# TENSOR GAUGE FIELDS OF DEGREE THREE*

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Starting from a generic second-order Lagrangian (that describes the dynamics of an Abelian tensor gauge field of degree three and depends on two arbitrary real constants) we perform the Dirac analysis. The procedure allows the determination of the number of degrees of freedom and also a generating set of gauge transformations for the initial theory.

*Key words*: gauge fields, constrained systems, Dirac algorithm.

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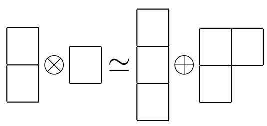
## 1. MOTIVATION

This paper offers a framework for the unitary approach of the tensor gauge fields of degree three that transforms under irreducible representations of the Lorentz group. It finds its roots in an old attempt to unify gravity with electromagnetism proposed by Einstein and developed by himself [1] and others [2,3]. The aforementioned unification scheme involves a tensor gauge field of degree two with no symmetry. In view of this, our starting point is represented by a tensor field of degree three $A_{\mu\nu\|\alpha}$ antisymmetric in the first two Lorentz indices ($A_{\mu\nu\|\alpha} = -A_{\nu\mu\|\alpha}$) that transforms under reducible representation

$A_{\mu\nu\|\alpha} \in$ 

of the Lorentz group. We postulate for the considered tensor field the gauge transformations

$$\delta_\epsilon A_{\mu\nu\|\alpha} = \partial_{[\mu}\epsilon_{\nu]\|\alpha}. \tag{1}$$

In the above, the bosonic gauge parameters $\epsilon_{\nu\|\alpha}$ have no symmetry. In view of these, the most general second-order Lagrangian density invariant under the gauge

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transformations (1) reads as

$$\mathcal{L}_0 = \frac{1}{6} F_{\mu\nu\rho\|\alpha} F^{\mu\nu\rho\|\alpha} + \frac{k_1}{4} F_{\mu\nu\rho\|\lambda} F^{\mu\nu\lambda\|\rho} + \frac{k_2}{4} F^{\mu\nu} F_{\mu\nu}, \tag{2}$$

where $k_1$ and $k_2$ are arbitrary real constants and $F_{\mu\nu\rho\|\alpha}$ is the field-strength of the tensor gauge field $A_{\mu\nu\|\alpha}$

$$F_{\mu\nu\rho\|\alpha} \equiv \partial_{[\mu} A_{\nu\rho]\|\alpha}. \tag{3}$$

By $F_{\mu\nu}$ we denoted the trace of the field-strength, $F_{\mu\nu} \equiv \sigma^{\alpha\beta} F_{\mu\nu\alpha\|\beta}$. Throughout the paper we work with the flat metric of 'mostly minus' signature $\sigma_{\mu\nu} = (+ - \cdots -)$. The notation $[\mu \ldots \nu]$ signifies full antisymmetry with respect to the indices between brackets without normalization factors (*i.e.* the independent terms appear only once and are not multiplied by overall numerical factors).

It has been shown [4] that if we decompose the gauge field $A_{\mu\nu\|\alpha}$ into its "irreducible" components

$$A_{\mu\nu\|\alpha} \equiv t_{\mu\nu|\alpha} + B_{\mu\nu\alpha} \equiv \left( A_{\mu\nu\|\alpha} - \frac{1}{3} A_{[\mu\nu\|\alpha]} \right) + \left( \frac{1}{3} A_{[\mu\nu\|\alpha]} \right), \tag{4}$$

then the Lagrangian action corresponding with the density (2) takes the form

$$\begin{aligned} S_0^{\mathrm{L}} \left[ t_{\mu\nu|\alpha}, B_{\mu\nu\rho} \right] = \int \mathrm{d}^D x & \left[ \frac{4 - 7k_1 - k_2}{48} H_{\mu\nu\rho\lambda} H^{\mu\nu\rho\lambda} \right. \\ & + \frac{2 + k_1 + k_2}{12} \left( \partial_\mu B_{\nu\rho\lambda} \right) \partial^\mu B^{\nu\rho\lambda} + \frac{2 + k_1}{12} \mathcal{F}_{\mu\nu\rho|\lambda} \mathcal{F}^{\mu\nu\rho|\lambda} \\ & \left. + \frac{k_2}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + \frac{2 + k_1 + k_2}{6} \mathcal{F}_{\mu\nu\rho|\lambda} \partial^\lambda B^{\mu\nu\rho} \right]. \end{aligned} \tag{5}$$

In the local functional (5) $H_{\mu\nu\rho\lambda}$ stands for the field-strength of the 3-form $B_{\mu\nu\rho}$ ($H_{\mu\nu\rho\lambda} \equiv \partial_{[\mu} B_{\nu\rho\lambda]}$), the tensor $\mathcal{F}_{\mu\nu\rho|\lambda}$ is defined by

$$\mathcal{F}_{\mu\nu\rho|\lambda} \equiv \partial_{[\mu} t_{\nu\rho]\|\alpha}$$

and $\mathcal{F}_{\mu\nu}$ is nothing but the trace of $\mathcal{F}_{\mu\nu\rho|\lambda}$ ($\mathcal{F}_{\mu\nu} \equiv \sigma^{\rho\lambda} \mathcal{F}_{\mu\nu\rho|\lambda}$). A quick look on the Lagrangian action (5) reveals two limit situations namely

$$k_1 = -2, \quad k_2 = 0, \tag{6}$$

and

$$k_1 = 1, \quad k_2 = -3 \tag{7}$$

For the choice (6) the gauge field $t_{\mu\nu|\alpha}$ with the mixed symmetry $(2,1)$ becomes a pure one (it does not appear in the Lagrangian action). On the other hand, if we set (7) into the local functional (5) then the 3-form $B_{\mu\nu\rho}$ becomes a pure gauge field.

The above analysis suggests the tensor gauge fields of degree three that transform under irreducible representations of the Lorentz group can be treated in unified manner through the gauge field $A_{\mu\nu\|\alpha}$ whose dynamics is governed by the Lagrangian density (2). This remark justifies the Dirac analysis of the gauge model (2).

## 2. RESULTS

In this section we perform the canonical analysis of the model subject to the Lagrangian density (2). In view of this, if we denote by $\pi_{\mu\nu\|\alpha}$ the canonical momenta associated with the fields $A^{\mu\nu\|\alpha}$, the definitions of the formers read as

$$\begin{aligned}\pi_{\mu\nu\|\alpha} &\equiv \frac{1}{2}\left(\frac{\partial\mathcal{L}_0}{\partial\dot{A}^{\mu\nu\|\alpha}} - \frac{\partial\mathcal{L}_0}{\partial\dot{A}^{\nu\mu\|\alpha}}\right) \\ &= F_{0\mu\nu\|\alpha} + \frac{k_1}{2}F_{\alpha[\mu\nu\|0]} + \frac{k_2}{2}\sigma_{\alpha[\mu}F_{\nu 0]},\end{aligned} \tag{8}$$

where by overdot we denoted the derivative in respect with the temporal coordinate $x^0$. From the definitions in the above, we infer the primary constraints

$$G_i^{(1)} \equiv \pi_{0i\|0} \approx 0, \quad G_{i\|j}^{(1)} \equiv \pi_{0i\|j} \approx 0 \quad i,j = \overline{1, D-1} \tag{9}$$

and also the relations

$$\pi_{ij\|0} = \frac{2+k_1+k_2}{2}F_{0ij\|0} + \frac{k_2}{2}F_{ijk\|}{}^{k}, \quad i,j = \overline{1, D-1}, \tag{10a}$$

$$\pi_{ij\|k} = F_{0ij\|k} + \frac{k_1}{2}F_{k[0i\|j]} + \frac{k_2}{2}\sigma_{k[i}F_{j]0l\|}{}^{l}, \quad i,j,k = \overline{1, D-1}. \tag{10b}$$

Starting with the equations (10b), only by algebraic computations, we derive

$$\pi_{ij\|}{}^{j} = \frac{2+k_1+(D-2)\,k_2}{2}F_{0ij\|}{}^{j}, \quad i = \overline{1, D-1}, \tag{11a}$$

$$\pi_{[ij\|k]} = (1-k_1)\,F_{0[ij\|k]} + \frac{3}{2}k_1 F_{ijk\|0}, \quad i,j,k = \overline{1, D-1}. \tag{11b}$$

The first step in the canonical analysis is completed by solving the system (10) in respect with some of the generalized velocities. In view of this, the results (11) lead to seven distinct situations (dictated by the coefficients $2+k_1+k_2$, $2+k_1+(D-2)\,k_2$

and $1-k_1$ of the *temporal components* of the field-strength in (10a) and (11), namely

$$k_1 = -2, \quad k_2 = 0; \tag{12}$$

$$k_1 = 1, \quad k_2 = -3; \tag{13}$$

$$k_1 = 1, \quad k_2 = -\frac{3}{D-2}; \tag{14}$$

$$k_1 = 1, \quad k_2 \equiv k \in \mathbb{R}\backslash\left\{-3, -\frac{3}{D-2}\right\}; \tag{15}$$

$$k_1 \equiv \bar{k} \in \mathbb{R}\backslash\{-2,1\}, \quad k_2 = -\frac{2+\bar{k}}{D-2}; \tag{16}$$

$$k_1 \equiv \tilde{k} \in \mathbb{R}\backslash\{-2,1\}, \quad k_2 = -\left(2+\tilde{k}\right); \tag{17}$$

$$k_1 + k_2 \neq -2 \neq k_1 + (D-2)k_2, \quad k_1 \in \mathbb{R}\backslash\{1\}. \tag{18}$$

In the remaining part of this section we will complete the canonical analysis of the model in each of the seven situations delimited in the above.

### 2.1. CASE I

In the situation (12) the definitions of the canonical momenta (8) lead to the independent primary constraints (9) and

$$\gamma_{ij}^{(1)} \equiv \pi_{ij\|0} \approx 0, \quad i,j = \overline{1,D-1}, \tag{19a}$$

$$\gamma_{ij|k}^{(1)} \equiv \pi_{ij\|k} - \frac{1}{3}\pi_{[ij\|k]} \approx 0, \quad i,j,k = \overline{1,D-1}, \tag{19b}$$

$$\gamma_{i}^{(1)} \equiv \pi_{ij\|}{}^{j} \approx 0, \quad i = \overline{1,D-1}. \tag{19c}$$

Expressing from the equations (8) (corresponding to the choice (12)) some of the generalized velocities, we get the canonical Hamiltonian (well defined only on the primary constraint surface)

$$H_0^{(I)}\left(x^0\right) = \frac{1}{3}\int \mathrm{d}^{D-1}\mathbf{x}\left[\frac{1}{6}\pi^{ij\|k}\left(\pi_{[ij\|k]} + 6F_{ijk\|0}\right) - \frac{1}{2}F_{ijk\|l}F^{[ijk\|l]} - 2A^{0i\|j}\left(\partial^k \pi_{[ij\|k]}\right)\right]. \tag{20}$$

By direct computation it simply verifies that the constraints (9) and (19) are Abelian so that the consistency of the primary constraints reduces only to the computation of the Poisson brackets between them and the canonical Hamiltonian. Based on these arguments, consistency of the primary constraints produces secondary constraints

$$G_{ij}^{(2)} \equiv \frac{1}{3}\partial^k \pi_{[ij\|k]} \approx 0, \quad i = \overline{1,D-1}, \tag{21}$$

that are second-order and off-shell reducible, with the reducibility functions

$$\left(Z^{ij}\right)_k = \partial^{[i}\delta_k^{j]}, \quad Z^k = \partial^k. \tag{22}$$

The consistency of the secondary constraints (21) does not reveal any new constraint as the constraints (9), (19), (21) are Abelian and

$$\left[G_{ij}^{(2)}, H_0^{(I)}\right] = 0. \tag{23}$$

The arguments in the above allow us to conclude that the canonical Hamiltonian is exactly the first-class Hamiltonian of the system. The irreducible character of the first-class constraints (9) and (19a), together with the first-order reducibilities among the constraints (19b)–(19c)

$$\frac{1}{2}\delta_l^{[i}\sigma^{j]k}\gamma_{ij|k}^{(1)} - \delta_l^i\gamma_i^{(1)} = 0$$

and the second-order ones corresponding to the constraints (21) allow us to compute the number of degrees of freedom for the model under study

$$N_{DOF}^{(I)} = \frac{(D-2)(D-3)(D-4)}{6}. \tag{24}$$

Finally, on behalf of the Dirac's conjecture (according to which any first-class constraint generates gauge transformations), if we pass again to the Lagrangian formulation (via extended action), we derive for the Lagrangian action corresponding to the choice (12) the generating set of gauge transformations

$$\delta_{\epsilon,\xi}^{(I)} A_{\mu\nu\|\alpha} = \partial_{[\mu}\epsilon_{\nu\alpha]} + \xi_{\mu\nu|\alpha}, \tag{25}$$

where the gauge parameters $\xi_{\mu\nu|\alpha}$ have the mixed symmetry $(2,1)$

$$\xi_{\mu\nu|\alpha} = -\xi_{\nu\mu|\alpha}, \quad \xi_{[\mu\nu|\alpha]} = 0. \tag{26}$$

To conclude with, this situation is nothing but the limit case where the tensor gauge field's part with the mixed symmetry $(2,1)$ becomes a pure gauge one.

### 2.2. CASE II

In this part, we complete the canonical analysis of the model (2) in which the real parameters $k_1$ and $k_2$ are taken as in (13). With this setting, the definitions of the canonical momenta (8) produce the independent primary constraints (9) and

$$\bar{\gamma}_{ij}^{(1)} \equiv \pi_{ij\|0} + \frac{3}{2}F_{ijk\|}{}^{k} \approx 0, \quad i,j = \overline{1,D-1}, \tag{27a}$$

$$\gamma_{ijk}^{(1)} \equiv \pi_{[ij\|k]} - \frac{3}{2}F_{ijk\|0} \approx 0, \quad i,j,k = \overline{1,D-1}. \tag{27b}$$

Expressing from the equations (8) some of the generalized velocities, we derive the canonical Hamiltonian (well defined only on the primary constraint surface)

$$\begin{aligned} H_0^{(II)}\left(x^0\right) = \int \mathrm{d}^{D-1}\mathbf{x} \Big[ & 2A^{0i\|0}\left(\partial^j \pi_{ij\|0}\right) - 2A^{0i\|j}\left(\partial^k \pi_{ki\|j}\right) \\ & -\frac{1}{4}F_{ijk\|l}F^{ijl\|k} + \frac{3}{4}F_{ijk\|}{}^{k}F^{ijl\|}{}_{l} + \frac{1}{3}\pi_{ij\|k}\pi^{ij\|k} \\ & -\frac{1}{6}F_{ijk\|\mu}F^{ijk\|\mu} - \frac{1}{6}\pi_{ij\|k}F^{ijk\|0} - \frac{2}{3(D-3)}\pi_{ij\|}{}^{j}\pi^{ik\|}{}_{k}\Big]. \end{aligned} \tag{28}$$

Simple computations lead to the Abelian character of the primary constraints (9) and (27) so that their consistency reduces only to the calculations of the Poisson brackets between canonical Hamiltonian and them. By direct computations one obtains

$$\left[G_i^{(1)}, H_0^{(II)}\right] = -\partial^j \pi_{ij\|0} \equiv G_i^{(2)} \approx 0\,, \tag{29a}$$

$$\left[G_{i\|j}^{(1)}, H_0^{(II)}\right] = \partial^k \pi_{ki\|j} \equiv G_{i\|j}^{(2)} \approx 0\,, \tag{29b}$$

$$\left[\bar{\gamma}_{ij}^{(1)}, H_0^{(II)}\right] = \frac{5}{6}\partial^k \gamma_{ijk}^{(1)} - G_{[i\|j]}^{(2)} \approx 0\,, \tag{29c}$$

$$\left[\gamma_{ij}^{(1)}, H_0^{(II)}\right] = 0. \tag{29d}$$

The results (29) display the secondary constraints

$$G_i^{(2)} \approx 0, \quad G_{i\|j}^{(2)} \approx 0, \tag{30}$$

that together with (9) and (27) constitute an Abelian set of constraints. The secondary constraints displayed in the above are off-shell first-order reducible with the reducibility functions

$$Z^i \equiv \partial^i, \quad \left(Z^{i\|j}\right)_k \equiv \partial^i \delta_k^j.$$

The consistency of the secondary constraints (30) does not produce any new constraint so the Dirac algorithm stops at this level.

At this stage, we can conclude that the canonical Hamiltonian (28) is exactly the first-class Hamiltonian of the system. The irreducible character of the first-class constraints (9) and (27) together with the first-order reducibilities of the secondary constraints (30) allow us to count the degrees of freedom for the model under study

$$N_{DOF}^{(II)} = \frac{D(D-2)(D-4)}{2}. \tag{31}$$

Finally, if we pass again to the Lagrangian formulation (via extended action), we derive for the corresponding functional associated with the local function (2) the generating set of gauge transformations

$$\delta_{\epsilon,\varepsilon}^{(II)} A_{\mu\nu\|\alpha} = \partial_{[\mu}\epsilon_{\nu]\alpha} + \partial_{[\mu}\varepsilon_{\nu]\alpha} - \partial_\alpha \varepsilon_{\mu\nu} + \varepsilon_{\mu\nu\alpha}, \tag{32}$$

where the gauge parameters $\epsilon_{\mu\nu}$ are symmetric while $\varepsilon_{\mu\nu}$ and $\varepsilon_{\mu\nu\rho}$ are completely antisymmetric.

To conclude with, this situation is nothing but the limit case where the antisymmetric part of the tensor gauge field becomes a pure gauge one.

### 2.3. CASE III

With the settings (14) of the real parameters $k_1$ and $k_2$, the definitions (8) of the canonical momenta furnish the primary constraints (9) and

$$\gamma_i^{(1)} \equiv \pi_{ij\|}{}^{j} \approx 0, \quad i = \overline{1, D-1}, \tag{33a}$$

$$\gamma_{ijk}^{(1)} \equiv \pi_{[ij\|k]} - \frac{3}{2} F_{ijk\|0} \approx 0, \quad i,j,k = \overline{1, D-1}. \tag{33b}$$

Performing, on the primary constraints surface, the Legendre transformation of the Lagrangian density (2) in respect with some of the generalized velocities (those that can be expressed from the definitions of the canonical momenta) we get the canonical Hamiltonian

$$\begin{aligned} H_0^{(III)}\left(x^0\right) = \int \mathrm{d}^{D-1}\mathbf{x} \Big[ & 2A^{0i\|0}\left(\partial^j \pi_{ij\|0}\right) - 2A^{0i\|j}\left(\partial^k \pi_{ki\|j}\right) \\ & - \frac{1}{4} F_{ijk\|l} F^{ijl\|k} + \frac{3}{4(D-3)} F_{ijk\|}{}^{k} F^{ijl\|}{}_{l} + \frac{1}{3} \pi_{ij\|k} \pi^{ij\|k} \\ & - \frac{1}{6} \pi_{ij\|k} F^{ijk\|0} - \frac{2}{3(D-3)} \pi_{ij\|}{}^{j} \pi^{ik\|}{}_{k} \\ & - \frac{1}{6} F_{ijk\|\mu} F^{ijk\|\mu} + \frac{D-2}{3(D-3)} \pi_{ij\|0} \pi^{ij\|0} + \frac{1}{D-3} \pi^{ij\|0} F_{ijk\|}{}^{k} \Big] \end{aligned} \tag{34}$$

As in the other two cases, the primary constraints (9) and (33) are Abelian so their consistency reduces to the computation of the Poisson brackets between them and the canonical Hamiltonian (34). By direct calculations we infer

$$\left[G_i^{(1)}, H_0^{(III)}\right] = -\partial^j \pi_{ij\|0} \equiv G_i^{(2)} \approx 0, \tag{35a}$$

$$\left[G_{i\|j}^{(1)}, H_0^{(III)}\right] = \partial^k \pi_{ki\|j} \equiv G_{i\|j}^{(2)} \approx 0, \tag{35b}$$

$$\left[\gamma_i^{(1)}, H_0^{(III)}\right] = G_i^{(2)} \approx 0, \tag{35c}$$

$$\left[\gamma_{ijk}^{(1)}, H_0^{(III)}\right] = -\left(\partial_{[i}\pi_{jk]\|0} - \frac{3}{2}\partial^m F_{ijk\|m}\right) \equiv \gamma_{ijk}^{(2)} \approx 0. \tag{35d}$$

The results (35) put into evidence the secondary constraints

$$G_i^{(2)} \approx 0, \quad G_{i\|j}^{(2)} \approx 0, \quad \gamma_{ijk}^{(2)} \approx 0, \quad i,j,k = \overline{1, D-1}. \tag{36}$$

that together with the primary constraints (9) and (33) constitute an Abelian set of constraints. This output combined with the property that the Poisson brackets between the canonical Hamiltonian and secondary constraints (36) vanish on-shell allow us to conclude that the Dirac algorithm stops at this level. Moreover, the canonical Hamiltonian is exactly the first-class Hamiltonian.

The concrete expressions of the first-class constraints (9), (33) and (36) evidence firstly, the constraints $\gamma^{(2)}_{ijk} \approx 0$ are off-shell reducible of order $(D-4)$ with the reducibility functions

$$\left(Z^{i_1\cdots i_k}\right)_{j_1\cdots j_{k+1}} = \partial_{[j_1}\delta^{i_1}_{j_2}\cdots\delta^{i_k}_{j_{k+1}]}, \quad k=\overline{3,D-2} \tag{37}$$

secondly, the constraints $\gamma^{(1)}_i \approx 0$, $G^{(2)}_i \approx 0$ and $G^{(2)}_{i\|j} \approx 0$ are of-shell first order reducible

$$\left(-\partial^i\right)\gamma^{(1)}_i + \left(\sigma^{ij}\right)G^{(2)}_{i\|j} = 0, \tag{38a}$$

$$\left(\partial^i\delta^j_k\right)G^{(2)}_{i\|j} = 0, \tag{38b}$$

$$\left(\partial^i\right)G^{(2)}_i = 0 \tag{38c}$$

and thirdly (9) and (33b) are irreducible. The aforementioned reducibilities of the first-class constraints imply that the system possesses the same number of physical degrees of freedom as in the previous situation (31).

If we return again to the Lagrangian formulation (via extended action), we derive for the functional corresponding to (2) the generating set of gauge transformations

$$\delta^{(III)}_{\epsilon,\varepsilon}A_{\mu\nu\|\alpha} = \sigma_{\alpha[\mu}\epsilon_{\nu]} + \partial_{[\mu}\epsilon_{\nu]\|\alpha} + \partial^\rho\varepsilon_{\mu\nu\alpha\rho}, \tag{39}$$

where the gauge parameters $\varepsilon_{\mu\nu\alpha\rho}$ are completely antisymmetric. In addition, the aforementioned generating set of gauge transformations is Abelian and off-shell reducible of order $(D-4)$.

A short look on the gauge transformations (39) reveals new '*conformal*' [8] and topological BF-like [9] behaviours of the massless tensor gauge field of degree three $A_{\mu\nu\|\alpha}$. Precisely, the first term in the right-hand side of (39) is nothing but a kind of flat-conformal gauge transformation for the analysed tensor gauge field while the last term mimic the gauge transformations of the 3-form in a topological BF-model.

## 2.4. CASE IV

From the dynamical point of view, this situation is quite similar to the previous one. With the choice (15), the definitions (8) of the canonical momenta put into evidence the primary constraints (9) and (33b) (that are Abelian) and also lead to the canonical Hamiltonian (well defined only on the primary constraint surface)

$$
\begin{aligned}
H_0^{(IV)}\left(x^0\right) = \int \mathrm{d}^{D-1}\mathbf{x} \Big[ & 2A^{0i\|0}\left(\partial^j \pi_{ij\|0}\right) - 2A^{0i\|j}\left(\partial^k \pi_{ki\|j}\right) \\
& - \frac{1}{4} F_{ijk\|l} F^{ijl\|k} - \frac{3k}{4(k+3)} F_{ijk\|}{}^{k} F^{ijl\|}{}_{l} + \frac{1}{3}\pi_{ij\|k}\pi^{ij\|k} \\
& - \frac{1}{6}\pi_{ij\|k} F^{ijk\|0} - \frac{2k}{3\left[3+k(D-3)\right]}\pi_{ij\|}{}^{j}\pi^{ik\|}{}_{k} \\
& - \frac{1}{6} F_{ijk\|\mu} F^{ijk\|\mu} + \frac{1}{k+3}\pi_{ij\|0}\pi^{ij\|0} - \frac{k}{k+3}\pi^{ij\|0} F_{ijk\|}{}^{k} \Big]
\end{aligned} \tag{40}
$$

The consistency of the primary constraints displays the same secondary constraints as in the previous situation (36) because the following Poisson brackets hold

$$
\left[G_i^{(1)}, H_0^{(IV)}\right] \equiv G_i^{(2)}, \quad \left[G_{i\|j}^{(1)}, H_0^{(IV)}\right] \equiv G_{i\|j}^{(2)}, \quad \left[\gamma_{ijk}^{(1)}, H_0^{(IV)}\right] \equiv \gamma_{ijk}^{(2)}. \tag{41}
$$

Concerning the consistency of the secondary constraints (36) this does not imply new constraints because firstly, the constraints (9), (33b) and (36) are Abelian and secondly, the Poisson brackets between the canonical Hamiltonian (40) and secondary constraints (36) weakly vanish.

As in the third situation, the canonical Hamiltonian (40) is exactly the first-class Hamiltonian. Also, the first-class constraints (9), (33b) and (36) possess the same reducibilities as in the previous case, namely (9) and (33b) are irreducible, $G_i^{(2)} \approx 0$ and $G_{i\|j}^{(2)} \approx 0$ are of-shell first-order reducible (the reducibility relations (38b)–(38c) hold) and the constraints $\gamma_{ijk}^{(2)} \approx 0$ are off-shell reducible of order $(D-4)$ with the reducibility functions (37). Based on these results we compute the number of degrees of freedom for the system under study

$$
N_{DOF}^{(IV)} = \frac{(D-1)(D-2)(D-3)}{3}. \tag{42}
$$

Finally, on behalf of the Dirac's conjecture, if we pass again to the Lagrangian formulation (via extended action), we derive the off-shell $(D-4)$-reducible generating set of gauge transformations

$$
\delta_{\epsilon,\varepsilon}^{(IV)} A_{\mu\nu\|\alpha} = \partial_{[\mu}\epsilon_{\nu]\|\alpha} + \partial^\rho \varepsilon_{\mu\nu\alpha\rho}, \tag{43}
$$

where the gauge parameters $\varepsilon_{\mu\nu\alpha\rho}$ are completely antisymmetric. As in the previous situation, the gauge transformations (43) reveals new topological BF-like [9] behaviour of the massless tensor gauge field of degree three $A_{\mu\nu\|\alpha}$. The aforementioned behaviour is due to the last term in the right-hand side of (43) that mimic the gauge transformations of the 3-form in a topological BF-model.

### 2.5. CASE V

We shall see that, as in the previous situation, the present case manifests similarities with the third situation. Here, the constants that parametrize (2) take the values (16). In this realm, the definitions of the canonical momenta (8) furnish the primary constraints (9) and (33a) (that are Abelian) and also produce the canonical Hamiltonian

$$\begin{aligned} H_0^{(V)}\left(x^0\right) = \int \mathrm{d}^{D-1}\mathbf{x} \Big[ & 2A^{0i\|0}\left(\partial^j \pi_{ij\|0}\right) - 2A^{0i\|j}\left(\partial^k \pi_{ki\|j}\right) \\ & - \frac{\bar{k}}{4} F_{ijk\|l} F^{ijl\|k} + \frac{\bar{k}+2}{4(D-3)} F_{ijk\|}{}^{k} F^{ijl\|}{}_{l} + \frac{1}{\bar{k}+2} \pi_{ij\|k} \pi^{ij\|k} \\ & - \frac{1}{6} F_{ijk\|\mu} F^{ijk\|\mu} + \frac{\bar{k}}{2\left(\bar{k}+2\right)\left(1-\bar{k}\right)} \pi^{ij\|k}\left(\pi_{[ij\|k]} - \left(\bar{k}+2\right) F_{ijk\|0}\right) \\ & + \frac{\bar{k}^2}{8\left(1-\bar{k}\right)} F_{ijk\|0} F^{ijk\|0} + \frac{\pi^{ij\|0}}{D-3}\left(\frac{D-2}{\bar{k}+2}\pi_{ij\|0} + F_{ijk\|}{}^{k}\right)\Big]. \end{aligned} \tag{44}$$

The requirement of preservation in time for the primary constraints (9) and (33a) leads to the secondary constraints $G_i^{(2)} \approx 0$ and $G_{i\|j}^{(2)} \approx 0$ respectively defined in (35a) and (35b).

By direct computations we infer the Abelian character of the constraints (9), (33a), $G_i^{(2)} \approx 0$ and $G_{i\|j}^{(2)} \approx 0$. Moreover, the consistency of the secondary constraints $G_i^{(2)} \approx 0$ and $G_{i\|j}^{(2)} \approx 0$ no longer produces tertiary constraints

$$\left[G_i^{(2)}, H_0^{(V)}\right] = 0 = \left[G_{i\|j}^{(2)}, H_0^{(V)}\right]. \tag{45}$$

The irreducible character of the first-class constraints (9), together with the first-order reducibilities (38) of the constraints (33a), $G_i^{(2)} \approx 0$ and $G_{i\|j}^{(2)} \approx 0$ allow us to compute the number of physical degrees of freedom

$$N_{Phys}^{(V)} = \frac{D(D-1)(D-4)}{2} + 2. \tag{46}$$

Using the same method as in the previous subsections, one can deduce for the functional Lagrangian action the off-shell first-order reducible generating set of gauge transformations

$$\delta_\epsilon^{(V)} A_{\mu\nu\|\alpha} = \sigma_{\alpha[\mu}\epsilon_{\nu]} + \partial_{[\mu}\epsilon_{\nu]\|\alpha}. \tag{47}$$

The first term in the right-hand side of the gauge transformations (47) puts into evidence a conformal-like [8] behaviour of the massless tensor gauge field of degree three in the present situation.

2.6. CASE VI

Here the real constants that label the local function (2) reads as in (17). Replacing the choice (17) in the definitions (8), the lasts lead to the set of primary constraints constituted by (9) and

$$\tilde{\gamma}^{(1)}_{ij} \equiv \pi_{ij\|0} + \frac{2+\tilde{k}}{2} F_{ijk\|}{}^{k} \approx 0, \quad i,j = \overline{1, D-1}. \tag{48}$$

Expressing from the equations (8) (written for the choice (17)) some of the generalized velocities, we get the canonical Hamiltonian (well defined only on the primary constraint surface)

$$\begin{aligned} H_0^{(VI)}\left(x^0\right) = \int \mathrm{d}^{D-1}\mathbf{x} \Big[ & 2A^{0i\|0}\left(\partial^j \pi_{ij\|0}\right) - 2A^{0i\|j}\left(\partial^k \pi_{ki\|j}\right) \\ & - \frac{1}{6} F_{ijk\|\mu} F^{ijk\|\mu} - \frac{\tilde{k}}{4} F_{ijk\|l} F^{ijl\|k} \\ & + \frac{\tilde{k}+2}{4} F_{ijk\|}{}^{k} F^{ijl\|}{}_{l} + \frac{1}{\tilde{k}+2} \pi_{ij\|k} \pi^{ij\|k} \\ & + \frac{\tilde{k}}{2\left(\tilde{k}+2\right)\left(1-\tilde{k}\right)} \pi^{ij\|k} \left(\pi_{[ij\|k]} - \left(\tilde{k}+2\right) F_{ijk\|0}\right) \\ & + \frac{\tilde{k}^2}{8\left(1-\tilde{k}\right)} F_{ijk\|0} F^{ijk\|0} - \frac{2}{\left(\tilde{k}+2\right)(D-3)} \pi_{ij\|}{}^{j} \pi^{ik\|}{}_{k} \Big]. \end{aligned} \tag{49}$$

It simply verifies that the constraints (9) and (48) are Abelian so that the consistency of the primary constraints reduces only to the computation of the Poisson brackets between them and the canonical Hamiltonian. By direct computation one obtains

$$\left[G^{(1)}_i, H_0^{(VI)}\right] = G^{(2)}_i, \quad \left[G^{(1)}_{i\|j}, H_0^{(VI)}\right] = G^{(2)}_{i\|j}, \quad \left[\tilde{\gamma}^{(1)}_{ij}, H_0^{(VI)}\right] = \tilde{\gamma}^{(2)}_{ij}, \tag{50}$$

where the functions in the right-hand sides are given in formulas (35a), (35b) and

$$\tilde{\gamma}^{(2)}_{ij} \equiv \partial^k \pi_{ij\|k} - \frac{\tilde{k}+2}{2} \partial^k F_{ijk\|0}. \tag{51}$$

These results derived in the above allow to display the secondary constraints possessed by the model under study

$$G^{(2)}_i \approx 0, \quad G^{(2)}_{i\|j} \approx 0, \quad \tilde{\gamma}^{(2)}_{ij} \approx 0. \tag{52}$$

It can be checked that the constraints (9), (48) and (52) are Abelian. Using this remark, the requirement of preservation in time for the secondary constraints reduces to the computation of the Poisson brackets between them and the canonical

Hamiltonian (49)

$$\left[G_i^{(2)}, H_0^{(VI)}\right] = 0, \quad \left[G_{i\|j}^{(2)}, H_0^{(VI)}\right] = 0 = \left[\tilde{\gamma}_{ij}^{(2)}, H_0^{(VI)}\right]. \tag{53}$$

The outputs (53) allow us to conclude that the model under study possesses no tertiary constraints and, in addition, the canonical Hamiltonian (52) coincides with the first-class Hamiltonian.

In view of the counting the degrees of freedom, one observes that: i) the first-class constraints (9) and (48) are irreducible and ii) the secondary constraints (52) are off-shell second-order reducible with the reducibility relations

$$\left(\partial^i\right) G_i^{(2)} = 0, \tag{54a}$$

$$\left(\partial^i \delta_k^j\right) G_{i\|j}^{(2)} = 0, \tag{54b}$$

$$\left(\delta_k^i \partial^j\right) G_{i\|j}^{(2)} + \left(\frac{1}{2}\delta_k^{[i}\partial^{j]}\right) \tilde{\gamma}_{ij}^{(2)} = 0 \tag{54c}$$

and respectively

$$\left(-\partial^k\right)\left(\partial^i \delta_k^j\right) + \left(\partial^k\right)\left(\delta_k^i \partial^j\right) = 0, \tag{55a}$$

$$\left(\partial^k\right)\left(\frac{1}{2}\delta_k^{[i}\partial^{j]}\right) = 0. \tag{55b}$$

Putting together the previous results we determine the number of degrees of freedom

$$N_{DOF}^{(VI)} = \frac{(D-1)(D-2)(D-4)}{2}. \tag{56}$$

Employing the same procedure as in the previous subsections, we derive for the Lagrangian action the off-shell second-order reducible generating set of gauge transformations

$$\delta_{\varepsilon,\epsilon}^{(VI)} A_{\mu\nu\|\alpha} = \partial_{[\mu}\varepsilon_{\nu\rho]} + \partial_{[\mu}\epsilon_{\nu]\|\alpha}, \tag{57}$$

where the bosonic gauge parameters $\varepsilon_{\mu\nu}$ are completely antisymmetric.

### 2.7. CASE VII

In the last situation, the real parameters $k_1$ and $k_2$ have the ranges defined by (18). Here, the definitions (8) of the canonical momenta lead to the Abelian primary constraints (9) and produce the canonical Hamiltonian

$$
\begin{aligned}
H_0^{(VII)}\left(x^0\right) = \int \mathrm{d}^{D-1}\mathbf{x}\Big[&2A^{0i\|0}\left(\partial^j \pi_{ij\|0}\right) - 2A^{0i\|j}\left(\partial^k \pi_{ki\|j}\right) \\
&- \frac{k_1}{4} F_{ijk\|l} F^{ijl\|k} + \frac{k_2\left(k_1+2\right)}{4\left(k_1+k_2+2\right)} F_{ijk\|}{}^{k} F^{ijl\|}{}_{l} + \frac{1}{k_1+2}\pi_{ij\|k}\pi^{ij\|k} \\
&+ \frac{k_1}{2\left(k_1+2\right)\left(1-k_1\right)}\pi^{ij\|k}\left(\pi_{[ij\|k]} - \left(k_1+2\right)F_{ijk\|0}\right) \\
&+ \frac{k_1^2}{8\left(1-k_1\right)} F_{ijk\|0} F^{ijk\|0} - \frac{2k_2}{\left(k_1+2\right)\left[2+k_1+k_2\left(D-2\right)\right]}\pi_{ij\|}{}^{j}\pi^{ik\|}{}_{k} \\
&- \frac{1}{6} F_{ijk\|\mu} F^{ijk\|\mu} + \frac{\pi^{ij\|0}}{k_1+k_2+2}\left(\pi_{ij\|0} - k_2 F_{ijk\|}{}^{k}\right)\Big].
\end{aligned} \tag{58}
$$

The consistency of the primary constraints (9) displays the secondary constraints $G_i^{(2)} \approx 0$ and $G_{i\|j}^{(2)} \approx 0$ (whose concrete expressions are respectively written in (35a) and (35b)) as the Poisson brackets hold

$$
\left[G_i^{(1)}, H_0^{(VII)}\right] = G_i^{(2)}, \quad \left[G_{i\|j}^{(1)}, H_0^{(VII)}\right] = G_{i\|j}^{(2)}. \tag{59}
$$

The Abelian character of the constraints (9), (35a) and (35b) supplemented with the Poisson brackets

$$
\left[G_i^{(2)}, H_0^{(VII)}\right] = 0 = \left[G_{i\|j}^{(2)}, H_0^{(VII)}\right] \tag{60}
$$

allow us to finalize the Dirac algorithm at this stage.

At this stage, we can conclude that the canonical Hamiltonian (58) is exactly the first-class Hamiltonian of the system. The irreducible character of the first-class constraints (9) supplemented with the first-order reducibilities (54a)–(54b) of the secondary first-class constraints $G_i^{(2)} \approx 0$ and $G_{i\|j}^{(2)} \approx 0$ allow us to count the degrees of freedom for the model under study

$$
N_{DOF}^{(VII)} = \frac{D\left(D-2\right)\left(D-3\right)}{2}. \tag{61}
$$

Finally, if we return to the Lagrangian formulation (via extended action), we conclude that in this situation (1) represents a generating set of gauge transformations for (2).

## 3. CONCLUSIONS

In this paper we have analysed the tensor gauge fields of degree three defined on an arbitrary Minkowski space-time. Starting with the most general second-order Lagrangian density for the tensor gauge fields of degree three, we have performed the Dirac analysis. The procedure has revealed seven distinct situations that can occur, cases dictated by the values of the constants that label the starting second-order Lagrangian density. It is remarkable that three of the seven cases manifest interesting conformal [8] and/or topological BF-like [9] behaviours of the massless tensor gauge field of degree three $A_{\mu\nu\|\alpha}$.

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