

ANOTHER RELATIONSHIP BETWEEN FIRST- AND SECOND-ORDER SYSTEMS*

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Another relationship between first- and second-order formulations of dynamics is proposed for a given class of first-order systems.

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In theoretical physics the equations that govern the dynamics of a given system are usually obtained from a variational principle based on a certain action. Such an action may be linear (first-order action) or quadratic (second-order action) in the time derivatives of the dynamic variables. It is easy too see that first-order actions lead to first-order equations of motion and second-order actions produce second-order equations of motion. Starting from a second-order action we can obtain a first-order action by introducing auxiliary variables. This is the standard relationship between the Lagrangian and Hamiltonian formalisms. It is well-known that for systems described by non-degenerate second-order Lagrangians the Euler–Lagrange [1] and Hamilton [2] equations are equivalent. In the case of constrained (degenerate) systems [3–5] the equivalence between the two sets of equations is no longer manifest and must be implemented via the introduction of Lagrange multipliers. For instance, the Klein–Gordon [6, 7], Maxwell [8–12], and Einstein [13] actions can be expressed in both second- and first-order form. Conversely, starting from a first-order action, it is not always possible to find an equivalent second-order action. Indeed, neither among the Schrödinger [14], Dirac [15], or chiral (self-dual) p -forms [16] actions allows an equivalent (local) second-order form. The previous discussion emphasizes that any second-order action allows an equivalent first-order form, but not conversely. Thus, there appears a lack of symmetry between first- and second-order formulations of dynamics. In this sense, first-order systems appear to be somehow privileged. An interesting viewpoint on first-order systems, including their quantization, can be found in [17, 18].

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In this paper, starting from a first-order system with a non-degenerate symplectic two-form, we emphasize a new type of relationship between first- and second-order formulations of its dynamics.

Let $z^A|_{A=1,\dots,2n}$ be a set of variables that parametrize the time-evolution of a dynamic system. For simplicity we take all these variables to be bosonic. Assume that the considered system is described by a first-order action

$$S_0[z^A] = \int_{t_1}^{t_2} dt (a_A(z) \dot{z}^A - V(z)) \equiv \int_{t_1}^{t_2} dt L_0(z, \dot{z}), \quad (1)$$

with $a_A(z)$ the one-form potential and $V(z)$ a given potential. In what follows we consider the case where the symplectic two-form

$$\omega_{AB}(z) = \frac{\partial a_B}{\partial z^A} - \frac{\partial a_A}{\partial z^B} = -\omega_{BA}(z) \quad (2)$$

is non-degenerate, *i.e.*, $\det \omega_{AB} \neq 0$ (locally). The non-degeneracy of the symplectic two-form leads to the existence of a bracket structure (Poisson or Dirac bracket), locally given by

$$[F_1, F_2] = \omega^{AB} \frac{\partial F_1}{\partial z^A} \frac{\partial F_2}{\partial z^B}, \quad (3)$$

in terms of which the Euler–Lagrange equations of motion, $\frac{\delta L_0}{\delta z^A} \equiv \frac{\partial L_0}{\partial z^A} - \frac{d}{dt} \left(\frac{\partial L_0}{\partial \dot{z}^A} \right) = 0$, deriving from the first-order variational principle based on (1), can be put in the Hamiltonian form

$$H^A \equiv \dot{z}^A - [z^A, V] = 0, \quad (4)$$

with ω^{AB} the inverse of ω_{AB} . The fixation of the integration constants in the general solution of equations (4) requires to impose some endpoint or initial conditions. We adopt the second version and choose the initial conditions

$$z^A(t_0) = z_0^A, \quad (5)$$

with $t_1 \leq t_0 \leq t_2$.

Now, we consider the second-order equations

$$E^A \equiv \ddot{z}^A - [[z^A, V], V] = 0, \quad (6)$$

in the presence of the initial conditions

$$z^A(t_0) = z_0^A, \quad \dot{z}^A(t_0) = [z^A, V]|_{z_0^A}, \quad (7)$$

which are mutually compatible with (5). Then, we have the following result: R_1) the second-order equations (6) in the presence of the initial conditions (7) possess the same solutions like the Hamilton equations (4) subject to the initial conditions (5). The proof will be given in [19].

Assume that there exists a constant, symmetric, and invertible matrix k_{AB} such that the relations

$$k_{AC} \frac{\partial [z^C, V]}{\partial z^B} = k_{BC} \frac{\partial [z^C, V]}{\partial z^A} \quad (8)$$

hold. Then, the second-order Lagrangian

$$\bar{L}_0(z, \dot{z}) = \frac{1}{2} k_{AB} (\dot{z}^A \dot{z}^B + [z^A, V] [z^B, V]) \quad (9)$$

leads to the relations

$$E^A = -k^{AB} \frac{\delta \bar{L}_0}{\delta z^B}, \quad (10)$$

with k^{AB} the inverse of k_{AB} . Therefore, we have proved that: R_2) if there exists a constant, symmetric, and invertible matrix k_{AB} such that relations (8) are fulfilled, then there exists the Lagrangian (9) such that the second-order equations (6) originate in some Euler–Lagrange equations of the type (10).

Conclusions R_1) and R_2) can be synthesized into the following theorem, which represents our main result.

Theorem 1 *Let $L_0(z, \dot{z}) = a_A(z) \dot{z}^A - V(z)$ be a first-order Lagrangian with a non-degenerate symplectic two-form. If there exists a constant, symmetric, and invertible matrix k_{AB} such that the relations $k_{AC} \frac{\partial [z^C, V]}{\partial z^B} = k_{BC} \frac{\partial [z^C, V]}{\partial z^A}$ hold, then there exists the second-order Lagrangian $\bar{L}_0(z, \dot{z}) = \frac{1}{2} k_{AB} \dot{z}^A \dot{z}^B + \frac{1}{2} k_{AB} [z^A, V] [z^B, V]$ such that*

$$\begin{cases} \dot{z}^A - [z^A, V] = \omega^{AB} \frac{\delta L_0}{\delta z^B} = 0, \\ z^A(t_0) = z_0^A, \end{cases} \Leftrightarrow \begin{cases} \ddot{z}^A - [[z^A, V], V] = -k^{AB} \frac{\delta \bar{L}_0}{\delta z^B} = 0, \\ z^A(t_0) = z_0^A, \dot{z}^A(t_0) = [z^A, V]|_{z_0^A}. \end{cases} \quad (11)$$

The above theorem emphasizes a new type of relationship between first- and second-order formulations of dynamics.

Let us consider a class of models described by the relations

$$z^1 = q, \quad z^2 = p, \quad V = \alpha \left(\frac{1}{2} qp^2 + \beta q^2 p + \frac{1}{3!} \gamma q^3 + \delta q \right), \quad 3\beta^2 - \gamma \neq 0, \quad (12)$$

where (q, p) are the canonical coordinates of a one-dimensional system

$$[q, q] = 0 = [p, p], \quad [q, p] = 1, \quad (13)$$

while α , β , γ , and δ are some arbitrary, non-vanishing, real constants. By direct computation we obtain the formulas

$$[q, V] = \alpha q(p + \beta q), \quad [p, V] = -\alpha \left(\frac{1}{2} p^2 + 2\beta qp + \frac{\gamma}{2} q^2 + \delta \right), \quad (14)$$

from which we find that the class of models defined by relations (12)–(13) satisfy (8), with

$$k_{11} = 4\beta^2 - \gamma, \quad k_{12} = k_{21} = \beta, \quad k_{22} = 1. \quad (15)$$

Substituting (14)–(15) in (9) we get the concrete form of the Lagrangian $\bar{L}_0(q, p, \dot{q}, \dot{p})$ for these models. Two different examples that satisfy (8) have been given in [20].

At the same time, relations (8) ensure that there exists a function $N(z)$ such that

$$[z^A, V] = -k^{AB} \frac{\partial N}{\partial z^B}. \quad (16)$$

If one defines a symmetric bracket [21, 22], locally expressed by

$$\{F_1, F_2\} = -k^{AB} \frac{\partial F_1}{\partial z^A} \frac{\partial F_2}{\partial z^B} = \{F_2, F_1\}, \quad (17)$$

from (16) we find that

$$[z^A, V] = \{z^A, N\}, \quad (18)$$

such that we arrive to the relations

$$[[z^A, V], V] = \frac{1}{2} \{z^A, \{N, N\}\}. \quad (19)$$

Then, equations (4) and (6) can be written in terms of the previously introduced symmetric bracket under the form

$$H^A \equiv \dot{z}^A - \{z^A, N\} = 0, \quad E^A \equiv \ddot{z}^A - \{\{z^A, N\}, N\} = 0, \quad (20)$$

while the Lagrangian (9) becomes

$$\bar{L}_0(z, \dot{z}) = \frac{1}{2} k_{AB} \dot{z}^A \dot{z}^B - \frac{1}{2} \{N, N\}. \quad (21)$$

Now, the initial conditions $\dot{z}^A(t_0) = [z^A, V]|_{z_0^A}$ for the second-order equations take the form $\dot{z}^A(t_0) = \{z^A, N\}|_{z_0^A}$. Formulas (20)–(21) show that we can express the first- and second-order formulation of dynamics in terms of the symmetric bracket (17) and the generator N . For the models (12)–(13) the symmetric bracket is given by

$$(q, q) = -\frac{1}{3\beta^2 - \gamma}, \quad (q, p) = \frac{\beta}{3\beta^2 - \gamma}, \quad (p, p) = -\frac{4\beta^2 - \gamma}{3\beta^2 - \gamma}, \quad (22)$$

and the generator N has the form

$$N = \alpha \left(\frac{1}{3!} p^3 + \frac{1}{2} \beta q p^2 - \left(\beta^2 - \frac{1}{2} \gamma \right) q^2 p - \beta \left(\frac{4}{3} \beta^2 - \frac{1}{2} \gamma \right) q^3 + \delta p + \delta \beta q \right). \quad (23)$$

Let us suppose that we can choose the matrix k_{AB} such that

$$\omega^{AC} k_{CD} \omega^{DB} = \tau k^{AB}, \quad (24)$$

where τ is a real constant. Taking into account (24) we find that the relationship between the symmetric and antisymmetric brackets is expressed by

$$\{F_1, F_2\} = -\frac{1}{\tau} k_{AB} [F_1, z^A] [z^B, F_2], \quad (25)$$

$$[F_1, F_2] = \tau \omega_{AB} \{F_1, z^A\} \{z^B, F_2\}. \quad (26)$$

Meanwhile, from (16)–(17), (19), (24), and (26) we deduce the relations

$$[z^A, V] = -\tau K_B^A \{z^B, V\}, \quad [[z^A, V], V] = -\tau \{\{z^A, V\}, V\}, \quad (27)$$

which allow us to express the first- and second-order equations of motion in terms of the symmetric bracket and of the generator V under the form

$$H^A \equiv \dot{z}^A + \tau K_B^A \{z^B, V\} = 0, \quad E^A \equiv \ddot{z}^A + \tau \{\{z^A, V\}, V\} = 0, \quad (28)$$

where we used the notation

$$K_B^A = k^{AC} \omega_{CB}. \quad (29)$$

In this case the initial conditions $\dot{z}^A(t_0) = [z^A, V]|_{z_0^A}$ for the second-order equations can be written as $\dot{z}^A(t_0) = -\tau K_B^A \{z^B, V\}|_{z_0^A}$, while the Lagrangian (21) reads as

$$\bar{L}_0(z, \dot{z}) = \frac{1}{2} k_{AB} \dot{z}^A \dot{z}^B + \frac{1}{2} \tau \{V, V\}. \quad (30)$$

Models (12)–(13) satisfy relations (24) for

$$\tau = -(3\beta^2 - \gamma). \quad (31)$$

To conclude with, in this paper we have shown that, given a first-order system with a non-degenerate symplectic two-form, we can find, under some suitable, supplementary conditions, an appropriate, non-degenerate, second-order Lagrangian formulation in terms of the same variables (meaning without introduction of auxiliary variables). The second-order equations of motion are initially expressed in terms of the antisymmetric bracket from the first-order formulation. Moreover, under some additional assumptions, we can express both first- and second-order formulations in terms of a symmetric bracket.

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REFERENCES

1. J.L. Lagrange, *Mécanique analytique* (Cambridge University Press, Cambridge, 2010).
2. W.R. Hamilton, *Phil. Trans. R. Soc. London* **125**, 95 (1835).
3. P.A.M. Dirac, *Can. J. Math.* **2**, 129 (1950).
4. P.A.M. Dirac, *Lectures on Quantum Mechanics* (Academic Press, New York, 1967).
5. M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992).
6. O. Klein, *Z. Phys.* **37**, 895 (1926).
7. W. Gordon, *Z. Phys.* **40**, 117 (1926).

8. J.C. Maxwell, *Philos. Mag.* **21**, 161 (1861).
9. J.C. Maxwell, *Philos. Mag.* **21**, 281 (1861).
10. J.C. Maxwell, *Philos. Mag.* **21**, 338 (1861).
11. J.C. Maxwell, *Philos. Mag.* **23**, 12 (1862).
12. J.C. Maxwell, *Philos. Mag.* **23**, 85 (1862).
13. A. Einstein, *Annalen Phys.* **354**, 769 (1916).
14. E. Schrödinger, *Annalen Phys.* **81**, 109 (1926).
15. P.A.M. Dirac, *Proc. R. Soc. A* **117**, 610 (1928).
16. M. Henneaux, C. Teitelboim, *Phys. Lett. B* **206**, 650 (1988).
17. L. Faddeev, R. Jackiw, *Phys. Rev. Lett.* **60**, 1692 (1988).
18. R. Jackiw, “(Constrained) Quantization Without Tears”, arXiv:hep-th/9306075 (1993).
19. C. Bizdadea, M.M. Barcan, E.M. Cioroianu, M.T. Miauta, S.O. Saliu, in preparation.
20. C. Bizdadea, M.M. Barcan, M.T. Miauta, S.O. Saliu, *Mod. Phys. Lett. A* **27**, 1250062 (2012).
21. P. Droz-Vincent, *Ann. Inst. Henri Poincaré Section A* **5**, 257 (1966).
22. W.H. Franke, A.J. Kalnay, *J. Math. Phys.* **11**, 1729 (1970).