{-\frac{\partial^2}{\partial Z^2}+\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right\}\Psi=0 \tag{26}$$

which is dictated by the Hamiltonian constraint of the theory. An initial investigation of this equation of Klein-Gordon type, has been conducted in [4]. Lack of exact form of the wave function of the Milne universe is one of the last obstacles to formulate the complete adelic Milne model.

5.2. CLASSICAL AND QUANTUM DYNAMICS IN p-ADIC AND ADELIC CASE

In foundations of standard quantum mechanics (over $\mathbb{R}$) one usually starts with a representation of the canonical commutation relation $[\hat{q},\hat{k}]=i\hbar$, where q is a spatial coordinate and k is the corresponding momentum. It is well-known that the procedure of quantization is not unique. In formulation of p-adic quantum mechanics [10] the multiplication $\hat{q}\psi\to x\psi$ has no meaning for $x\in\mathbb{Q}_p$ and $\psi(x)\in\mathbb{C}$. Also, there is no possibility to define p-adic "momentum" or "Hamiltonian" operator.

Dynamics of a p-adic quantum model is described by a unitary operator of evolution $U(t)$ without using the Hamiltonian. Instead of that, the evolution operator has been formulated in terms of its kernel $\mathcal{K}_t(x,y)$

$$U_p(t)\psi(x)=\int_{\mathbb{Q}_p}\mathcal{K}_t(x,y)\psi(y)dy. \tag{27}$$

In this way p-adic quantum mechanics is given by a triple $(L_2(\mathbb{Q}_p),W_p(z_p),U_p(t_p))$. Keeping in mind that standard quantum mechanics can be also given as the corresponding triple, ordinary and p-adic quantum mechanics can be unified in the form of adelic quantum mechanics [13] $(L_2(\mathcal{A}),W(z),U(t))$. $L_2(\mathcal{A})$ is the Hilbert space on $\mathcal{A}$, $W(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathcal{A})$ and $U(t)$ is a unitary representation of the evolution operator on $L_2(\mathcal{A})$. The evolution operator $U(t)$ is defined by

$$U(t)\psi(x)=\int_{\mathcal{A}}\mathcal{K}_t(x,y)\psi(y)dy=\prod_v\int_{\mathbb{Q}_v}\mathcal{K}_t^{(v)}(x_v,y_v)\psi^{(v)}(y_v)dy_v. \tag{28}$$

The eigenvalue problem for $U(t)$ reads

$$U(t)\psi_{\alpha\beta}(x)=\chi(E_\alpha t)\psi_{\alpha\beta}(x), \tag{29}$$

where $\psi_{\alpha\beta}$ are adelic eigenfunctions, $E_\alpha = (E_\infty, E_2, ..., E_p, ...)$ is the corresponding energy, indices α and β denote energy levels and their degeneration. A suitable way to calculate p-adic propagator $\mathcal{K}_p(x'', t''; x', t')$ is to use Feynman's path integral method, *i.e.*

$$\mathcal{K}(x'', t''; x', t') = \int_{x',t'}^{x'',t''} \chi_p \left(-\frac{1}{h} \int_{t'}^{t''} L(\dot{q}, q, t) dt \right) \mathcal{D}q. \quad (30)$$

It has been evaluated [20] for quadratic Lagrangians in the same way for real and p-adic case and the following exact general expression is obtained:

$$\mathcal{K}_v(x'', t''; x', t') = \lambda_v \left(-\frac{1}{2h} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \right) \left| \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \right|_v^{\frac{1}{2}} \chi_v(-\frac{1}{h} \bar{S}(x'', t''; x', t')). \quad (31)$$

When one has a system with more than one dimension with uncoupled spatial coordinates, then the total propagator is the product of the corresponding one-dimensional propagators. AQM takes into account ordinary as well as p-adic quantum effects and may be regarded as a starting point for construction of a more complete M-theory. In the low-energy limit adelic quantum mechanics becomes the ordinary one [21].

5.2.1. Mini Super-Space Models in p-Adic and Adelic Quantum Mechanics

In this approach we investigate conditions under which quantum-mechanical p-adic ground state exists in the form of Ω-function and some other eigenfunctions. This approach leads to the desired result and it enables adelization of all exactly soluble mini super-space cosmological models, usually with some restrictions on the parameters of the models. One can speculate, but also continue a study, that nonarchimedean geometry or "nonarchimedean phase" in evolution of the Universe restricts a set of initial conditions and a set of Lagrangians related to a realistic dynamics of our Universe [22]. The necessary condition for the existence of an adelic model is an existence of p-adic quantum-mechanical ground state $\Omega(|x|_p)$, *i.e.*

$$\int_{|x'|_p \leq 1} \mathcal{K}_p(x'', T; x', 0) dx' = \Omega(|x''|_p), \quad (32)$$

(T is connected with the lapse function N which plays a role of the Lagrange multiplier in the standard way $T = N(t'' - t')$). Analogously, if a system is in the state $\Omega(p^\nu |x|_p)$, where $\Omega(p^\nu |x|_p) = 1$ if $|x|_p \leq p^{-\nu}$ and $\Omega(p^\nu |x|_p) = 0$ if $x|_p > p^{-\nu}$, then its kernel must satisfy equation

$$\int_{|x'|_p \leq p^{-\nu}} \mathcal{K}_p(x'', T; x', 0) dx' = \Omega(x^\nu |x''|_p). \quad (33)$$

If p-adic ground state is of the form of the δ-function, where δ-function is defined as $\delta(a - b) = 1$ if $a = b$ and $\delta(a - b) = 0$ if $a \neq b$, then the corresponding kernel of the model has to satisfy equation

$$\int_{\mathbb{Q}_p} \mathcal{K}_p(x'',T;x',0)\delta(p^\nu - |x'|_p)dx' = \chi_p(ET)\delta(p^\nu - |x''|_p), \tag{34}$$

with zero energy $E = 0$. In the following, we apply (32) and (34) to the our mini super-space model.

5.3. MILNE UNIVERSE IN p-ADIC AND ADELIC QUANTUM COSMOLOGY FOR $p \neq 2$

Due to (24) and (31), the corresponding quantum-mechanical propagator may be written as product

$$\mathcal{K}_p(X'',T;X',0) = \prod_{\mu=0}^{2} \mathcal{K}_p^{(\mu)}(X''^\mu,T;X'^\mu,0). \tag{35}$$

Propagator's μ component in compact form is presented as

$$\mathcal{K}_p^{(\mu)}(X''^\mu,T;X'^\mu,0)=\frac{\lambda_p((-1)^{\delta_0^\mu}2T)}{|T|_p^{1/2}}\chi_p\left(-(-1)^{\delta_0^\mu}\frac{(X''^\mu - X'^\mu)^2}{2T}+(-1)^{\delta_0^\mu}2\Lambda T\right) \tag{36}$$

where $\delta_0^\mu = 1$ if $\mu = 0$ and 0 otherwise. In Eq. (35) X at the left-hand side is "Minkowskian" 3-vector whose square is $X^2 = -(X^0)^2 + (X^1)^2 + (X^2)^2$. Actually, because we have $1+2$ dimensional problem, equation (32) reads

$$\int_{Q_p^3} \mathcal{K}_p(X'',T;X',0)\Omega(|X'|_p)d^3X' = \Omega(|X''|_p), \tag{37}$$

where, for the 3-vector $X \in Q_p^3$, p-adic norm is $|X|_p = \max_{0\leq\mu\leq 2}\{|X^\mu|\}$, and

$$\mathcal{K}_p(X'',T;X',0) = \frac{\lambda_p(2T)}{|T|_p^{3/2}}\chi_p\left(-\frac{(X''-X')^2}{2T} + 2\Lambda T\right). \tag{38}$$

After substituting (38) into (37), we get

$$\frac{\lambda_p(2T)}{|T|_p^{3/2}}\chi_p\left(-\frac{X''^2}{2T} + 2\Lambda T\right)\int_{|X'^0|_p\leq 1}\chi_p\left(\frac{(X'^0)^2}{2T} - \frac{X''^0}{T}X'^0\right)dX'^0$$
$$\times\prod_{i=1}^{2}\int_{|X'^i|_p\leq 1}\chi_p\left(-\frac{(X'^i)^2}{2T} + \frac{X''^i}{T}X'^i\right)dX'^i = \Omega(|X''|_p). \tag{39}$$

Using lower part of the integral (7) for $\nu = 0$ to calculate integrals in (5.3) for each coordinate X'^μ, $\mu = 0,1,2$, we obtain

$$\chi_p(2\Lambda T)\prod_{\mu=0}^{2}\Omega(|X''^\mu|_p) = \Omega(|X''|_p), \quad |T|_p < 1. \tag{40}$$

If we take into account that $\prod_{\mu=0}^{2}\Omega(|X''^{\mu}|_p) = \Omega(|X''|_p)$ is an identity, equation (40) is equivalent to

$$|2\Lambda T|_p \leq 1, \quad |T|_p < 1. \tag{41}$$

which, for $p \neq 2$, implies simply $|\Lambda|_p \leq 1$. From the upper part of the (7) we get

$$\frac{\lambda_p(2T)}{|T|_p^{3/2}}\chi_p\left(-\frac{X''^2}{2T}+2\Lambda T\right)\prod_{\mu=0}^{2}\Omega\left(\left|\frac{X''^{\mu}}{T}\right|\right) = \Omega(|X''|_p), \quad |2T|_p \geq 1. \tag{42}$$

Thus, we obtained eigenstate

$$\Psi_p(X,T) = \Omega(|X|_p), \quad |\Lambda|_p \leq 1, \qquad |T|_p = 1. \tag{43}$$

In a similar way it can be shown existence of the following eigenfunctions

$$\Psi_p(X,T) = \Omega(p^{\nu}|X|_p), \qquad |\Lambda|_p \leq p^{2\nu}, \qquad |T|_p = p^{-2\nu}, \tag{44}$$

$$\Psi_p(X,T) = \delta(p^{\nu} - |X|_p), \qquad |\Lambda|_p \leq p^{2-2\nu}, \qquad |T|_p = p^{2\nu-2}. \tag{45}$$

5.4. MILNE UNIVERSE IN p-ADIC QUANTUM COSMOLOGY FOR $p = 2$

We present now completely new and original results for the corresponding Milne model for $p = 2$. We start with calculation of the simplest vacuum (ground) state in p-adic quantum case, *i.e.* in the form $\Omega\left(|X''|_2\right)$:

In our case we get:

$$\frac{\lambda_p(2T)}{|T|_2^{3/2}}\chi_2\left(-\frac{(X'')^2}{2T}+2\Lambda T\right)\int\limits_{|X'^0|_2\leq 1}\chi_2\left(\frac{\left(X'^0\right)^2}{2T}-\frac{X''^0}{T}X'^0\right)dX'^0$$
$$\times\prod_{j=1}^{2}\int\limits_{|X'^j|_2\leq 1}\chi_2\left(-\frac{\left(X'^j\right)^2}{2T}+\frac{X''^j}{T}X'^j\right)dX'^j = \Omega\left(|X''|_2\right). \tag{46}$$

Integrals in the above case are Gauss' p-adic integrals in particular case $p = 2$, and their solutions are of the form:

$$\int\limits_{|x|_2\leq 2^{\gamma}}\chi_2\left(\alpha x^2+\beta x\right)dx = 2^{\gamma}\Omega\left(2^{\gamma}|\beta|_2\right), \quad \text{for} \quad |\alpha|_2 2^{2\gamma} \leq 1, \tag{47}$$

$$\int\limits_{|x|_2\leq 2^{\gamma}}\chi_2\left(\alpha x^2+\beta x\right)dx = \frac{\lambda_2(\alpha)}{|2\alpha|_2^{1/2}}\chi_2\left(-\frac{\beta^2}{4\alpha}\right)\delta\left(|\beta|_2-2^{1-\gamma}\right), \text{ for } |\alpha|_2 2^{2\gamma}=2 \tag{48}$$

$$\int_{|x|_2\leq 2^\gamma} \chi_2\left(\alpha x^2+\beta x\right)dx=\frac{\lambda_2(\alpha)}{|2\alpha|_2^{1/2}}\chi_2\left(-\frac{\beta^2}{4\alpha}\right)\Omega\left(2^\gamma|\beta|_2\right),\ \ |\alpha|_2 2^{2\gamma}=4, \tag{49}$$

$$\int_{|x|_2\leq 2^\gamma} \chi_2\left(\alpha x^2+\beta x\right)dx=\frac{\lambda_2(\alpha)}{|2\alpha|_2^{1/2}}\chi_2\left(-\frac{\beta^2}{4\alpha}\right)\Omega\left(2^{-\gamma}\left|\frac{\beta}{2\alpha}\right|_2\right),\ \ |\alpha|_2 2^{2\gamma}\geq 8. \tag{50}$$

Then (for the $\gamma=0$), we consider the next 4 cases:

a) For solutions of form (50) we get:

$$\int_{|X'^0|_2\leq 1} \chi_2\left(\frac{(X'^0)^2}{2T}-\frac{X''^0}{T}X'^0\right)dX'^0=\lambda_2\left(\frac{1}{2T}\right)|T|_2^{1/2}\chi_2\left(-\frac{(X''^0)^2}{2T}\right)\Omega(|X''^0|_2), \tag{51}$$

and

$$\int_{|X'^j|_2\leq 1} \chi_2\left(-\frac{(X'^j)^2}{2T}+\frac{X''^j}{T}X'^j\right)dX'^j=\lambda_2(-\frac{1}{2T})\,|-T|_2^{1/2}\chi_2\left(\frac{(X''^j)^2}{2T}\right)\Omega(|-X''^j|_2). \tag{52}$$

Replacing (51) and (52) in (33), after some straightforward calculation one gets:

$$\chi_2(2\Lambda T)\prod_{\mu=0}^{2}\Omega\left(|X''^\mu|_2\right)=\Omega\left(|X''|_2\right). \tag{53}$$

From a standard requirement:

$$\prod_{\mu=0}^{2}\Omega\left(|X''^\mu|_2\right)=\Omega\left(|X''|_2\right), \tag{54}$$

one get important “2-adic“ conditions on Λ and T for existence of our Milne model for $p=2$, what is also necessary to construct a full adelic model, *i.e.*:

$$\chi_2(2\Lambda T)=1\Rightarrow\{2\Lambda T\}_2=0\Rightarrow|2\Lambda T|_2\leq 1\Rightarrow|2|_2|\Lambda|_2|T|_2\leq 1\Rightarrow|\Lambda|_2|T|_2\leq 2 \tag{55}$$

An additional constraint follows from (50):

$$|\alpha|_2 2^{2\gamma}\geq 8\Rightarrow\left|\frac{1}{2T}\right|_2 2^0\geq 8\Rightarrow|T|_2\leq\frac{1}{4}. \tag{56}$$

b) In case represented by integral (49) the vacuum state is

$$\Omega\left(2^{-\gamma}\left|\frac{\beta}{2\alpha}\right|_2\right)=\Omega\left(|X''^\mu|_2\right),\quad \text{i.e.}\quad \Omega\left(2^\gamma|\beta|_2\right)=\Omega\left(\left|\frac{X''^\mu}{T}\right|_2\right). \tag{57}$$

After a careful calculation we get a quite simple and possibly very interesting constraint: $|\Lambda|_2 \leq 4$.

c) Let us find the vacuum state which follows from integral (4), somehow as an analogue to the previous cases:

$$\chi_2\left(2\Lambda T\right)\prod_{\mu=0}^{2}\delta\left(\left|\frac{X''^{\mu}}{T}\right|_2 - 2\right) = \Omega\left(\left|X''\right|_2\right). \tag{58}$$

All particular constraints on parameters can be presented in a simple form: $|\Lambda|_2 \leq 2$!

d) Finally, in case represented by integral (47), for the $\gamma = 0$, after substitution into (33) we get

$$\frac{\lambda_2(2T)}{|T|_2^{3/2}}\chi_2\left(-\frac{(X'')^2}{2T} + 2\Lambda T\right)\prod_{\mu=0}^{2}\Omega\left(\left|\frac{X''^{\mu}}{T}\right|_2\right) = \Omega(|X''|_2). \tag{59}$$

It is valid if: $|1/(2T)|_2 2^0 \leq 1 \Rightarrow |T|_2 \geq 2$.

Vacuum state of the form $\delta\left(2^{\gamma} - |x|_2\right)$ will satisfy the following condition:

$$\int_{Q_2} K_2(x'',T;x',0)\delta\left(2^{\gamma} - \left|x'\right|_2\right)dx' = \chi\left(ET\right)\delta\left(2^{\gamma} - \left|x''\right|_2\right). \tag{60}$$

Let us remind that for us vacuum state means also $E = 0$ and after some calculation we find a starting condition to determine allowed range of the parameters for our Milne model, *i.e.* we get the equality

$$\chi_2\left(2\Lambda T\right) = \delta\left(2^{\gamma} - \left|X''\right|_2\right). \tag{61}$$

It is valid, if

$$|4\alpha|_2 \geq 2^{2-2\gamma} \Rightarrow \left|4\frac{1}{2T}\right|_2 = \left|\frac{2}{T}\right|_2 = \frac{|2|_2}{|T|_2} = \frac{1}{2|T|_2} \geq 2^{2-2\gamma}, \tag{62}$$

i.e. we get a particular constraint on parameter T in the form: $|T|_2 \leq 2^{2\gamma-3}$.

The final range of allowed values of Λ in our Milne model and for existence of the vacuum state presented in the form of the δ function for $p = 2$ is

$$|\Lambda|_2 2^{2\gamma-3} \leq 2 \Rightarrow |\Lambda|_2 \leq 2^{-2\gamma+4}. \tag{63}$$

6. CONCLUSION

In this paper we present the complete solution for the p-adic Milne model, both $p \neq 2$ and $p = 2$. The “complete“ or more precise the explicit adelic Milne model is still missing, but after this result it should be just problem of finding the ground wave function in real quantum cosmology. We find a set of constraints for Λ factor which is important to define range of validity of the Milne model. After completion of the adelic wave function it would be possible to see what range, if any, for Λ is allowed in this scenario and conclude about the quantum origin of this or any similar "Milne" scenario on Planckian or trans-Planckian region. This investigation, combined with those one presented references [23, 24] could shed more light on existence/absence of tachyons in the Milne universe.

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