REVISITING EIGHT-MANIFOLD FLUX COMPACTIFICATIONS OF M-THEORY USING GEOMETRIC ALGEBRA TECHNIQUES*

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Motivated by open problems in F-theory, we reconsider warped compactifications of M-theory on 8-manifolds to AdS_3 spaces in the presence of a non-trivial field strength of the M-theory 3-form, studying the most general conditions under which such backgrounds preserve $\mathcal{N}=2$ supersymmetry in three dimensions. In contrast with previous studies, we allow for the most general pair of Majorana generalized Killing pinors on the internal 8-manifold, without imposing any chirality conditions on those pinors. We also show how such pinors can be lifted to the 9-dimensional metric cone over the compactification 8-manifold. We describe the translation of the generalized Killing pinor equations for such backgrounds to a system of differential and algebraic constraints on certain form-valued pinor bilinears and develop techniques through which such equations can be analysed efficiently.

Key words: string theory compactifications, M-theory, supergravity, supersymmetry, differential geometry.

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1. INTRODUCTION

Consider eleven-dimensional supergravity on a background $\tilde{M}$ endowed with a spinnable Lorentzian metric $\tilde{g}$ of 'mostly plus' signature. The fields of the theory are the three-form potential $\tilde{C}$ with four-form field strength $\tilde{G}$, the gravitino $\tilde{\Psi}_M$ and the metric. As in $[1,2]$, we consider compactifications down to an AdS_3 space of cosmological constant $\Lambda=-8\kappa^2$, where κ is a positive real parameter — this includes the Minkowski case as the limit $\kappa\rightarrow 0$. Thus $\tilde{M}=N\times M$, where N is an oriented 3-manifold diffeomorphic with $\mathbb{R}^3$ and carrying the AdS_3 metric while M is an oriented Riemannian eight-manifold with metric denoted by g. The metric $\tilde{g}$ is a warped product with the warp factor Δ. For the field strength $\tilde{G}$, we use the ansatz:

$$\tilde{G}=e^{3\Delta}G \quad \text{with} \quad G=\mathrm{vol}_3\wedge f+F \ ,$$

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where $f = f_m e^m \in \Omega^1(M)$, $F = \frac{1}{4!}F_{mnpq}e^{mnpq} \in \Omega^4(M)$ and vol_3 is the volume form of N.
For the eleven-dimensional supersymmetry generator $\tilde{\eta}$, we use the ansatz:

$$\tilde{\eta} = e^{\frac{\Delta}{2}}\eta \quad \text{with} \quad \eta = \psi \otimes \xi \ ,$$

where ξ is a real pinor of spin $1/2$ on the internal space M and ψ is a real pinor on the AdS_3 space N. As in [1,2] (and in contradistinction with [3]) *we do not require that ξ has definite chirality* *. Mathematically, ξ is a section of the pinor bundle of M, which is a real vector bundle of rank 16 defined on M, carrying a fiberwise representation of the Clifford algebra $\mathrm{Cl}(8,0)$. The corresponding morphism $\gamma : (\wedge T^*M, \diamond) \to (\mathrm{End}(S), \circ)$ of bundles of algebras is an isomorphism. As in [4], we have set $\gamma^m \stackrel{\text{def.}}{=} \gamma(e^m)$ and $\gamma^{(9)} \stackrel{\text{def.}}{=} \gamma^1 \circ \ldots \circ \gamma^8$. Assuming that ψ is a Killing pinor on the AdS_3 space, the supersymmetry condition $\delta_{\tilde{\eta}}\tilde{\Psi}_M = 0$ amounts to the following *constrained generalized Killing (CGK) pinor equations* [5] for ξ:

$$D_m\xi = 0 \ , \ Q\xi = 0 \ , \tag{1}$$

where D_m is a linear connection on S and $Q \in \Gamma(M, \mathrm{End}(S))$ is a globally-defined endomorphism of the vector bundle S, given explicitly by:

$$D_m = \nabla_m^S + A_m \ , \ A_m = \frac{1}{4}f_p\gamma_m{}^p\gamma^{(9)} + \frac{1}{24}F_{mpqr}\gamma^{pqr} + \kappa\gamma_m\gamma^{(9)} \ , \tag{2}$$

$$Q = \frac{1}{2}\gamma^m\partial_m\Delta - \frac{1}{288}F_{mpqr}\gamma^{mpqr} - \frac{1}{6}f_p\gamma^p\gamma^{(9)} - \kappa\gamma^{(9)} \ . \tag{3}$$

The space of solutions to (1) is a finite-dimensional $\mathbb{R}$-linear subspace $\mathcal{K}(D,Q)$ of the space $\Gamma(M,S)$ of smooth globally-defined sections of S. The problem of interest is to find those metrics and fluxes on M for which some fixed amount of supersymmetry is preserved in three dimensions, *i.e.* for which the space $\mathcal{K}(D,Q)$ has some given non-vanishing dimension, which we denote by s. The case $s = 1$ (which corresponds to $\mathcal{N} = 1$ supersymmetry in three dimensions) was studied in [1,2] and reconsidered in [5] by using geometric algebra techniques. The case $s = 2$ (which leads to $\mathcal{N} = 2$ supersymmetry in three dimensions) was studied in [3], but considering only Majorana-*Weyl* solutions of (1), *i.e.* solutions ξ which also satisfy the supplementary constraint $\gamma^{(9)}\xi = \pm\xi$. Here, we consider the case when no such chirality constraint is imposed on the solutions of (1).

2. THE GEOMETRIC ALGEBRA APPROACH

Since the procedure used is explained in detail in [5,6] (being also summarized in [4]) in what follows we give only a brief overview.

*As we shall see in a moment, this seemingly trivial generalization has drastic consequences, leading to a problem which is technically much harder than that solved in the celebrated work of [3].

The geometric product. Following an idea originally due to Chevalley and Riesz [7, 8], we identify $\mathrm{Cl}(T^*M)$ with the exterior bundle $\wedge T^*M$, thus realizing the Clifford product as the *geometric product*, which is the fiberwise associative, unital and bilinear binary composition $\diamond : \wedge T^*M \times_M \wedge T^*M \to \wedge T^*M$ given on sections by the expansion:

$$\omega \diamond \eta = \sum_{k=0}^{\left[\frac{d}{2}\right]} \frac{(-1)^k}{(2k)!} \omega \wedge_{2k} \eta + \sum_{k=0}^{\left[\frac{d-1}{2}\right]} \frac{(-1)^{k+1}}{(2k+1)!} \pi(\omega) \wedge_{2k+1} \eta \ , \tag{4}$$

where π is the *grading automorphism* defined through:

$$\pi(\omega) \stackrel{\text{def.}}{=} \sum_{k=0}^{d} (-1)^k \omega^{(k)} \ , \ \forall \omega = \sum_{k=0}^{d} \omega^{(k)} \in \Omega(M) \ , \ \text{where } \omega^{(k)} \in \Omega^k(M) \ . \tag{5}$$

The Clifford bundle is thus identified with the bundle of algebras $(\wedge T^*M, \diamond)$, which is known as the *Kähler-Atiyah bundle* of (M,g). The binary $\mathcal{C}^\infty(M,\mathbb{R})$-bilinear operations $\wedge_k$ which appear in the expansion above are the (action on sections of the) *contracted wedge products*, defined iteratively through:

$$\omega \wedge_0 \eta \stackrel{\text{def.}}{=} \omega \wedge \eta \ , \ \omega \wedge_{k+1} \eta \stackrel{\text{def.}}{=} g^{ab}(e_a \lrcorner \omega) \wedge_k (e_b \lrcorner \eta) = g_{ab}(\iota_{e^a}\omega) \wedge_k (\iota_{e^b}\eta) \ ,$$

where ι denotes the so-called *interior product* (see [5]). We will mostly use, instead, the so-called *generalized products* $\triangle_k$, which are defined by rescaling the contracted wedge products:

$$\triangle_k \stackrel{\text{def.}}{=} \frac{1}{k!} \wedge_k \ . \tag{6}$$

The Kähler-Atiyah bundle also admits an involutive anti-automorphism τ (known as *the main anti-automorphism* or as *reversion*), which is given by:

$$\tau(\omega) \stackrel{\text{def.}}{=} (-1)^{\frac{k(k-1)}{2}} \omega \ , \ \forall \omega \in \Omega^k(M) \ . \tag{7}$$

3. LIFTING THE CGK EQUATIONS TO THE METRIC CONE OVER THE COMPACTIFICATION SPACE

The CGK pinor equations can be lifted to the metric cone $(\hat{M}, g_{\text{cone}})$ over M as explained in [6] and outlined in [4], to which we refer the reader for the notations used below. As in loc. cit., we work with an admissible [9, 10] bilinear pairing $\mathscr{B}$ on the pin bundle S of M which is symmetric and with respect to which all γ_m are self-adjoint. We let $\hat{\mathscr{B}}$ denote the pull-back of $\mathscr{B}$ to the pin bundle $\hat{S}$ of the cone. We work with that pinor representation γ_{cone} on the cone which has signature $+1$.

The basic form-valued bilinears on the cone. Recall that s denotes the dimension of the space of solutions to the CGK pinor equations. Choosing a basis $(\hat{\xi}_i)_{i=1\ldots s}$ of such solutions on the cone, we set (see [5,6]):

$$\check{E}^{\rm cone}_{ij} \stackrel{\rm def.}{=} \check{E}^{\rm cone}_{\hat{\xi}_i,\hat{\xi}_j} = \frac{1}{2^{\left[\frac{d+1}{2}\right]}} \check{\boldsymbol{E}}^{\rm cone}_{ij} \in \Omega^{+,{\rm cone}}(\hat{M}) ~~,$$

and:

$$\check{\boldsymbol{E}}^{\rm cone}_{ij} = \sum_{k=0}^{d} \check{\boldsymbol{E}}^{(k),{\rm cone}}_{ij} =_U \sum_{k=0}^{d} \frac{1}{k!} \check{\boldsymbol{E}}^{(k),{\rm cone}}_{a_1\ldots a_k}(\hat{\xi}_i,\hat{\xi}_j) e^{a_1\ldots a_k}_{{\rm cone},+} ~~,$$
$$\check{\boldsymbol{E}}^{(k),{\rm cone}}_{a_1\ldots a_k}(\hat{\xi}_i,\hat{\xi}_j) \stackrel{\rm def.}{=} \hat{\mathscr{B}}(\gamma_{a_k\ldots a_1}\hat{\xi}_i,\hat{\xi}_j) = \hat{\mathscr{B}}(\hat{\xi}_i,\gamma_{a_1\ldots a_k}\hat{\xi}_j) ~~.$$

The symbol $\Omega^{+,{\rm cone}}(\hat{M})$ denotes the space of twisted self-dual forms on the cone , which is a subalgebra of the Kähler-Atiyah algebra of $(\hat{M}, g_{\rm cone})$ (see [5,6] and [4]).

CGK equations on the cone. As explained in [6], it is computationally convenient to replace the algebra $(\Omega^{+,{\rm cone}}(\hat{M}), \diamond^{\rm cone})$ of twisted selfdual forms of the cone (which is the effective domain of definition of $\gamma_{\rm cone}$) with a certain isomorphic model $(\Omega^{<}(\hat{M}), \blacklozenge^{\rm cone}_{+})$ whose precise definition is given in loc. cit. When $s=2$ ($\mathcal{N}=2$ supersymmetry in three dimensions), the CGK pinor equations on $\hat{M}$ admit *two* linearly independent solutions ξ_1 and ξ_2. We have $\Omega^{<}(\hat{M}) = \oplus_{k=0}^{4}\Omega^k(\hat{M})$, so we are interested in pinor bilinears $\check{\boldsymbol{E}}^{(k)}_{\hat{\xi}_1,\hat{\xi}_2}$ with $k=0\ldots 4$ for two independent solutions $\hat{\xi}_1,\hat{\xi}_2 \in \Gamma(\hat{M},\hat{S})$ of the CGK pinor equations lifted to the cone (which are equivalent with the original CGK pinor equations on M):

$$\hat{D}_a\hat{\xi} = \hat{Q}\hat{\xi} = 0 ~~, \tag{8}$$

where the definition of $\hat{D}_a = \nabla^{\hat{S},{\rm cone}}_a + A^{\rm cone}_a$, where $a = 1\ldots 9$, and $\hat{Q}$ can be found in [6] and [4].

To extract the translation of these equations into constraints on differential forms, we implemented certain procedures within the package `Ricci` [11] for tensor computations in `Mathematica`®. The dequantizations:

$$\check{A}^{\rm cone}_a = \gamma^{-1}_{\rm cone}(A^{\rm cone}_a) \in \Omega^{<}(\hat{M}) ~~,$$
$$\check{Q}^{\rm cone} = \gamma^{-1}_{\rm cone}(\hat{Q}) \in \Omega^{<}(\hat{M}) ~~,$$

of $A^{\rm cone}$ and $\hat{Q}$ are given by $\check{A}^{\rm cone}_9 = 0$ and:

$$\check{A}^{\rm cone}_m = \frac{1}{4}\iota^{\rm cone}_{e^{\rm cone}_m} F_{\rm cone} + \frac{1}{4}(e^{\rm cone}_m)\wedge f_{\rm cone}\wedge\theta ~~\in \Omega^{<}(\hat{M}) ~~,~~ \forall m = 1\ldots 8 ~~,$$
$$\check{Q}^{\rm cone} = \frac{1}{2} r{\rm d}\Delta - \frac{1}{6} f_{\rm cone}\wedge\theta - \frac{1}{12}F_{\rm cone} - \kappa\theta ~~\in \Omega^{<}(\hat{M}) ~~,$$

while the $\hat{\mathscr{B}}$-transpose of $\hat{Q}$ dequantizes to the cone reversion of $\check{Q}^{\rm cone}$:

$$\hat{\tau}(\check{Q}^{\rm cone}) = \frac{1}{2} r{\rm d}\Delta + \frac{1}{6} f_{\rm cone}\wedge\theta - \frac{1}{12}F_{\rm cone} - \kappa\theta ~~.$$

The forms $f_{\rm cone}$ and $F_{\rm cone}$ above are the *cone lifts* (see [6]) of f and F respectively, while Δ stands for the pullback $\Pi^*(\Delta)=\Delta\circ\Pi$ of the warp factor through the natural projection $\Pi:\hat{M}\rightarrow M$, even though the notation does not show this explicitly. The one-form θ is defined through:

$$\theta \stackrel{\rm def.}{=} \mathrm{d}r\in\Omega^1(\hat{M}) ~,$$

where r is the radial coordinate along the metric cone $\hat{M}$.

A basis for the space spanned by the forms $\frac{1}{k!}\hat{\mathscr{B}}(\hat{\xi}_1,\hat{\gamma}_{a_1\ldots a_k}\hat{\xi}_2)e_{\rm cone}^{a_1\ldots a_k}\in\Omega^<(\hat{M})$ (of rank $k\leq 4$) which can be constructed on the cone from $\hat{\xi}_1$ and $\hat{\xi}_2$ is given by (where we have raised all indices using the cone metric to avoid notational clutter) :

$$\begin{gathered}V_1^a=\hat{\mathscr{B}}(\hat{\xi}_1,\hat{\gamma}^a\hat{\xi}_1) ~,~ V_2^a=\hat{\mathscr{B}}(\hat{\xi}_2,\hat{\gamma}^a\hat{\xi}_2) ~,~ V_3^a=\hat{\mathscr{B}}(\hat{\xi}_1,\hat{\gamma}^a\hat{\xi}_2)~,\\ K^{ab}=\hat{\mathscr{B}}(\hat{\xi}_1,\hat{\gamma}^{ab}\hat{\xi}_2) ~,~ \Psi^{abc}=\hat{\mathscr{B}}(\hat{\xi}_1,\hat{\gamma}^{abc}\hat{\xi}_2)~,\\ \Phi_1^{abce}=\hat{\mathscr{B}}(\hat{\xi}_1,\hat{\gamma}^{abce}\hat{\xi}_1) ~,~ \Phi_2^{abce}=\hat{\mathscr{B}}(\hat{\xi}_2,\hat{\gamma}^{abce}\hat{\xi}_2) ~,~ \Phi_3^{abce}=\hat{\mathscr{B}}(\hat{\xi}_1,\hat{\gamma}^{abce}\hat{\xi}_2)~.\end{gathered}$$

To arrive at these bilinears we used the identity:

$$\mathscr{B}(\hat{\xi}_i,\hat{\gamma}^{a_1\ldots a_k}\hat{\xi}_j)=(-1)^{\frac{k(k-1)}{2}}\mathscr{B}(\hat{\xi}_j,\hat{\gamma}^{a_1\ldots a_k}\hat{\xi}_i) ~,$$

which follows from the fact that $\gamma_a^t=\gamma_a$ and implies that certain of the forms $\check{\boldsymbol{E}}^{(k),{\rm cone}}_{\hat{\xi}_i,\hat{\xi}_j}$ vanish identically.

Here and below, we have taken $\hat{\xi}_1$ and $\hat{\xi}_2$ to form a $\hat{\mathscr{B}}$-orthonormal basis of the $\mathbb{R}$-vector space $\mathcal{K}(\hat{D},\hat{Q})$ of solutions to the CGK equations on the cone:

$$\hat{\mathscr{B}}(\hat{\xi}_i,\hat{\xi}_j)=\delta_{ij} ~,~ \forall i,j=1,2 ~,$$

and we noticed that the pairing $\hat{\mathscr{B}}=\Pi^*(\mathscr{B})$ on $\hat{S}=\Pi^*(S)$ has the same symmetry and type properties as the admissible pairing $\mathscr{B}$ on S — namely, both $\mathscr{B}$ and $\hat{\mathscr{B}}$ are symmetric and nondegenerate (and they can be taken to be positive-definite) and make the eight- (respectively nine-) dimensional gamma 'matrices' γ^m and $\hat{\gamma}^a$ into self-adjoint operators. From now on — in order to avoid notational clutter — we shall suppress the superscripts and subscripts "cone". In particular, we shall denote the cone lifts $F_{\rm cone}$ and $f_{\rm cone}$ simply by F and f. With these notations and conventions, the truncated model $(\check{\mathcal{K}}^{<,{\rm cone}}(\hat{D},\hat{Q}),\blacklozenge_+^{\rm cone})$ of the flat Fierz algebra on the cone admits the basis:

$$\begin{gathered}\check{E}_{12}^<=\frac{1}{32}(V_3+K+\Psi+\Phi_3) ~,~ \check{E}_{21}^<=\frac{1}{32}(V_3-K-\Psi+\Phi_3) ~,\\ \check{E}_{11}^<=\frac{1}{32}(1+V_1+\Phi_1) ~,~ \check{E}_{22}^<=\frac{1}{32}(1+V_2+\Phi_2)\end{gathered}$$

and can be generated by two elements (see Subsection 5.10 of [5]), which we choose to be:

$$\check{E}^{<}_{12} = \frac{1}{32}(V_3 + K + \Psi + \Phi_3)\ \ ,\ \ \check{E}^{<}_{21} = \hat{\tau}(\check{E}^{<}_{12}) = \frac{1}{32}(V_3 - K - \Psi + \Phi_3)\ .$$

The overall coefficient $\frac{1}{32}$ comes from the prefactor $\frac{1}{2^{[\frac{d+1}{2}]}}$ when $d = 9$. As explained in [5], the Fierz relations for the inhomogeneous forms $\check{E}_{ij}$ (which in this case are twisted selfdual $\check{E}_{ij} = \check{E}^{<}_{ij} + \tilde{*}\check{E}^{<}_{ij}$, with $\check{E}^{<}_{ij} \in \Omega^{<}(\hat{M})$) are given by:

$$\check{E}_{ij} \diamond \check{E}_{kl} = \delta_{jk}\check{E}_{il}\ \ ,\ \ \forall i,j,k,l = 1,2\ \ . \tag{9}$$

Since the associative and unital multiplication $\blacklozenge_+$ is defined through:

$$\omega \blacklozenge_+ \eta = 2P_<(P_+(\omega) \diamond P_+(\eta))\ \in \Omega^{<}(M)\ ,$$

we find the following relations when taking $\omega = \check{E}^{<}_{ij}$ and $\eta = \check{E}^{<}_{kl}$ (with indices i,j,k,l fixed):

$$\begin{aligned}\check{E}^{<}_{ij} \blacklozenge_+ \check{E}^{<}_{kl} &= 2P_<(P_+(\check{E}^{<}_{ij}) \diamond P_+(\check{E}^{<}_{kl})) = 2P_<(\frac{1}{2}(\check{E}^{<}_{ij} + \tilde{*}\check{E}^{<}_{ij}) \diamond (\frac{1}{2}(\check{E}^{<}_{kl} + \tilde{*}\check{E}^{<}_{kl}))\\ &= \frac{1}{2}P_<(\check{E}_{ij} \diamond \check{E}_{kl}) = \frac{1}{2}P_<(\delta_{jk}\check{E}_{il}) = \frac{1}{2}\delta_{jk}\check{E}^{<}_{il}\ \ ,\end{aligned}$$

where we made use of (9). We thus obtain the Fierz relations for the truncated model in our case:

$$\check{E}^{<}_{ij} \blacklozenge_+ \check{E}^{<}_{kl} = \frac{1}{2}\delta_{jk}\check{E}^{<}_{il}\ \ ,\ \ \forall i,j,k,l = 1,2\ \ . \tag{10}$$

In order to avoid notational clutter, we shall henceforth use $\blacklozenge$ instead of $\blacklozenge_+$.
On the other hand, the algebraic constraints in (8) amount [5] to the following two relations for $\check{E}^{<}_{12}$:

$$\check{Q} \blacklozenge \check{E}^{<}_{12} \mp \check{E}^{<}_{12} \blacklozenge \hat{\tau}(\check{Q}) = 0\ \ , \tag{11}$$

while the differential constraints in (8) give $\check{D}^{\mathrm{ad}}_a \check{E}^{<}_{12} = 0 \Longleftrightarrow \check{D}^{\mathrm{ad}}_a \check{E}^{<}_{21} = 0$, which in turn imply:

$$\mathrm{d}\check{E}^{<}_{12} = e^a \wedge \nabla_a \check{E}^{<}_{12} = -e^a \wedge [\check{A}_a, \check{E}^{<}_{12}]_{-,\blacklozenge}\ \ . \tag{12}$$

As explained in Subsection 5.10 of [5], it is enough to consider relations (11) and (12) for the generators $\check{E}^{<}_{12}$ and $\check{E}^{<}_{21} = \hat{\tau}(\check{E}^{<}_{12})$, since the corresponding constraints for $\check{E}^{<}_{11} = 2\check{E}^{<}_{12} \blacklozenge \check{E}^{<}_{21}$ and $\check{E}^{<}_{22} = 2\check{E}^{<}_{21} \blacklozenge \check{E}^{<}_{12}$ follow from those upon reversion.

Algebraic constraints. Using the procedures which we have implemented and the package `Ricci` [11] for tensor computations in `Mathematica`®, we find that the first equation (that with the minus sign) in (11) amounts to the following system when separated on ranks:

$$
\begin{aligned}
&\iota_{f\wedge\theta}K=0,\\
&r\iota_{\mathrm{d}\Delta}K+\frac{1}{3}\iota_{f\wedge\theta}\Psi-\frac{1}{6}\iota_{\Psi}F-2\kappa\,\iota_{\theta}K=0,\\
&\frac{1}{3}\iota_{f\wedge\theta}\Phi_3-\frac{1}{6}F\bigtriangleup_3\Phi_3+r(\mathrm{d}\Delta)\wedge V_3+2\kappa\,V_3\wedge\theta=0,\\
&r\iota_{\mathrm{d}\Delta}\Phi_3-\frac{1}{3}V_3\wedge f\wedge\theta+\frac{1}{6}\iota_{V_3}F-\frac{1}{6}*(F\bigtriangleup_1\Phi_3)+\frac{1}{3}*(f\wedge\theta\wedge\Phi_3)-2\kappa\,\iota_\theta\Phi_3=0,\\
&r(\mathrm{d}\Delta)\wedge\Psi-\frac{1}{3}f\wedge\theta\wedge K-\frac{1}{6}K\bigtriangleup_1F-\frac{1}{3}*(f\wedge\theta\wedge\Psi)+\frac{1}{6}*(\Psi\bigtriangleup_1F)+2\kappa\,\Psi\wedge\theta=0,
\end{aligned}
$$

while the second equation (that with the plus sign) in (11) amounts to:

$$
\begin{aligned}
&-\frac{1}{6}\iota_F\Phi_3+r\iota_{\mathrm{d}\Delta}V_3-2\kappa\,\iota_\theta V_3=0,\\
&\frac{1}{3}\iota_{V_3}(f\wedge\theta)-\frac{1}{6}*(F\wedge\Phi_3)=0,\\
&r\iota_{\mathrm{d}\Delta}\Psi+\frac{1}{3}(f\wedge\theta)\bigtriangleup_1K+\frac{1}{6}\iota_KF+\frac{1}{6}*(F\wedge\Psi)-2\kappa\,\iota_\theta\Psi=0,\\
&\frac{1}{3}(f\wedge\theta)\bigtriangleup_1\Psi+\frac{1}{6}\Psi\bigtriangleup_2F+\frac{1}{6}*(K\wedge F)+r(\mathrm{d}\Delta)\wedge K-2\kappa\,K\wedge\theta=0,\\
&\frac{1}{3}(f\wedge\theta)\bigtriangleup_1\Phi_3+\frac{1}{6}F\bigtriangleup_2\Phi_3-\frac{1}{6}*(F\wedge V_3)+*(r(\mathrm{d}\Delta)\wedge\Phi_3)-2\kappa*(\Phi_3\wedge\theta)=0.
\end{aligned}
$$

Differential constraints. Using the same `Mathematica`® package, we find that the differential constraints (12), when separated on ranks, amount to:

$$
\begin{aligned}
\mathrm{d}V_3&=\frac{1}{2}\Phi_3\bigtriangleup_3F+\iota_{f\wedge\theta}\Phi_3,\\
\mathrm{d}K&=(f\wedge\theta)\bigtriangleup_1\Psi+\Psi\bigtriangleup_2F,\\
\mathrm{d}\Psi&=\frac{3}{2}F\bigtriangleup_1K-\frac{1}{2}F\bigtriangleup_3*\Psi+2*(f\wedge\theta\wedge\Psi)-f\wedge\theta\wedge K,\\
\mathrm{d}\Phi_3&=-2F\wedge V_3+\frac{1}{2}e^m\wedge*((\iota_{e^m}F)\bigtriangleup_1\Phi_3)-\frac{1}{2}e^m\wedge*(((e_m)\wedge f\wedge\theta)\bigtriangleup_1\Phi_3).
\end{aligned}
$$

According to our notational conventions, e^m in the equations above stands for the cone lifts e^m_{cone} etc. Furthermore, $*\stackrel{\mathrm{def.}}{=}*_{\mathrm{cone}}$ is the ordinary Hodge operator of (M,g_{cone}) and ι stands for ι^{cone}. The generalized products $\bigtriangleup_p\stackrel{\mathrm{def.}}{=}\bigtriangleup_p^{\mathrm{cone}}$ are constructed with the cone metric on $\hat{M}$.

Fierz relations. Let us consider the Fierz identities (10) for the basis elements $\check{E}^{<}_{ij}$ $(i,j=1,2)$ of the truncated model of the flat Fierz algebra on the cone:

$$\begin{aligned}
&(\mathrm{F1}) : \check{E}^{<}_{12}\blacklozenge\check{E}^{<}_{12} = 0\,, && (\mathrm{F2}) : \check{E}^{<}_{12}\blacklozenge\check{E}^{<}_{21} = \frac{1}{2}\check{E}^{<}_{11}\,,\\
&(\mathrm{F3}) : \check{E}^{<}_{12}\blacklozenge\check{E}^{<}_{22} = \frac{1}{2}\check{E}^{<}_{12}\,, && (\mathrm{F4}) : \check{E}^{<}_{12}\blacklozenge\check{E}^{<}_{11} = 0\,,\\
&(\mathrm{F5}) : \check{E}^{<}_{11}\blacklozenge\check{E}^{<}_{11} = \frac{1}{2}\check{E}^{<}_{11}\,, && (\mathrm{F6}) : \check{E}^{<}_{11}\blacklozenge\check{E}^{<}_{12} = \frac{1}{2}\check{E}^{<}_{12}\,,\\
&(\mathrm{F7}) : \check{E}^{<}_{11}\blacklozenge\check{E}^{<}_{21} = 0\,, && (\mathrm{F8}) : \check{E}^{<}_{11}\blacklozenge\check{E}^{<}_{22} = 0\,,\\
&(\mathrm{F9}) : \check{E}^{<}_{21}\blacklozenge\check{E}^{<}_{12} = \frac{1}{2}\check{E}^{<}_{22}\,, && (\mathrm{F10}) : \check{E}^{<}_{21}\blacklozenge\check{E}^{<}_{11} = \frac{1}{2}\check{E}^{<}_{21}\,,\\
&(\mathrm{F11}) : \check{E}^{<}_{21}\blacklozenge\check{E}^{<}_{21} = 0\,, && (\mathrm{F12}) : \check{E}^{<}_{21}\blacklozenge\check{E}^{<}_{22} = 0\,,\\
&(\mathrm{F13}) : \check{E}^{<}_{12}\blacklozenge\check{E}^{<}_{11} = 0\,, && (\mathrm{F14}) : \check{E}^{<}_{22}\blacklozenge\check{E}^{<}_{12} = 0\,,\\
&(\mathrm{F15}) : \check{E}^{<}_{22}\blacklozenge\check{E}^{<}_{21} = \frac{1}{2}\check{E}^{<}_{21}\,, && (\mathrm{F16}) : \check{E}^{<}_{22}\blacklozenge\check{E}^{<}_{22} = \frac{1}{2}\check{E}^{<}_{22}\,.
\end{aligned}$$

We note that some of these conditions are equivalent through reversion (namely (F1)⇔(F11), (F3)⇔(F15), (F4)⇔(F7), (F6)⇔(F10), (F8)⇔(F13) and (F12)⇔(F14)), while relations (F2), (F5), (F9), (F16) are selfdual under reversion. Expanding for example equation (F1):

$$(V_3+K+\Psi+\Phi_3)\blacklozenge(V_3+K+\Psi+\Phi_3)=0$$

gives the following conditions when separating rank components:

$$\begin{aligned}
-||K||^2+||\Phi_3||^2-||\Psi||^2+||V_3||^2=0\,,\\
-2\iota_K\Psi+*(\Phi_3\wedge\Phi_3)=0\,,\\
\iota_{V_3}\Psi-*(\Phi_3\wedge\Psi)-\iota_K\Phi_3=0\,,\\
K\wedge V_3-*(K\wedge\Phi_3)-\Psi\,\triangle_2\,\Phi_3=0\,,\\
\Psi\,\triangle_1\,\Psi-\Phi_3\,\triangle_2\,\Phi_3+2*(K\wedge\Psi)+2*(V_3\wedge\Phi_3)+K\wedge K=0\,.
\end{aligned}$$

Similarly, all independent Fierz relations given above can be expanded into rank components and studied by elimination. Such a detailed analysis and various implications are taken up in a forthcoming publication.

4. CONCLUSIONS

The geometric algebra approach developed in [5, 6] provides a synthetic and computationally efficient method for translating generalized Killing (s)pinor equations into conditions on differential forms constructed as (s)pinor bilinears. This approach is highly amenable to implementation in various symbolic computation systems specialized in tensor algebra — and we touched upon two such implementations which we have carried out using `Ricci` [11]. It affords a more unified and systematic description of flux compactifications and generally of supergravity and string compactifications. We illustrated our techniques with the most general flux compactifications of M-theory preserving $\mathcal{N} = 2$ supersymmetry in three dimensions, a class of compactifications which had not been studied in full generality before — showing on the one hand how to obtain a complete description of the differential and algebraic constraints on pinor bilinears and on the other hand how to write all Fierz identities between the form bilinears. A detailed analysis of the resulting equations, geometry and physics is the subject of ongoing work. The methods introduced in [5, 6] have much wider applicability, leading to promising new directions in the study of supergravity and string theory backgrounds and actions.

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REFERENCES

1. D. Martelli, J. Sparks, Phys. Rev. **D 68**, 085014 (2003).
2. D. Tsimpis, JHEP **0604**, 027 (2006).
3. K. Becker, M. Becker, Nucl. Phys. **B 477**, 155–167 (1996).
4. C. I. Lazaroiu, "*Geometric algebra and M–theory compactifications*", Proceedings of the 8-th edition of QFTHS, 19-22 Sept. 2012, Craiova, Romania.
5. C. I. Lazaroiu, E.M. Babalic, I.A. Coman, "*Geometric algebra techniques in flux compactifications (I)*", `arXiv:1212.6766[hep-th]` (2012).
6. C.I. Lazaroiu, E.M. Babalic, "*Geometric algebra techniques in flux compactifications (II)*", `arXiv:1212.6918[hep-th]` (2012).
7. C. Chevalley, "*The Algebraic Theory of spinors and Clifford Algebras*", Collected works, vol. 2, eds. P. Cartier and C. Chevalley (Springer, 1996).
8. M. Riesz, "*Clifford Numbers and spinors: with Riesz's Private Lectures to E. Folke Bolinder and a Historical Review by Pertti Lounesto*" (Kluwer, 1993).
9. D.V. Alekseevsky, V. Cortés, Commun. Math. Phys. **183**, 477–510 (1997).
10. D.V. Alekseevsky, V. Cortés, C. Devchand, A.V. Proyen, Commun. Math. Phys. **253**, 385–422 (2005).
11. J.M. Lee, D. Lear, J. Roth, J. Coskey, L. Nave, "*Ricci — A Mathematica package for doing tensor calculations in differential geometry* ", `http://www.math.washington.edu/~lee/Ricci/`.