ASYMPTOTIC SYMMETRIES AT NULL INFINITY AND LOCAL CONFORMAL PROPERTIES OF SPIN COEFFICIENTS*

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Received February 20, 2013

We show that the symmetry algebra of asymptotically flat four dimensional spacetimes at null infinity in the sense of Newman and Unti is isomorphic to the direct sum of the abelian algebra of infinitesimal conformal rescalings with $\mathfrak{bms}_4$. We then work out the local conformal properties of the relevant Newman-Penrose coefficients, as well as the surface charges and their algebra.

Key words: Gauge gravity correspondance, Gauge symmetries, Surface charges, Newman–Penrose formalism, GHP formalism.

1. INTRODUCTION

This conference proceedings summarizes the results of paper [1] to which we refer for detailed computations and discussions.

The definitions of asymptotically flat four dimensional space-times at null infinity by Bondi-Van der Burg-Metzner-Sachs [2, 3] (BMS) and Newman-Unti (NU) [4] in 1962 merely differ by the choice of the radial coordinate. Such a change of gauge should not affect the asymptotic symmetry algebra if, as we contend, this concept is to have a major physical significance. The problem of comparing the symmetry algebra in both cases is that, besides the difference in gauge, the very definitions of these algebras are not the same. Indeed, NU allow the leading part of the metric induced on Scri to undergo a conformal rescaling. When this generalization is considered in the BMS setting, it turns out that the symmetry algebra is the direct sum of the BMS algebra $\mathfrak{bms}_4$ [5] with the abelian algebra of infinitesimal conformal rescalings [6], [7].

In this note we show that, as expected, the asymptotic symmetry algebra in the NU framework is again the direct sum of $\mathfrak{bms}_4$ with the abelian algebra of infinitesimal conformal rescalings of the metric on Scri and thus coincides, as it should,

*Paper presented at "The 8^{th} Workshop on Quantum Field Theory and Hamiltonian Systems", September 19-22, 2012, Craiova, Romania.

Rom. Journ. Phys., Vol. 58, Nos. 5-6, P. 423–427, Bucharest, 2013

with the generalized symmetry algebra in the BMS approach. We then discuss the transformation properties of the Newman-Penrose coefficients parametrizing solution space in the NU approach, focussing on the inhomogeneous terms in the transformation laws that contain the information on the central extensions of the theory, and we finally study the associated surface charges and their algebra by following the analysis in the BMS gauge [8].

2. NU METRIC ANSATZ AND ASYMPTOTIC SYMMETRIES

The metric ansatz of NU can be written as

$$ds^2 = W du^2 - 2dr du + g_{AB}(dx^A - V^A du)(dx^B - V^B du)\,, \tag{1}$$

with coordinates u, r, x^A and where

$$g_{AB}dx^A dx^B = r^2 \bar{\gamma}_{AB} dx^A dx^B + r C_{AB} dx^A dx^B + o(r)\,,$$

with $\bar{\gamma}_{AB}$ conformally flat. Below, we will use standard stereographic coordinates $\zeta = \cot\frac{\theta}{2}e^{i\phi}, \bar{\zeta}$, $\bar{\gamma}_{AB}dx^A dx^B = e^{2\widetilde{\varphi}} d\zeta d\bar{\zeta}$, $\widetilde{\varphi} = \widetilde{\varphi}(u,x)$. There is also an additional condition, related to the fixing of the origin of the affine parameter of the null geodesic generators of the null hypersurfaces used to build the metric [4], which yields here $C^A_A = 0$ [1].

In the following we denote by $\bar{D}_A$ the covariant derivative with respect to $\bar{\gamma}_{AB}$ and by $\bar{\Delta}$ the associated Laplacian. The fall-off conditions are $V^A = O(r^{-2})$ and $W = -2r\partial_u\widetilde{\varphi} + \bar{\Delta}\widetilde{\varphi} + O(r^{-1})$, where $\bar{\Delta}\widetilde{\varphi} = 4e^{-2\widetilde{\varphi}}\partial\bar{\partial}\widetilde{\varphi}$ with $\partial = \partial_\zeta, \bar{\partial} = \partial_{\bar{\zeta}}$.

The infinitesimal NU transformations are defined as those infinitesimal transformations that leave the form of the metric and the fall-off conditions invariant, up to a rescaling of the conformal factor $\delta\widetilde{\varphi}(u,x^A) = \widetilde{\omega}(u,x^A)$, and are in this case generated by

$$\begin{cases} \xi^u = f, \\ \xi^A = Y^A + I^A, \quad I^A = -\partial_B f \int_r^\infty dr' g^{AB}, \\ \xi^r = -r\partial_u f + Z + J, \quad J = \partial_A f \int_r^\infty dr' V^A, \end{cases} \tag{2}$$

with $\partial_r f = 0 = \partial_r Y^A = \partial_r Z$, $Z = \frac{1}{2}\bar{\Delta} f$, $\partial_u Y^A = 0$, with Y^A a conformal Killing vector of $\bar{\gamma}_{AB}$, *i.e.* $Y^\zeta \equiv Y = Y(\zeta)$, $\quad Y^{\bar{\zeta}} \equiv \bar{Y} = \bar{Y}(\bar{\zeta})$ in the coordinates $(\zeta,\bar{\zeta})$, and also with

$$f = e^{\widetilde{\varphi}}\big[\widetilde{T} + \frac{1}{2}\int_0^u du' e^{-\widetilde{\varphi}}\widetilde{\psi}\big], \quad \widetilde{T} = \widetilde{T}(\zeta,\bar{\zeta}), \tag{3}$$

with $\psi = \bar{D}_A Y^A$ and $\widetilde{\psi} = \psi - 2\widetilde{\omega}$. Asymptotic Killing vectors thus depend on $Y^A, \widetilde{T}, \widetilde{\omega}$ and the metric, $\xi = \xi[Y, \widetilde{T}, \widetilde{\omega}; g]$. For such metric dependent vector fields, consider the suitably modified Lie bracket taking the metric dependence of the spacetime vectors into account, $[\xi_1, \xi_2]_M = [\xi_1, \xi_2] - \delta^g_{\xi_1}\xi_2 + \delta^g_{\xi_2}\xi_1$, where $\delta^g_{\xi_1}\xi_2$ denotes the variation in ξ_2 under the variation of the metric induced by ξ_1, $\delta^g_{\xi_1} g_{\mu\nu} = \mathcal{L}_{\xi_1} g_{\mu\nu}$. Consider now the extended $\mathfrak{bms}_4$ algebra, *i.e.*, the semi-direct sum of the algebra of conformal Killing vectors of the Riemann sphere with the abelian ideal of infinitesimal supertranslations, trivially extended by infinitesimal conformal rescalings of the conformally flat degenerate metric on Scri. The commutation relations are given by $[(Y_1,\widetilde{T}_1,\widetilde{\omega}_1),(Y_2,\widetilde{T}_2,\widetilde{\omega}_2)] = (\widehat{Y},\widehat{\widetilde{T}},\widehat{\widetilde{\omega}})$ where

$$\begin{cases} \widehat{Y}^A = Y_1^B \partial_B Y_2^A - Y_2^B \partial_B Y_1^A, \\ \widehat{\widetilde{T}} = Y_1^A \partial_A \widetilde{T}_2 - Y_2^A \partial_A \widetilde{T}_1 + \frac{1}{2}(\widetilde{T}_1 \partial_A Y_2^A - \widetilde{T}_2 \partial_A Y_1^A), \\ \widehat{\widetilde{\omega}} = 0 . \end{cases} \tag{4}$$

In these terms, one can show the following:

Theorem 2.1 *The spacetime vectors $\xi[Y,\widetilde{T},\widetilde{\omega};g]$ realize the extended $\mathfrak{bms}_4$ algebra in the modified Lie bracket,*

$$\Big[\xi[Y_1,\widetilde{T}_1,\widetilde{\omega}_1;g],\xi[Y_2,\widetilde{T}_2,\widetilde{\omega}_2;g]\Big]_M = \xi[\widehat{Y},\widehat{\widetilde{T}},\widehat{\widetilde{\omega}};g], \tag{5}$$

in the bulk of an asymptotically flat spacetime in the sense of Newman and Unti.

Note in particular that for two different choices of the conformal factor $\widetilde{\varphi}$ which is held fixed, $\widetilde{\omega} = 0$, the asymptotic symmetry algebras are isomorphic to $\mathfrak{bms}_4$, which is thus a gauge invariant statement.

3. EXPLICIT RELATIONS BETWEEN THE NU AND THE BMS GAUGES AND LOCAL CONFORMAL TRANSFORMATION LAWS OF THE NU COEFFICIENTS

The choice of the radial coordinate in the definition of asymptotically flat space-times in the BMS [2], [3], [5] and the NU [4] approaches differs but the relation between the two radial coordinates does not involve constant terms [1] and is of the form $r' = r + O(r^{-1})$. This change of coordinates only affects lower order terms in the asymptotic expansion of the metric that play no role in the definition of asymptotic symmetries and explains a posteriori why the asymptotic symmetry algebras in both approaches are isomorphic.

In the BMS set-up, the general solution to Einstein's field equations is parameterized by some functions [2], [3], [7] among which are the mass and angular momentum aspects, and the news tensors. In the NU case instead [4], the free data characterizing solution space are described in terms of the spin coefficient σ^0 and its time derivative, and also in terms of the Ψ^0_α (with $\alpha = 0,1,2,3,4$), five complex scalars representing all the components of the Weyl tensor. The explicit relations between the free data characterizing asymptotic solution space in both approaches were established for instance in [1].

Using the "eth" operators [9] defined for a field η^s of spin weight s according to the conventions of [10] through $\eth\eta^s = P^{1-s}\bar{\partial}(P^s\eta^s)$, $\bar{\eth}\eta^s = P^{1+s}\partial(P^{-s}\eta^s)$ with

$P=\sqrt{2}e^{-\widetilde{\varphi}}$, where $\eth,\bar{\eth}$ raise respectively lower the spin weight by one unit and let $\mathcal{Y}=P^{-1}\bar{Y}$ and $\bar{\mathcal{Y}}=P^{-1}Y$. The conformal Killing equations and the conformal factor then become $\eth\bar{\mathcal{Y}}=0=\bar{\eth}\mathcal{Y}$ and $\psi=(\eth\mathcal{Y}+\bar{\eth}\bar{\mathcal{Y}})$. Using the notation $S=(Y,\widetilde{T},\widetilde{\omega})$, we have $-\delta_S\bar{\gamma}_{AB}=2\widetilde{\omega}\bar{\gamma}_{AB}$ for the background metric.

To work out the transformation properties of the NU coefficients characterizing asymptotic solution space, one needs to evaluate the subleading terms in the Lie derivative of the metric on-shell. This can also be done by translating the results from the BMS gauge, using the dictionary of [1], which yields in this case

$$\begin{aligned}
-\delta_S\sigma^0 &= [f\partial_u+\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+\frac{3}{2}\eth\mathcal{Y}-\frac{1}{2}\bar{\eth}\bar{\mathcal{Y}}-\widetilde{\omega}]\sigma^0-\eth^2 f\,,\\
-\delta_S\dot{\sigma}^0 &= [f\partial_u+\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+2\eth\mathcal{Y}-2\widetilde{\omega}]\dot{\sigma}^0-\frac{1}{2}\eth^2\widetilde{\psi}\,,\\
-\delta_S\Psi^0_i &= [f\partial_u+\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+\frac{5-i}{2}\eth\mathcal{Y}+\frac{1+i}{2}\bar{\eth}\bar{\mathcal{Y}}-3\widetilde{\omega}]\Psi^0_i+(4-i)\eth f\Psi^0_{i+1}\,,
\end{aligned}\tag{6}$$

with $i=1,2,3,4$.

4. SURFACE CHARGE ALGEBRA

In this section, $\widetilde{\omega}=0$ so that $f=T+\frac{1}{2}u\psi$ and we use the notation $s=(\mathcal{Y},\bar{\mathcal{Y}},T)$ for elements of the symmetry algebra, which is given in these terms by $[s_1,s_2]=\widehat{s}$ where

$$\begin{gathered}
\widehat{\mathcal{Y}}=\mathcal{Y}_1\eth\mathcal{Y}_2-(1\leftrightarrow 2),\qquad \widehat{\bar{\mathcal{Y}}}=\bar{\mathcal{Y}}_1\bar{\eth}\bar{\mathcal{Y}}_2-(1\leftrightarrow 2),\\
\widehat{T}=(\mathcal{Y}_1\eth+\bar{\mathcal{Y}}_1\bar{\eth})T_2-\frac{1}{2}\psi_1T_2-(1\leftrightarrow 2)\,.
\end{gathered}\tag{7}$$

The translation of the charges, the non-integrable piece due to the news and the central charges computed in [8] gives here

$$\begin{aligned}
Q_s[\mathcal{X}] &= -\frac{1}{8\pi G}\int d^2\Omega^{\varphi}\Big[\big(f(\Psi^0_2+\sigma^0\dot{\bar{\sigma}}^0)+\mathcal{Y}(\Psi^0_1+\sigma^0\eth\bar{\sigma}^0+\frac{1}{2}\eth(\sigma^0\bar{\sigma}^0))\big)+\text{c.c.}\Big],\\
\Theta_s[\delta\mathcal{X},\mathcal{X}] &= \frac{1}{8\pi G}\int d^2\Omega^{\varphi}\, f\big[\dot{\bar{\sigma}}^0\delta\sigma^0+\text{c.c.}\big]\,,\\
K_{s_1,s_2}[\mathcal{X}] &= \frac{1}{8\pi G}\int d^2\Omega^{\varphi}\Big[\big(\frac{1}{4}f_1\eth f_2\bar{\eth}\bar{R}+\frac{1}{2}\bar{\sigma}^0 f_1\eth^2\psi_2-(1\leftrightarrow 2)\big)+\text{c.c.}\Big]\,.
\end{aligned}\tag{8}$$

We recognize all the ingredients of the surface charges described in [11]. More precisely the angular (super-)momentum that we get is

$$\begin{aligned}
Q_{\mathcal{Y},0,0}=-\frac{1}{8\pi G}\int d^2\Omega^{\varphi}\,\mathcal{Y}\Big[&\Psi^0_1+\sigma^0\eth\bar{\sigma}^0+\frac{1}{2}\eth(\sigma^0\bar{\sigma}^0)\\
&-\frac{u}{2}\eth\big(\Psi^0_2+\bar{\Psi}^0_2+\partial_u(\sigma^0\bar{\sigma}^0)\big)\Big]\,.
\end{aligned}\tag{9}$$

and differs from Q_{η_c} given in equation (4) of [11] by the explicitly u-dependent term of the second line. It thus has a similar structure to Penrose's angular momentum as described in equations (11), (12), and (17a) of [11] in the sense that it also differs by a specific amount of linear supermomentum, but the amount is different and explicitly u-dependent, $Q_{\mathcal{Y},0,0} = Q^{u=0}_{\mathcal{Y},0,0} + \frac{1}{2} u Q_{0,0,\bar{\eth}\mathcal{Y}}$.

The main result derived in [8] states that if one is allowed to integrate by parts, and if one defines the "Dirac bracket" through $\{Q_{s_1}, Q_{s_2}\}^*[\mathcal{X}] = -\delta_{s_2} Q_{s_1}[\mathcal{X}] + \Theta_{s_2}[-\delta_{s_1}\mathcal{X}, \mathcal{X}]$, then the charges define a representation of the $\mathfrak{bms}_4$ algebra, up to a field dependent central extension, $\{Q_{s_1}, Q_{s_2}\}^* = Q_{[s_1,s_2]} + K_{s_1,s_2}$, where K_{s_1,s_2} satisfies the generalized cocycle condition $K_{[s_1,s_2],s_3} - \delta_{s_3} K_{s_1,s_2} + \text{cyclic}(1,2,3) = 0$. This representation theorem can be verified directly in the present context [1].

To the best of our knowledge, except for the previous analysis in the BMS gauge, the above representation result does not exist elsewhere in the literature.

A major issue in these considerations is whether one uses the globally well-defined version of the $\mathfrak{bms}_4$ algebra or a local version which contains the Virasoro algebra and involves an expansion in terms of Laurent series. The formulas presented above generally apply to both cases, except for divergences in the charges that appear in the second case and have to be handled properly. This is discussed in more details in [1].

Acknowledgments. The authors thank Cédric Troessaert for useful discussions. G.B. is Research Director of the Fund for Scientific Research-FNRS (Belgium). This work is supported in part by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P6/11, by IISN-Belgium, by "Communauté française de Belgique - Actions de Recherche Concertées" and by Fondecyt Projects No. 1085322 and No. 1090753.

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