EXACT SPECTRAL PROBLEM SOLUTION FOR A LATTICE MANY-BODY SYSTEM WITH A FLOW*

BOYKA ANEVA

INRNE, Bulgarian Academy of Sciences, BG-1784 Sofia, Bulgaria
E-mail: blan@inrne.bas.bg

Received February 20, 2013

The matrix product state approach to interacting many-body systems was inspired by the quantum inverse scattering method and developed to describe the stationary behaviour. Driven diffusive systems have the intriguing feature that the properties of the steady state strongly depend on the boundary rates. We have shown that the boundary conditions of the asymmetric simple exclusion process on a lattice define the boundary symmetry as a generalized Onsager algebra, a coideal subalgebra of the bulk quantum affine $U_q((\hat{su}(2))$. We implement algebraic Bethe Ansatz based on the zeros of the Askey-Wilson polynomials to diagonalize the transition rate matrix of the process and find the complete spectrum. The condition for the constructed irreducible finite-dimensional representation of the boundary algebra can be related to the Gallavotti-Cohen symmetry.

PACS: 02.30.Ik, 05.70.Ln, 11.30.-j, 03.70.+k.

1. INTRODUCTION

The open asymmetric simple exclusion process (ASEP) is an interacting many-body system with wide range of applications [1–3]. It is described by the probability distribution $P(s_i,t)$ of a stochastic variable $s_i = 0,1$, at a site $i = 1,2,....L$ of a linear chain. In the set of occupation numbers $(s_1, s_2, ..., s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, or $s_i = 1$ if the site i is occupied. On successive sites particles hop with probability $g_{01}dt$ to the left, and $g_{10}dt$ to the right. The event of hopping occurs if out of two adjacent sites one is a vacancy and the other is occupied by a particle. The process is totally asymmetric if all jumps occur in one direction only and symmetric if $g_{01} = g_{10} = g$. In the case of open systems, additional processes can take place at the boundaries. At the left boundary $i = 1$ a particle can be added with probability αdt and removed with probability γdt, and at the right boundary $i = L$ it can be removed with probability βdt and added with probability δdt. Without loss of generality we choose the left probability rate $g_{01} = q$ and the

*Paper presented at "The 8$^{\rm th}$ Workshop on Quantum Field Theory and Hamiltonian Systems", September 19-22, 2012, Craiova, Romania.

right one $g_{10} = 1$. The totally asymmetric process and the symmetric one correspond to $q = 0$ and $q = 1$ respectively. The time evolution of the system is governed by the master equation $\frac{dP(s,t)}{dt} = \sum_{s'} \Gamma(s,s')P(s',t)$ with $\Gamma(s,s) = -\sum_{s'} \Gamma(s,s')$ due to probability conservation. By using 'ket' notation one writes $P(s,t)$ as a vector of probabilities $|P(t)\rangle$ of dimension 2^L which is the number of configurations on a chain of L sites. The master equation can formally be mapped to a Schroedinger equation in imaginary time $\frac{d|P(t)\rangle}{dt} = \Gamma|P(t)\rangle$ where $-\Gamma = H$ is referred to as a quantum "Hamiltonian" with nearest-neighbour interaction in the bulk and single-site boundary terms. The ground state of this, in general, non-Hermitian "Hamiltonian", corresponds to the steady state of the stochastic dynamics.

The formal relation to the integrable spin $1/2$ XXZ $U_q(su(2))$ invariant quantum spin chain [5] with anisotropy Δ and with added general non diagonal boundary terms is known [4]. Through the mapping Bethe Ansatz (BA) solution was possible for the ASEP [6]. It is based on the XXZ Hamiltonian integrability condition [7, 8], which in terms of the ASEP notations reads (k is an integer such that $|k| \leq L/2$)

$$(q^{L+2k} - 1)(\alpha\beta - q^{L-2k-2}\gamma\delta) = 0 \qquad (1)$$

The ASEP BA solution is valid for a chain of even number of sites only.

Despite the formal equivalence to the XXZ chain there are important distinguishing features due to the ASEP stochastic nature (for details see [2]). An independent treatment of the stochastic process is worth considering.

The master equation can be formally solved $|P(t)\rangle = \exp(\Gamma t)|P(0)\rangle$. As noted in [2] if one can diagonalize the transition rate matrix then one can obtain all probabilities at all times. Γ is non-Hermitian, in general, and has different left and right eigenvectors. As a stochastic matrix, it has a trivial 'bra' eigenvector $\langle 0|\Gamma = 0$ and a nontrivial one which is the stationary state. One has $\Gamma|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle$, $\langle\phi_\mu|\Gamma = \mu\langle\phi_\mu|$, with the bi-orthogonality condition $\langle\phi_\mu|\psi_\lambda\rangle = \delta_{\mu,\lambda}$. Assuming that they form a complete system $1 = \sum_\lambda |\psi_\lambda\rangle\langle\phi_\lambda|$, then $\Gamma = \sum_\lambda \lambda|\psi_\lambda\rangle\langle\phi_\lambda|$. Hence $|P(t)\rangle = \exp(\lambda t)|\psi_\lambda\rangle\langle\phi_\lambda|P(0)\rangle$. Note that the transition matrix is a positive Markov matrix with real eigenvalues or, if complex, they appear in conjugate pairs. According to Perron-Frobenius theorem it has one maximum eigenvalue zero and negative real parts of all the eigenvalues.

In this paper we find a solution to the transition rate matrix spectral problem. We propose a diagonalization procedure in the auxiliary space, which is the representation space of the boundary algebra of the process. The boundary operators of the ASEP are shifted generators of the generalized Onsager algebra. We implement the algebraic Bethe ansatz based on the Bethe equation for the zeros of the Askey-Wilson polynomials. We find a complete set of 2^L eigenvectors with distinct eigenvalues including the unique ground state of eigenvalue zero.

2. THE MODEL WITHIN THE TRIDIAGONAL ALGEBRA APPROACH

2.1. MATRIX PRODUCT STATE ANSATZ

It states that the steady state properties of the ASEP are exactly obtained in terms of matrices obeying a quadratic algebra [9, 10]. For a given configuration $(s_1, s_2, ..., s_L)$ the stationary probability is defined by the expectation value $P(s) = \frac{\langle w|D_{s_1}D_{s_2}...D_{s_L}|v\rangle}{Z_L}$, where $D_{s_i} = D_1$ if a site $i = 1,2,...,L$ is occupied and $D_{s_i} = D_0$ if a site i is empty and $Z_L = \langle w|(D_0 + D_1)^L|v\rangle$ is the normalization factor to the stationary probability distribution. The operators $D_i, i = 0,1$ satisfy the quadratic (bulk) algebra

$$D_1D_0 - qD_0D_1 = x_1D_0 - D_1x_0, \qquad x_0 + x_1 = 0 \tag{2}$$

with boundary conditions of the form

$$(\beta D_1 - \delta D_0)|v\rangle = x_0|v\rangle \tag{3}$$

$$\langle w|(\alpha D_0 - \gamma D_1) = -\langle w|x_1. \tag{4}$$

The exact solution in the stationary state was related to the Askey-Wilson polynomials [11].

We emphasize the one parameter dependence of the MPA defining relations due to $x_0 = -x_1 = \zeta$, with $0 < \zeta < \infty$. In most known applications it is restricted to the choice $\zeta = 1$. The relation $x_0 + x_1 = 0$ implies an abelian symmetry with a conserved quantity $D_0 + D_1$, following from

$$D_0 \to D_0 + x_0, \qquad D_1 \to D_1 + x_1 \tag{5}$$

The matrix $D_0 + D_1$ enters the expressions for all the relevant physical quantities and is referred to as the transfer matrix. In the steady state the current through a bond between the sites i and $i+1$ is site independent and has the form

$$J_L = \zeta\frac{\langle w|(D_0 + D_1)^{L-1}|v\rangle}{\langle w|(D_0 + D_1)^L|v\rangle} = \zeta\frac{Z_{L-1}}{Z_L} \tag{6}$$

The current vanishes, $J = 0$, if $\zeta = 0$.

Conjecture: The abelian transformation with parameter ζ may be interpreted as Gallavotti-Cohen symmetry [12]

$$e^{\zeta} \to \frac{\gamma\delta}{\alpha\beta}q^{L-1}e^{-\zeta} \tag{7}$$

2.2. THE TRIDIAGONAL BOUNDARY SYMMETRY OF THE ASEP

We have recently shown [13] that the quantum affine $U_q(\hat{su}(2))$ [14, 15] is the bulk hidden symmetry of the asymmetric simple exclusion process. We are led to the formulation of the tridiagonal (generalized Onzager) algebra approach to the asymmetric exclusion process:

1. The matrices of the MPA which determine the weights of each configuration in the steady state of the ASEP obey the level zero q-Serre relations of quantum affine $U_q(\hat{su}(2))$.

2. The boundary operators $D^L = \alpha D_0 - \gamma D_1$ and $D^R = \beta D_1 - \delta D_0$ are coideal elements of the bulk quantum affine $U_q(\hat{su}(2))$. Namely, $D^L = A^* + \alpha - \gamma$, $D^R = A + \beta - \delta$, where A^*, A generate the Askey-Wilson algebra [16, 17] (as a coideal subalgebra)

$$[A,[A,A^*]_q,]_{q^{-1}} = \rho A^* + \omega A + \eta \tag{8}$$

$$[A^*,[A^*,A]_q]_{q^{-1}} = \rho^* A + \omega A^* + \eta^* \tag{9}$$

3. The shifted boundary operators $A^* = D^L - \alpha + \gamma$ and $A = D^R - \beta + \delta$ generate the tridiagonal algebra [18, 19] (generalized Onsager algebra) which follows from the AW algebra, through the natural homomorphism, in the form of deformed Dolan-Grady relations

$$[A,[A,[A,A^*]_q]_{q^{-1}}] = \rho[A,A^*] \tag{10}$$
$$[A^*,[A^*,[A^*,A]_q]_{q^{-1}}] = \rho^*[A^*,A]$$

The above points can be formalized through the chain of homomorphisms

$$TD \to AW \to U_q(\hat{su}(2)) \tag{11}$$

The boundary AW symmetry is the algebraic property behind the exact steady state solution to the ASEP in terms of the AW polynomials although it was obtained [11] without reference to it. The tridiagonal method is a means for the exact solvability of the simple exclusion process, analogously to the quantum inverse scattering method to integrable models. The q-Serre relations that follow from the bulk quadratic algebra of the MPA imply the existence of an universal R matrix satisfying the Yang-Baxter equation and the RTT relations. The consequence of the boundary AW algebra is the explicit construction of the operator-valued reflection K-matrix [21] solving the boundary Yang-Baxter equation. The Dolan-Grady relations [20] yield the exact spectrum of the transition rate matrix (*i.e.* "Hamiltonian") and imply the existence of a set of nonlocal charges in involution which in the language of the QISM is due to the commutativity of the one-parameter family of transfer matrices. In [13] the set of nonlocal conserved charges are constructed for the symmetric process $q \to 1$, where the boundary symmetry is the Onsager algebra.

In [13] the explicit form of the structure constants of the AW (and TD) algebra ($x_1 = -x_0$) was found in terms of the ASEP model parameters. For the TD algebra they read

$$\rho = x_0^2 \beta\delta q^{-1}(q^{1/2}+q^{-1/2})^2, \qquad \rho^* = x_0^2 \alpha\gamma q^{-1}(q^{1/2}+q^{-1/2})^2 \tag{12}$$

We specially stress the difference in the boundary algebra of the XXZ and the ASEP as an important point in the study of the two models. The boundary operators of the ASEP are shifted generators of the AW (and TD) algebra in the infinite-dimensional representation where A^* is the second order difference operator for the AW polynomials and A is multiplication by x. In this representation all the structure constants in (7) and (8) are nonzero with no relations among them. The AW algebra of the XXZ spin chain, in [22], is a particular case of the ASEP boundary AW algebra due to $\rho = \rho^*$, $\eta = \eta^* = 0$. The TD algebra as a coideal subalgebra of the $U_q(\hat{s}l(2))$, explored for the exact spectrum of the XXZ chain with general boundary terms in [23] has equal structure constants $\rho = \rho^*$, depending on the boundary parameters at the left end of the chain. The ASEP and the XXZ spin chain are formally equivalent through a similarity transformation but they describe different physics. A relation among the structure constants is unacceptable for a model of nonequilibrium physics because it will restrict the physics of the system.

3. EXACT SPECTRAL PROBLEM SOLUTION

We use the unique solution of the Bethe-Ansatz equation for the AW zeros

$$\prod_{\nu=1}^{4} \frac{y_k - w_\nu}{w_\nu y_k - 1} = \prod_{l=1, l\neq k}^{L} \frac{(qy_k - y_l)(qy_k y_l - 1)}{(y_k - qy_l)(y_k y_l - q)} \tag{13}$$

where $w_1 = a$, $w_2 = b$, $w_3 = c$, $w_4 = d$ to construct a finite-dimensional representation of the boundary algebra in the space of Laurent polynomials of a given degree. For comments and details we refer to [24, 25].

3.1. TRUNCATION OF THE THREE-TERM RECURRENCE RELATION FOR THE AW POLYNOMIALS

The operator $Ap_n = xp_n$ is represented by a tridiagonal matrix $\mathcal{A}$ with matrix elements from the three-term recurrence relation for the AW polynomials.

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \qquad p_{-1} = 0.$$

A natural way to obtain the finite-dimensional representations of the AW algebra is to set $b_n = 0$. This is achieved by the vanishing of one of the factors $(1 - abq^n)$, $(1 - acq^n)$, $(1 - adq^n)$ or $(1 - abcdq^{n-1})$ in the explicit formula for b_n (see [31] for details). (Note the identification of the AW parameters a, b, c, d with the boundary parameters $abcd = \frac{\gamma\delta}{\alpha\beta}$.) For the ASEP,we have to terminate the sequence at p_L without constraints on the parameters a, b, c, d and q^L that would restrict the physics of the stochastic system.

We make use of the parameter $\zeta \equiv x_0$, associated with the hidden abelian symmetry in the bulk. We obtain a discrete set of AW polynomials ($p[y] = p[y^{-1}]$) due to $p_L[y] = 0$ in the following steps:
1. Expand p_L as a product of its zeros obtaining the BA equation (11) for the L zeros $y_0, y_1, ...y_{L-1}$.
2. Then rescale

$$a \to e^{-\zeta/2}q^{1/4}a, \quad b \to e^{-\zeta/2}q^{1/4}b, \quad c \to e^{-\zeta/2}q^{1/4}c, \quad d \to e^{-\zeta/2}q^{1/4}d \qquad (14)$$

With $\zeta \geq 1$ we have a representation in terms of a', b', c', d', which enter the LHS of the Bethe equations.
3. Set $b_n = 0$, *e.g.* the factor $((1 - e^{-2\zeta}qabcdq^{n-1}) = 0$. The condition to terminate the AW algebra ladder representation due to $b_n = 0$, for $n = L-1$ becomes

$$e^{-2\zeta}abcdq^{L-1} = 1 \qquad (15)$$

The condition (13), for truncating the recurrence relations for the AW polynomials, is preserved by the Gallavotti-Cohen symmetry (6). In the ASEP parameter space it defines a fixed point $q^{(L-1)/2}\sqrt{\frac{\gamma\delta}{\alpha\beta}}$ with respect to the transformation (6).

Remark: The relation of the parameter ζ to the Gallavotti-Cohen symmetry is merely a conjecture. It has no effect on the exact result for the spectrum, obtained due to the rescaling the parameters of the AW polynomials. We can rescale a, b, c, d in (12) directly by ζ. If we choose the factor $(1 - abq^n)$ to set $b_n = 0$, then the condition is $\zeta^2 abq^{L-1} = 1$, which seems simpler for applications.

We thus obtain a discrete set of AW orthogonal polynomials (see [26–28] for details) $p_n(x_k, a, b, c, d|, q)$, $n = 0, 1, \ldots, L-1$ with rescaled parameters, the basis for an irreducible (IR) finite-dimensional representation W of the tridiagonal algebra determined by eq.(13). The matrices, representing A, A^*, in the tridiagonal and diagonal representation are finite $L^2 \times L^2$ square matrices. They are block-tridiagonal and block-diagonal respectively, where each block is an $L \times L$ square matrix. In the representation W of the TD on the space with basis, the discrete set of AW polynomials, the spectrum of the diagonal operator A^* is degenerate. Each eigenvalue λ^* has an eigenspace $p_n(x_k)$, with $k = 0, ..., L-1$ of dimension L.

For each fixed x_k, however, there is a finite-dimensional subrepresentation V, with basis $p_n(x_k)$, $n = 0, ..., L-1$, which is not an invariant subspace of W. The vectors $|\nu_n\rangle = |p_n\rangle$ form an orthogonal basis for this representation $\langle \nu_m|\nu_n\rangle = \delta_{mn}$. The tridiagonal matrix representing A is irreducible tridiagonal, while the diagonal is such that each eigenvalue λ_n has dimension one. These finite-dimensional matrices obey the relations (8) with ρ, ρ^* given by (10). We relate the representation V to a highest weight IR finite-dimensional evaluation representation of $U_q\hat{s}u(2)$ with deformation parameter q. (Note the change from $q^{1/2}$ to q. The AW algebra is originally defined as a coideal subalgebra of $U_{q^{1/2}}\hat{s}u(2)$ [21], but we relate the corresponding TD algebra finite-dimensional representation with a discrete set of AW polynomials to $U_q\hat{s}u(2)$.)

3.2. THE EIGENVALUES OF A AND A^*

To obtain a compete set of 2^n eigenvectors with 2^n eigenvalues for any finite n, $0 \le n \le L$, we associate with each lattice site i a basis vector $p_0(x_i)$ if a site is empty (occupation number $s_i = 0$) or $p_1(x_i)$ if there is a particle on the site (occupation number $s_i = 1$). Let $|\psi(x_1,x_2,...,x_L)\rangle$ denote the basis in the configuration space of the ASEP on the lattice of L sites, depending on the set of occupation numbers $s_{i_1}, s_{i_2}, ..., s_{i_L}$. The ASEP configuration space is identified with the $U_q(\hat{s}u(2))$ irreducible tensor product evaluation representation $V_1(x_1) \otimes ... \otimes V_1(x_L)$, with a highest weight vector generating the $2j = L$ subrepresentation.

By definition the highest weight vector of the tensor product obeys $E^+\Omega = 0$, with $\Omega = p_0(x_1)p_0(x_2)...p_0(x_L)$. The discrete set of AW polynomials satisfy the three-term recurrence relation with $p_0(x) = 1$ for $x = x_k$. Hence the highest weight vector Ω is a constant vector $\Omega = 1$ and is an eigenvector of the operator A^*

$$A^*\Omega = (1 + abcdq^{-1})\Omega \tag{16}$$

Hence, a corresponding shift of A^* produces the unique ground state of the system with eigenvalue zero.

By construction the state $|\psi(x_i)\rangle$ becomes an eigenvector of the operator A to be interpreted as the transition matrix in the auxiliary space of the system. It acts on it by means of the coproduct which plays the role of a counting function by taking into account all the admissible permutations to produce the correct number of states. We thus obtain $2^L - 1$ distinct eigenvalues of A in the different spin sectors. The eigenvalue equation with the corresponding eigenvalues reads

$$A\psi(x_1,x_2,...,x_L) = (\sum_{i=1}^{L} x_i + \sum_{i<j} x_i x_j + ... + x_1 x_2 ... x_L)|\psi(x_1,x_2,...,x_L)\rangle. \tag{17}$$

With the interpretation of the operator A as the ”Hamiltonian” eq.(15) yields the nonzero ”energy” eigenvalues.

3.3. THE MAIN RESULT

The transition rate matrix of the ASEP is diagonalized in the auxiliary space of the finite-dimensional representation of the deformed Onsager boundary algebra.

The basis in this representation is equivalent to the $U_q(\hat{s}u(2))$ irreducible highest weight tensor product evaluation module which forms the complete set of eigenvectors for the transition rate matrix. The ground state vector of the ASEP is identified with the unique highest weight vector.

We identify the transition rate matrix Γ_M of the open ASEP in the auxiliary space of symmetric Laurent polynomials p_n, $0 \leq n \leq L-1$ with the representation (and the dual one) of the right boundary operator $A+\beta-\delta$ and the left boundary operator $A^*+\alpha-\gamma$. For a chain of L sites there is a representation of dimension 2^L for any finite L. In this representation the transition rate matrix Γ_M has a unique eigenstate $(\Omega,0,0,...0)$ of eigenvalue zero which is the eigenstate of the left boundary operator, to be identified with the ASEP stationary state, and 2^L-1 eigenstates of the right boundary operator with nonzero distinct eigenvalues given by

$$-\Gamma_M = \beta+\delta+\left((1-q)\sum_{i=1}^{L}\hat{x}_i+(1-q)^2\sum_{i<j}\hat{x}_i\hat{x}_j+...+(1-q)^L\hat{x}_1\hat{x}_2...\hat{x}_L\right), \tag{18}$$

where $\hat{x}_i^{-1}=\hat{y}_i+\hat{y}_i^{-1}$. The zeros $\hat{y}_i$ satisfy the Bethe-Ansatz equation (11) with $w_1 = k'_+(\alpha,\gamma)$, $w_2 = k'_+(\beta,\delta)$, $w_3 = k'_-(\alpha,\gamma)$, $w_4 = k'_-(\beta,\delta)$ and $k'_\pm(u,v) = e^{-\zeta/2}k_\pm(u,v)$; $e^{-\zeta} = q^{(1-L)/2}\sqrt{\frac{\alpha\beta}{\gamma\delta}}$; $k_\pm(u,v) = \frac{v-u+(1-q)\pm\sqrt{(u-v-(1-q))^2+4uv}}{2u}$.

There is a dual representation in the auxiliary space of symmetric Laurent polynomials p_n, $0 \leq n \leq L-1$ of dimension 2^L for any finite L. In this basis the transition matrix Γ_M has a unique eigenstate $(p_0,0,0,...0)^t$ of eigenvalue zero which is the eigenstate of the right boundary operator and 2^L-1 nonzero eigenvalues, the eigenvalues of the left boundary operator given by

$$-\Gamma_M = \alpha+\gamma+\left((1-q)\sum_{i=1}^{L}(\hat{x}_i+(1-q)^2\sum_{i<j}\hat{x}_i\hat{x}_j+...+(1-q)^L\hat{x}_1\hat{x}_2...\hat{x}_L\right). \tag{19}$$

We note that the result in [6], obtained for even L only and for the energy eigenvalues in the one spin sector, is in a different basis. According to [30] the BA equation in [6] will arise from a solution of a certain form of an AW difference equation for a polynomial $F_n(x) = F_n[z]\left(\frac{(q^{3/2}z;q^2)_\infty(q^{3/2}z^{-1};q^2)_\infty}{(q^{1/2}z;q^2)_\infty(q^{1/2}z^{-1};q^2)_\infty}\right)^{n+1}\prod_{i=1}^{n}(x-x_i)$. In view of the importance of the AW polynomials for the steady state solution of the ASEP, they form the most efficient basis for the exact description of the fundamental model of non-equilibrium physics.

Acknowledgments. The author would like to thank the organizers for the invitation to participate in the workshop and give a talk, and for the warm atmosphere during the stay in Craiova. The financial support of UNESCO is acknowledged.

REFERENCES

1. G.M. Schuetz, "*Phase Trans. and Crit.Phen.*" **19** (Academic Press, London, 2000).
2. R.A. Blythe, M.R. Evans, J. Phys. A **40**, R333-R441 (2007).
3. J. Krug, H. Spohn, in "*Solids far from equilibrium*", C. Godreche (eds.) (Cambridge University Press, 1991).
4. F.H.L. Essler, V.Rittenberg, J. Phys. A **29**, 3375-3407 (1996).
5. V. Pasquier, H. Saleur, Nucl. Phys. B **330**, 523-556 (1990).
6. J. de Gier, F.H. Essler, Phys. Rev. Lett. **95**, 240601 (2005); J. Stat. Mech., P12011 (2006); J. Phys. A: Math. Theor. **41**, 485002 (2008).
7. R.I. Nepomechie, J. Stat. Phys. **111**, 1363 (2003); J. Phys. A **37**, 433 (2004).
8. J. Cao, H.-Q. Lin, K.-J. Shi, Y. Wang, Nucl. Phys. B **663**, 487 (2003).
9. B. Derrida, Phys. Rep. **301**, 65-83 (1998).
10. B. Derrida, M.R. Evans, V. Hakim, V. Pasquier, J. Phys. A **26**, 1493-1517 (1993).
11. M. Uchliama, T. Sasamoto, M. Wadati, J. Phys. A **37**, 4985-5002 (2004).
12. G. Gallavotti, E.G.D. Cohen, Phys. Rev. Lett. **74**, 2694 (1995).
13. B. Aneva, J. Phys. A **40**, 11677 (2007); SIGMA **4**, 056 (2008).
14. M. Jimbo, Lett. Math. Phys. **10**, 63 (1985); **11**, 246 (1986).
15. V. Chari, A. Pressley, Comm. Math. Phys. **142**, 261-283 (1991).
16. A.S. Zhedanov, Theor. Math. Fiz. **89**, 190 (1991).
17. Y.A. Granovskii, A.S. Zhedanov, J. Phys. A **26**, L357-L359 (1993).
18. P. Terwilliger, arXiv:math.QA/0307016 (2003); Lec. Notes in Math. **1883**, 225 (2006).
19. T. Ito, P. Terwilliger, arXiv:math.QA/0310042 (2003); arXiv:math.QA/0602199 (2006); arXiv:math.QA/0807.0271 (2008).
20. L. Dolan, M. Grady, Phys. Rev. D **25**, 1587-1604 (1982).
21. B. Aneva, M. Chaichian, P.P. Kulish, J. Phys. A **41**, 135201 (2008).
22. P. Baseilhac, Nucl. Phys. B **705**, 605-619 (2005); **709**, 491-521 (2005); J. Phys. A **39**, 11773 (2006).
23. P. Baseilhac, K. Koizumi, J. Stat. Mech. P09006 (2007).
24. P.B. Wiegmann, A.V. Zabrodin, Nucl. Phys. B **451**, 699-724 (1995).
25. M.E.H. Ismail, S.S. Lin, S.S. Roan, arXiv:math-ph/0407033 (2004).
26. T.S. Chihara, "*An introduction to Orthogonal polynomials*", Math. Appl. **13** (Gordon and Breach, 1978).
27. G. Gaspar, M. Rahman, "*Basic Hypergeometric Series*" (Cambridge University Press, 2004).
28. F.V. Atkinson, "*Discrete and Continuous Boundary Problems*" (Academic Press, New York, 1964).
29. T. Koornwinder, SIGMA **3**, 063 (2007).
30. T. Koornwinder, private communication (2007).
31. R.A. Askey, J.A. Wilson, Mem. Amer. Math. Soc. **54**(319), iv+55 (1985).