ON A NEW LATTICE VOLTERRA SYSTEM*

NICOLETA CORINA BABALIC

University of Craiova, Faculty of Exact Sciences, Department of Physics
13 Al. I. Cuza Str., 200585 Craiova, Romania
E-mail: b_coryna@yahoo.com

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We present an integrable discretization of a general differential-difference bi-component Volterra system using the Hirota bilinear formalism.

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1. INTRODUCTION

Constructing integrable discretizations to a given partial differential or differential-difference equation is one of the main difficulties in the topic of integrable systems. Because the lattice equations have in general complicated forms it is not always easy to apply integrability criteria like complexity growth [1] or singularity confinement [2].

One of the most powerful methods that can be used in finding integrable discretizations is the Hirota bilinear method. It consists in three steps. Starting from a differential/differential-difference bilinear form, in a first step, one has to replace the differential Hirota operators with discrete ones preserving gauge invariance. In the second step, one has to prove the integrability constructing the multisoliton solution [3]. In a third step, which is rather complicated, the nonlinear form is recovered. For that it is possible to introduce an auxiliary function as Hirota has shown in [4]. Another possible approach in order to get soliton solutions is given by the symmetry method [5], but this approach will be not tackled in our paper.

In this paper we are going to give one integrable discretization of a general two component bidirectional Volterra system (the dispersion relation has two branches). This system has been formulated for the first time by Hirota and Satsuma [6] but with constraints on parameters and various solutions (rational, white and dark solitonic) have been obtained [7], [8], [9]. Although the system is rather old, the fully

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discretization has not been studied so far. The lattice Volterra system can be formulated as a coupled differential-difference focusing or defocusing mKdV system with non-zero boundary conditions. In this paper we present one fully discrete bilinear form, show the structure of the three soliton solution and then recover the nonlinear form.

2. DISCRETIZATION OF GENERAL VOLTERRA SYSTEM

The general differential-difference bicomponent Volterra system is:

$$\dot{Q}_n = (a + bQ_n + cQ_n^2)(R_{n+1} - R_{n-1}) \tag{1}$$
$$\dot{R}_n = (a + bR_n + cR_n^2)(Q_{n+1} - Q_{n-1})$$

In [7] we have shown that considering the following scalings and translations,

$$u_n = \frac{2c}{\sqrt{4ac - b^2}}(Q_n + \frac{b}{2c})$$
$$v_n = \frac{2c}{\sqrt{4ac - b^2}}(R_n + \frac{b}{2c})$$
$$t \to t\left(\frac{4ac - b^2}{4c}\right)$$

and imposing

$$\alpha_0 = \frac{b}{\sqrt{4ac - b^2}}$$

we can cast the system in the following form:

$$\dot{u}_n = (1 + u_n^2)(v_{n+1} - v_{n-1}) \tag{2}$$
$$\dot{v}_n = (1 + v_n^2)(u_{n+1} - u_{n-1}),$$

where $u_n,\, v_n \to \alpha_0$ as $n \to \pm\infty$.

In case of $4ac - b^2 \leqslant 0$ the system is changed in a defocusing form, but from the point of view of integrability this is not crucial.

$$\dot{u}_n = (1 - u_n^2)(v_{n+1} - v_{n-1}) \tag{3}$$
$$\dot{v}_n = (1 - v_n^2)(u_{n+1} - u_{n-1})$$

In order to construct the Hirota bilinear form of system (2) we use the nonlinear substitutions:

$$u_n = \alpha - \frac{i}{2}\frac{\partial}{\partial t}\ln\frac{f_n}{g_n}$$
$$v_n = \alpha - \frac{i}{2}\frac{\partial}{\partial t}\ln\frac{f_n'}{g_n'}$$

where f_n/g_n and $f_n'/g_n' \to const$ while $n \to \infty$.

The bilinear Hirota form of the system under consideration is:

$$\dot{f}_n g_n - f_n \dot{g}_n = (1+\alpha^2)(f'_{n+1}g'_{n-1} - g'_{n+1}f'_{n-1}) \tag{4}$$
$$\dot{f}'_n g'_n - f'_n \dot{g}'_n = (1+\alpha^2)(f_{n+1}g_{n-1} - g_{n+1}f_{n-1}) \tag{5}$$
$$(1+i\alpha)f'_{n+1}g'_{n-1} + (1-i\alpha)f'_{n-1}g'_{n+1} = 2f_n g_n \tag{6}$$
$$(1+i\alpha)f_{n+1}g_{n-1} + (1-i\alpha)f_{n-1}g_{n+1} = 2f'_n g'_n \tag{7}$$

Using the Hirota bilinear operators D_n and D_t, the system can be cast in the following form:

$$D_t f_n \bullet g_n = 2(1+\alpha_0^2)\sinh D_n f'_n \bullet g'_n$$

$$D_t f'_n \bullet g'_n = 2(1+\alpha_0^2)\sinh D_n f_n \bullet g_n$$

$$(\cosh D_n + i\alpha_0 \sinh D_n) f'_n \bullet g'_n = f_n g_n$$

$$(\cosh D_n + i\alpha_0 \sinh D_n) f_n \bullet g_n = f'_n g'_n$$

where

$$D_x^m a \bullet b = (\partial_x - \partial_y)^m a(x) b(y)|_{x=y}$$

In order to construct an integrable discretization, we replace time derivatives in (4) and (5) with finite differences (and replace t with δm)

$$\dot{f}_n \to \frac{1}{\delta}(f(n,(m+1)\delta) - f(n,\delta m))$$

and impose the bilinear gauge invariance, in other words the invariance of the resulting bilinear equation with respect to multiplication with $\exp(\mu n + \nu m)$ for any μ,ν.

The new lattice Volterra system consists of the fully discrete gauge invariant bilinear equations:

$$\tilde{f}_n g_n - f_n \tilde{g}_n = \delta(1+\alpha^2)(\tilde{f}'_{n+1}g'_{n-1} - \tilde{g}'_{n+1}f'_{n-1}) \tag{8}$$
$$\tilde{f}'_n g'_n - f'_n \tilde{g}'_n = \delta(1+\alpha^2)(\tilde{f}_{n+1}g_{n-1} - \tilde{g}_{n+1}f_{n-1}) \tag{9}$$
$$(1+i\alpha)\tilde{f}'_{n+1}g'_{n-1} + (1-i\alpha)f'_{n-1}\tilde{g}'_{n+1} = \tilde{f}_n g_n + f_n \tilde{g}_n \tag{10}$$
$$(1+i\alpha)\tilde{f}_{n+1}g_{n-1} + (1-i\alpha)f_{n-1}\tilde{g}_{n+1} = \tilde{f}'_n g'_n + f'_n \tilde{g}'_n, \tag{11}$$

where $\tilde{f}_n = f(n,(m+1)\delta)$, $\tilde{g}_n = g(n,(m+1)\delta)$, etc.

This can be written with Hirota as:

$$\sinh(\frac{\delta}{2}D_m)f_n \bullet g_n = \delta(1+\alpha^2)\sinh(\frac{\delta}{2}D_m + D_n)f'_n \bullet g'_n \tag{12}$$

$$\sinh(\frac{\delta}{2}D_m)f'_n \bullet g'_n = \delta(1+\alpha^2)\sinh(\frac{\delta}{2}D_m + D_n)f_n \bullet g_n \tag{13}$$

$$\left(\cosh(\frac{\delta}{2}D_m + D_n) + i\alpha\sinh(\frac{\delta}{2}D_m + D_n)\right) f_n \bullet g_n = \cosh(\frac{\delta}{2}D_m)f'_n \bullet g'_n \tag{14}$$

$$\left(\cosh(\frac{\delta}{2}D_m + D_n) + i\alpha\sinh(\frac{\delta}{2}D_m + D_n)\right) f'_n \bullet g'_n = \cosh(\frac{\delta}{2}D_m)f_n \bullet g_n. \tag{15}$$

In order to check the integrability one has to compute at least the three soliton solution of the above system. It turns out that indeed the three soliton solution exists and is given by:

$$f_n = 1 + \sum_{i=1}^{3} a_i p_i^n q_i^{m\delta} + \sum_{i<j}^{3} a_i a_j A_{ij}(p_i p_j)^n (q_i q_j)^{m\delta} + A_{12}A_{13}A_{23}a_1 a_2 a_3 (p_1 p_2 p_3)^n (q_1 q_2 q_3)^{m\delta},$$

$$f'_n = 1 + \sum_{i=1}^{3} a'_i p_i^n q_i^{m\delta} + \sum_{i<j}^{3} a'_i a'_j A_{ij}(p_i p_j)^n (q_i q_j)^{m\delta} + A_{12}A_{13}A_{23}a'_1 a'_2 a'_3 (p_1 p_2 p_3)^n (q_1 q_2 q_3)^{m\delta},$$

$$g_n = 1 + \sum_{i=1}^{3} b_i p_i^n q_i^{m\delta} + \sum_{i<j}^{3} b_i b_j A_{ij}(p_i p_j)^n (q_i q_j)^{m\delta} + A_{12}A_{13}A_{23}b_1 b_2 b_3 (p_1 p_2 p_3)^n (q_1 q_2 q_3)^{m\delta},$$

$$g'_n = 1 + \sum_{i=1}^{3} b'_i p_i^n q_i^{m\delta} + \sum_{i<j}^{3} b'_i b'_j A_{ij}(p_i p_j)^n (q_i q_j)^{m\delta} + A_{12}A_{13}A_{23}b'_1 b'_2 b'_3 (p_1 p_2 p_3)^n (q_1 q_2 q_3)^{m\delta},$$

where $\epsilon_j = \pm 1$ represents the propagation direction of the soliton j, $k_j = \log p_j$ the wave number and $\omega_j = \log q_j$ the angular frequency, $i,j = \overline{1,3}$.

The dispersion relation, the phase factors and the interaction term have the following forms:

$$q_i = \left(\frac{p_i + \delta\epsilon_i(1+\alpha^2)}{p_i + p_i^2\epsilon_i\delta(1+\alpha^2)} \right)^{1/\delta}$$

$$a_i = \frac{i\alpha(-1+\frac{\epsilon_i}{2}(p_i+p_i^{-1})}{(p_i-p_i^{-1})(1+\delta+\delta\alpha^2)} - \frac{\epsilon_i}{2}, \quad b_i = \frac{i\alpha(-1+\frac{\epsilon_i}{2}(p_i+p_i^{-1})}{(p_i-p_i^{-1})(1+\delta+\delta\alpha^2)} + \frac{\epsilon_i}{2}$$

$$a_i' = \frac{i\alpha\epsilon_i(-1+\frac{\epsilon_i}{2}(p_i+p_i^{-1})}{(p_i-p_i^{-1})(1+\delta+\delta\alpha^2)} + \frac{1}{2}, \quad b_i' = \frac{i\alpha\epsilon_i(-1+\frac{\epsilon_i}{2}(p_i+p_i^{-1})}{(p_i-p_i^{-1})(1+\delta+\delta\alpha^2)} - \frac{1}{2}$$

$$A_{ij} = \left(\frac{\epsilon_i p_i - \epsilon_j p_j}{1-\epsilon_i\epsilon_j(p_i+p_j)} \right)^2.$$

Now that we have proven the system's integrability, in the third step we have to construct the nonlinear form. Dividing (10) by (8) and (11) by (9) and taking into account that

$$\tan\left(\frac{i}{2}\log\frac{G}{F}\right) = i\frac{G-F}{G+F}$$

we obtain the following system:

$$\tan(\tilde{Q}_n - Q_n) = \frac{\delta(1+\alpha^2)\tan(\tilde{R}_{n+1} - R_{n-1})}{1+\alpha\tan(\tilde{R}_{n+1} - R_{n-1})}$$

$$\tan(\tilde{R}_n - R_n) = \frac{\delta(1+\alpha^2)\tan(\tilde{Q}_{n+1} - Q_{n-1})}{1+\alpha\tan(\tilde{Q}_{n+1} - Q_{n-1})},$$

where $Q_n = \frac{i}{2}\ln(f_n/g_n), R_n = \frac{i}{2}\ln(f_n'/g_n')$.

3. CONCLUSIONS

Using the Hirota formalism and the Hirota three steps for constructing an integrable discretization of a given partial differential or differential-difference equation we found in this paper a new lattice form of the general differential-difference bi-component Volterra system.

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