CONSERVATION LAWS AND ASSOCIATED LIE SYMMETRIES FOR THE $2D$ RICCI FLOW MODEL*

RODICA CIMPOIASU

University of Craiova, 13 A.I. Cuza, 200585 Craiova, Romania,
E-mail: rodicimp@yahoo.com

Received February 20, 2013

The paper presents a connection between Lie symmetries and conservation laws for the $2D$ Ricci flow model.The procedure starts by obtaining a set of multipliers which generates conservation laws.Then, taking into account the most general form of multiplier and making use of a relation which connects Lie symmetries and conservation laws for any dynamical system, one determines associated symmetry generators. On this basis, new group invariant solutions of the model, not yet discussed in literature, are highlighted.

Key words: conservation laws, Lie symmetries, group invariant solutions, Ricci flow model.

PACS: 11.30.Na, 04.20.Jb, 04.70.Bw.

1. INTRODUCTION

There are many reasons for computing symmetries and conservation laws corresponding to systems described by differential equations. In recent years, a remarkable number of mathematical models occurring in various research domains have been studied from the point of view of the symmetry groups' theory [1, 2]. The concept of symmetry is for example fundamental in the study of constrained dynamical systems, where a global symmetry, called the BRST symmetry, could define geometric structures suitable for describing the evolution [3]. The most adequate technique able to find classes of analytical solutions, the so-called Lie symmetry method, investigates integrability starting from the invariance of evolutionary equations under some linear transformations of the variables which define the so-called Lie group of symmetries.

There are many interesting results concerning for the correspondence between symmetries and conservation laws. For example, one can establish [4] a nice relationship between symmetries and conservation laws for self-adjoint differential equations, identity which does not depend on the use of a Lagrangian. Another interesting

*Paper presented at "The 8^{th} Workshop on Quantum Field Theory and Hamiltonian Systems", September 19-22, 2012, Craiova, Romania.

result which will be used here concerns a direct link between the components of a conserved vector for an *arbitrary* partial differential equation and the Lie-Bäcklund symmetry generator of the equation, which is associated to the conserved vector's components [5].

This paper will tackle with the connection between symmetries and conservation laws by using the property of the Euler differential operator to annihilate whatever expression having the mathematical form of a divergence. By applying Euler operator on a combination of evolutionary equations, one can determine a set of multipliers which generate conservation laws. We will apply this procedure for the $2D$ Ricci flow, a very interesting model coming from the gravity theory. Some applications of the Ricci flow model are discussed in [6, 7]. The Lie symmetry problem and some invariant solutions of this model, using the standard approach, have been already discussed [8]. Making use of the procedure mentioned above, new Lie symmetry generators and group invariant solutions of the model, not yet discussed in literature, will be highlighted.

The paper is organized as follows: after this introductive section, the mathematical formulation of the problem of constructing conserved currents for a general differential equation and associated Lie symmetries, will be analyzed in the second section. The conservation law multipliers' method and a close connection between symmetry generators and conserved vectors will be used. In the third section, these methods will be applied to the $2D$ Ricci flow model. A set of conservation laws corresponding to a multiplier $\Lambda(x,t,y,U)$, that does not depend on the derivatives of the dependent variable U, the associated Lie symmetry operators and some new group invariant solutions, will be pointed out . Some concluding remarks will end the paper.

2. CONNECTION BETWEEN LIE SYMMETRIES AND CONSERVATION LAWS

For each partial differential equation (PDE) or for each PDE system there is a local group of transformations (called symmetry group) that acts on the space of its independent and dependent variables, with the property that it maps the set of all analytical solutions to itself, and so it leaves the form of the equation (or system) unchanged. The widely applicable method to find the symmetry group associated with a PDE (or PDE system) is called the *classical Lie method*. Consequently, the knowledge of Lie point symmetries allows us to construct group-invariant solutions. Two solutions are equivalent if there is a symmetry transformation which transforms one into another. Moreover, by applying the symmetry group to a known solution, a family of new solutions may be generated.

Let us consider a system of q partial differential equations (PDEs):

$$\Delta = \{\Delta^{\nu}(x,u(x),u^{(n)}(x))\}_{\nu=1}^{q} = 0 \tag{1}$$

defined on a domain $M \subset R^p$ (*i.e.* a connected open subset of R^p) with at most $n-$th order partial derivatives of $u(x) = (u^1(x), ..., u^q(x))$ with respect to $x = (x^1, ..., x^p)$. The notation $u^{(n)}(x)$ designates the set of partial derivatives of $u(x)$ with respect to x, up to the n-th order.

Let us denote by $X^{(n)}$ the extension of $n-$th order of the Lie infinitesimal symmetry operator with the general form:

$$X = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{2}$$

The Lie symmetry method [9], requires to impose the following invariance condition for the system (1):

$$X^{(n)}(\Delta^\nu)\,|_{\Delta=0} = 0,\ \nu = \overline{1,q} \tag{3}$$

A systematical procedure able to find conservation laws, called the direct method, has been developed [10]. The direct method consists of two main steps:

(i) to determine a set of conservation law multipliers $\{\Lambda_\nu : M \times V^{(n)} \to R\}_{\nu=1}^{q}$, so that a linear combination of the PDEs with the conservation law multipliers yields a divergence expression:

$$\Lambda_\nu[x,U(x),U^{(n)}(x)]\Delta^\nu[x,U(x),U^{(n)}(x)] = Div\ \mathbf{P}[x,U(x),U^{(n)}(x)] \tag{4}$$

In the previous statement, I used the notations: $V^{(n)} = V_1 \times ... \times V_n$ for the $n-$th prolonged space of the Euclidean spaces $V_1, ..., V_n$, a simplified version of the $n-$th order jet space [9]; $U(x)$ for all possible functions, not only for the solutions $u(x)$ of the system (1); $\mathbf{P} \equiv \{P_i : M \times V^{(n)} \to R\}_{i=1}^{p}$ for a vectorial smooth function defined everywhere on $M \times V^{(n)}$.

It is known that conservation law multipliers could be found using the method [10] of Euler operator. Thereby, we may solve the following system of PDEs:

$$E_\rho\left[\Lambda_\nu[x,U(x),U^{(n)}(x)]\Delta^\nu[x,U(x),U^{(n)}(x)]\right] = 0,\ \forall\ \rho = \overline{1,q} \tag{5}$$

for the unknown functions $\{\Lambda_\nu,\ \nu = \overline{1,q}\}$.

(ii) find the corresponding conserved current [11] $\mathbf{P} \equiv \{P_i : M \times V^{(n)} \to R\}_{i=1}^{p}$ in order to obtain the conservation law:

$$\begin{aligned} Div\ \mathbf{P}\ [x,U(x),U^{(n)}(x)]\ _{|U(x)=u(x)} &= \\ \Lambda_\nu[x,U(x),U^{(n)}(x)]\Delta^\nu[x,U(x),U^{(n)}(x)]_{|U(x)=u(x)} &= \\ \Lambda_\nu[x,u(x),u^{(n)}(x)]\Delta^\nu[x,u(x),u^{(n)}(x)] &= 0 \end{aligned} \tag{6}$$

When comparing the conserved law (6) to its alternative definition [12], it should be clear that $\mathbf{P} = (\rho, \mathbf{J})$, $n = \max\{l,m\}$, where $\rho\ [x,u(x),u^{(l)}(x)]$ is the conserved density of order l and $\mathbf{J}\ [x, u(x), u^{(m)}(x)]$ is the associated flux of order m.

An important relation [5] was derived between the symmetry operator X and the components P_i of the conserved current $\mathbf{P}$.It has the form:

$$X^{(r)}(P^i) + P^i D_k(\xi^k) - P^k D_k(\xi^i) = 0,\ i = \overline{1,p} \tag{7}$$

where X is applied in the $r-$th extended form.

The conditions (7) with X known, joined to the conservation law $Div\ \mathbf{P} = 0$, might be viewed as a system of linear partial differential equations which could be solved for the components P^i, $i = \overline{1,p}$ of the conserved vector. Yet, (7) could be used as well in order to obtain the symmetry operators X associated to a given conserved current $\mathbf{P}$. In the next section, the latter way will be applied.

3. APPLICATION TO THE 2D RICCI FLOW MODEL

3.1. THE 2D RICCI FLOW MODEL

The Ricci flow represents a model of geometrical evolution equations generated by a continuous deformation of the metric's components on a Riemannian manifold [14]. Its evolution is described by non-linear parabolic differential equations and they are expressed in the terms of an intrinsic Ricci curvature tensor according to:

$$\frac{\partial}{\partial t} g_{\mu\nu} = -R_{\mu\nu} \tag{8}$$

The Ricci flow equations on two-dimensional manifolds have attracted a considerable attention in the physics' literature in connection with two-dimensional black holes' geometry, exact solutions of the renormalization group equations which describe the decay of singularities in non-compact spaces, etc. [15].

In two-dimensions it is useful to consider a local system of conformally flat coordinates $\{X, Y\}$ for which the metric of the space has the form:

$$ds^2 = \frac{1}{2}\exp\{\Phi(X, Y, t)\}(dX^2 + dY^2) \tag{9}$$

It has been proved [16] that, if passing to complex variables of the form $x = \frac{1}{2}(X + iY), y = \frac{1}{2}(X - iY)$, the equations (8) become**:**

$$\frac{\partial}{\partial t} e^{\Phi(x,y,t)} = \partial_{xy}\Phi(x, y, t) \tag{10}$$

When introducing the field $u(x, y, t)$ by the substitution:

$$u(x, y, t) = e^{\Phi(x,y,t)} \tag{11}$$

the equation (10) takes the form:

$$u_t = (\ln u)_{xy} \tag{12}$$

An equivalent form for the previous equation, which will be used in this paper is:

$$u_t = \frac{u_{xy}}{u} - \frac{u_x u_y}{u^2} \tag{13}$$

Among the main results concerning (13) we might mention: (i) it could be obtained as a particular case of the $3D$ Ricci flow equations which accepts a Killing vector [15] ;(ii) by linearization, it presents various classes [16] of solutions, these depending on the "sector" where it is defined; (iii) using the standard method, a study of its Lie symmetries and invariant solutions was performed [8].

3.2. CONSERVATION LAW MULTIPLIERS OF THE MODEL

In the following considerations, my aim will be to obtain conservation law multipliers for the $2D$ Ricci flow model described above. They are useful because the associated conservation laws for the analyzed model will be generated.

A multiplier Λ of the equation (13) has the property:

$$\Lambda \left[U_t - \frac{U_{xy}}{U} + \frac{U_x U_y}{U^2} \right] = Div\, \mathbf{P}\, [t, x, y, U, U^{(2)}] \tag{14}$$

for all functions $U(t,x,y)$**,** not only for the solutions $u(t,x,y)$ of (13)**.**

Let us consider a multiplier of the form $\Lambda = \Lambda(t,x,y,U)$. Others, which depend on the first and higher order partial derivatives of U could also be considered. Yet, in two dimensions, calculations do rapidly become more complicated. Therefore, computer assisted calculations may lead to further conservation laws.

The right hand side of (14) is a divergence expression. Hence, this expression vanishes when applying the Euler operator which, in two dimensions, takes the form:

$$E_U = \frac{\partial}{\partial U} - D_t \frac{\partial}{\partial U_t} - D_x \frac{\partial}{\partial U_x} - D_y \frac{\partial}{\partial U_y} + D_x^2 \frac{\partial}{\partial U_{2x}} + D_y^2 \frac{\partial}{\partial U_{2y}} + D_x D_y \frac{\partial}{\partial U_{xy}} + \ldots \tag{15}$$

Consequently, the equation (14) becomes:

$$E_U \left[\Lambda \left(U_t - \frac{U_{xy}}{U} + \frac{U_x U_y}{U^2} \right) \right] = 0 \tag{16}$$

which, by expansion, takes the following expression :

$$-\frac{2\Lambda_U}{U} U_{xy} + \left(\frac{\Lambda_U}{U^2} - \frac{\Lambda_{2U}}{U} \right) U_x U_y - \Lambda_{yU} U_x - \Lambda_{xU} U_y - \left(\Lambda_t + \frac{\Lambda_{xy}}{U} \right) = 0 \tag{17}$$

Because (17) is satisfied for all functions $U(t,x,y)$ and because the multiplier Λ is chosen as suitable in order not to depend on derivatives of U, the coefficient functions of various derivatives of U must vanish. Thereby, a determining system for multiplier $\Lambda(t,x,y,U)$ is generated:

$$\Lambda_U = \Lambda_t = \Lambda_{xy} = 0 \tag{18}$$

By solving the differential system (18), a multiplier with a general form results:

$$\Lambda = \alpha f(x) + \beta g(y) \neq 0 \tag{19}$$

with arbitrary functions $f(x)$, $g(y)$ and arbitrary constants α, β.

3.3. CONSERVATION LAWS AND ASSOCIATED LIE SYMMETRY GENERATORS

We will continue our investigations on the $2D$ Ricci flow model by determining in this subsection the conserved currents associated with the multiplier (19). From (14) and (19), through elementary manipulations, we obtain:

$$\begin{aligned}[\alpha f(x) + \beta g(y)]\left[U_t - \frac{U_{xy}}{U} + \frac{U_x U_y}{U^2}\right] &= D_t\left[(\alpha f(x) + \beta g(y))U\right] + \\ +D_x\left[-\beta g(y)\frac{U_y}{U}\right] + D_y\left[-\alpha f(x)\frac{U_x}{U}\right] &= Div\,\mathbf{P}\end{aligned} \tag{20}$$

For the case when $U(t,x,y)$ is a solution of (13), the previous expression leads to the following set of conservation laws:

$$D_t\left[(\alpha f(x) + \beta g(y))u\right] + D_x\left[-\beta g(y)\frac{u_y}{u}\right] + D_y\left[-\alpha f(x)\frac{u_x}{u}\right] = 0 \tag{21}$$

for arbitrary functions $f(x)$, $g(y)$ and arbitrary constants α, β. As a result, the conserved current $\mathbf{P}$ has the components:

$$P^1 = (\alpha f(x) + \beta g(y))u, \quad P^2 = -\beta g(y)\frac{u_y}{u}, \quad P^3 = -\alpha f(x)\frac{u_x}{u}, \tag{22}$$

It is important to remark that conserved currents (22) might have quite arbitrary coefficients in independent variables. It is also important to underline the fact that the previous conserved quantities are very important in generating various classes of solutions for Ricci-type flow equations. Such solutions, soliton-like ones, but not only them, are mentioned for example in [16].

The next aim of the paper is to illustrate how we might find the point symmetries associated to the conservation law (21). These symmetries are generated by Lie operators which define the Lie differential group. They are extremely important in finding the class of invariant solutions. Such solutions for the Ricci flow, using similarity reduction, were obtained in [8]. Here, when combining the use of the differential Lie group to the attached conservation laws, new invariant solutions will be generated.

The starting point of computations is clearly the relation (7), in which P^i are considered as known components of the conserved current and X as an unknown symmetry operator. The symmetry conditions for (22) are:

$$\begin{aligned} X^{(1)}(P^1)+P^1D_x(\xi)+P^1D_y(\eta)-P^2D_x(\varphi)-P^3D_y(\varphi)=0\\ X^{(1)}(P^2)+P^2D_t(\varphi)+P^2D_y(\eta)-P^1D_t(\xi)-P^3D_y(\xi)=0\\ X^{(1)}(P^3)+P^3D_t(\varphi)+P^3D_x(\xi)-P^1D_t(\eta)-P^2D_x(\eta)=0 \end{aligned} \tag{23}$$

where the Lie symmetry operator $X=\varphi\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\phi\frac{\partial}{\partial u}$ is of type (2) and admits the first extension:

$$X^{(1)}(t,x,y,u)=X+\phi^t\frac{\partial}{\partial u_t}+\phi^x\frac{\partial}{\partial u_x}+\phi^y\frac{\partial}{\partial u_y} \tag{24}$$

The coefficient functions ϕ^t, ϕ^x, ϕ^y will be calculated according to the Lie symmetry theory [9]:

$$\begin{aligned} \phi^t=&D_t(\phi-\varphi u_t-\xi u_x-\eta u_y)+\varphi u_{2t}+\xi u_{xt}+\eta u_{yt}\\ \phi^x=&D_t(\phi-\varphi u_t-\xi u_x-\eta u_y)+\varphi u_{tx}+\xi u_{2x}+\eta u_{yx}\\ \phi^y=&D_t(\phi-\varphi u_t-\xi u_x-\eta u_y)+\varphi u_{ty}+\xi u_{xy}+\eta u_{2y} \end{aligned}$$

From now on, with no loss of generality, the choice $\varphi=c=const.$ in (24) will be analyzed. By expanding the determining equations (23) and separating various monomials in derivatives of u,the following differential system is generated:

$$u\left[\alpha\xi\frac{df(x)}{dx}+\beta\eta\,\frac{dg(y)}{dy}+[\alpha f(x)+\beta g(y)](\xi_x+\eta_y)\right]+[\alpha f(x)+\beta g(y)]\phi=0 \tag{25a}$$

$$u\beta\eta\,\frac{dg(y)}{dy}-\beta g(y)[\phi-u\phi_u]=0 \tag{25b}$$

$$u\alpha\xi\,\frac{df(x)}{dx}-\alpha f(x)[\phi-u\phi_u]=0 \tag{25c}$$

$$u^2[\alpha f(x)+\beta g(y)]\xi_t+\beta g(y)\phi_y=0 \tag{25d}$$

$$u^2[\alpha f(x)+\beta g(y)]\eta_t+\alpha f(x)\phi_x=0, \tag{25e}$$

with as unknown functions $f(x)$, $g(y)$ and coefficient functions $\xi(x,t)$, $\eta(y,t)$, $\phi(t,x,y,u)$ which appear in the Lie operator X.

By solving the previous differential system, the following solutions are obtained:

Symmetry I**:** For $f(x)=0$ and $\forall\ g(y)$, the symmetry operator admits the expression:

$$X^I = \frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x} - \xi_x u \frac{\partial}{\partial u},\ \forall \xi(x) \tag{26}$$

Symmetry II**:** For $g(y) = 0$ and $\forall\ f(x)$, the symmetry generator takes the form:

$$X^{II} = \frac{\partial}{\partial t} + \eta(y)\frac{\partial}{\partial x} - \eta_y u \frac{\partial}{\partial u}, \forall \eta(y) \tag{27}$$

Symmetry III**:** For $\forall\ f(x) \neq 0$, $\forall\ g(y) \neq 0$, the Lie operator is:

$$X^{III} = \frac{\partial}{\partial t} \tag{28}$$

3.4. GROUP INVARIANT SOLUTIONS OF THE MODEL

Let us find the group invariant solutions generated respectively by the operators obtained before.

(I) Let us derive the invariant solution generated by the Lie symmetry generator (26). The function $u^I = \psi^I(t,x,y)$ is a group invariant solution of (13) provided that:

$$X^I[u^I - \psi^I(t,x,y)]\big|_{u=\psi^I} = 0 \tag{29}$$

This condition is equivalent to the partial differential equation:

$$\psi^I_t + \xi(x)\psi^I_x + \xi_x \psi^I = 0 \tag{30}$$

The solution of (30) is given by the expression:

$$\psi^I(t,x,y) = \frac{H(\left(t - \int \frac{1}{\xi(x)}dx,\ y\right)}{\xi(x)} \tag{31}$$

Introducing the notation $z = t - \int \frac{1}{\xi(x)}dx$ and substituting (31) into (13), the following partial differential equation results for $H(z,y)$:

$$H_z H^2 + H H_{zy} - H_z H_y = 0 \tag{32}$$

This reduced differential equation admits the solutions:

$$\forall\ H(y) \text{ and } H(z,y) = \frac{e^{(y+z+c)}}{-1 + e^{(y+z+c)}},\ \forall c = const \tag{33}$$

Thereby, the invariant solutions associated with the symmetry operator (26) are given by the expressions:

$$u^I_1 = \frac{H(y)}{\xi(x)}, \forall\ H(y),\ \forall\ \xi(x) \tag{34}$$

$$u^I_2 = \frac{e^{(y+z+c)}}{\left[-1 + e^{(y+z+c)}\right]\xi(x)},\ \forall c = const \tag{35}$$

Remark: For the linear sector of invariance ($\xi(x) = -ax + b$ has linear form), the invariant condition (30) leads to the differential equation:

$$a\psi^I_{lin} - \left(\psi^I_{lin}\right)_t + (ax - b)\left(\psi^I_{lin}\right)_x = 0 \tag{36}$$

with a,b arbitrary constants.

It provides the invariant solution:

$$u^I_{lin} = G(y,w)e^{at}, \forall\, G(y,w) \tag{37}$$

$$w = \frac{e^{at}(ax - b)}{a} \tag{38}$$

In this particular situation, the reduced equation becomes:

$$awG_wG^2 + aG^3 - GG_{yw} + G_yG_w = 0$$

and generates the following exact solution:

$$u^I_{lin} = \frac{aM(y)}{e^{at}(ax - b)},\ \forall M(y) \tag{39}$$

(II)**:** Applying the same approach, similar results could be obtained for the symmetry operator (27), by operating the changes: $x \to y$, $\xi(x) \to \eta(y)$.

(III)**:** The time translational operator (28) leads to a stationary invariant solution with general expression:

$$u^{III} = u_1(x)u_2(y), \forall u_1(x),\ \forall u_2(y) \tag{40}$$

4. CONCLUDING REMARKS

The advantage of the method reported in this paper is that it may be applied for arbitrary differential equations, with no regard to their Lagrangian formulation.

The paper approached, as a concrete example of evolutionary equation, the $2D$ Ricci flow model. The most general form of multiplier and the corresponding conservation laws were determined in the forms (19), (21) which depend on two arbitrary functions $f(x)$ and $g(y)$. By making use of symmetry conditions (23), I arrived to differential system. Solving it using Maple program, one can generated three Lie symmetry operators. Two of them, namely (26) and (27) implies the arbitrary functions $\xi(x)$ and $\eta(y)$, respectively. The third operator (28) is simpler In fact it is the time translation operator. New invariamt solutions (34), (35), (39) and (40), for Ricci flow model, associated to the mentioned Loe symmetry group have been pointed out.

These results have been obtained by the choice of a conservation law multiplier $\Lambda = \Lambda(t,x,y,U)$ which only depends on independent variables (t,x,y) and on the dependent one $U(t,x,y)$. It should be interesting to see how the results reported by this paper might be extended for more general conservation law multipliers $\Lambda = \Lambda(t,x,y,U,U^{(n)})$, $n \geq 1$ which , further more, could depend up on first or higher order partial derivatives of U. Calculations will become more complicated and might lead to further conservation laws, consequently to more associated symmetries and invariant solutions. This open problem will be studied in forthcoming works.

REFERENCES

1. G.W. Bluman, S. Kumei, "*Symmetries and Differential Equations*" (Springer, New York, 1989).
2. P.E. Hydon, "*Symmetry Methods for Differential Equations*" (Cambridge University Press, 2000).
3. R. Constantinescu, J. Math. Phys. **38**, 2786–2794 (1997).
4. S.C. Anco, G. Bluman, J. Math. Phys. **37**, 2361–2375 (1996).
5. A.H. Kara, F.M. Mahomed, IJTP **39**, 23–40 (2000).
6. M.T. Anderson, Notices of the AMS **51**, 184–193 (2004).
7. T. Oliynyk, V. Suneeta, E. Woolgar, Phys. Lett.B **610**, 115–121 (2005).
8. R. Cimpoiasu, R. Constantinescu, J. Nonlin. Math. Phys. **13**, 285–292 (2006).
9. P.J. Olver, "*Applications of Lie Groups to Differential Equations*" (Springer, New York, 1986).
10. G.W. Bluman, A.F. Cheviakov, S.C. Anco, "*Applications of Symmetry Methods to Partial Diferential Equations*" (Springer, New York, 2010).
11. A.M. Vinogradov, Acta Appl. Math. **15**, 161–209 (1989).
12. M.J. Ablowitz, P.A. Clarkson, "*Solitons, Nonlinear Evolution Equations and Inverse Scattering*" (Cambridge University Press, Cambridge, 1991).
13. D. Grumiller, W. Kummer, D.V. Vassilievich, Phys. Rep. **369**, 327–430 (2002).
14. I. Bakas, JHEP **0308**, 013 (2003).
15. I. Bakas, Comptes Rendus Physique **6**, 175–184 (2005).
16. S. Carstea, M. Visinescu, Mod. Phys.Lett. A **20**, 2993–2795 (2005).