FRACTIONAL CAPUTO HEAT EQUATION WITHIN THE DOUBLE LAPLACE TRANSFORM

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The heat equation and its fractional generalization are used in various applications in science and engineering. In this paper firstly we introduce the double Laplace transform of the partial fractional integrals and derivatives which can be used to solve partial differential equations with Caputo fractional derivatives. Secondly, the fractional heat equation was investigated in details with the help of this new generalized transform

Key words: Fractional integral, Caputo fractional derivative, Laplace transform.

1. INTRODUCTION

The topic of partial differential equations is one of the most important subjects in mathematics and other sciences. However, there are no general methods to solve such equations. One of the most known method to solve partial differential equations is the integral transform method [1, 2]. In [3, 4] the double Laplace and Sumudu transforms were used to solve wave and Poisson equations.

The fractional calculus, which is as old as the classical one, is the generalization the ordinary integrals and derivatives of integer orders to arbitrary ones. This kind of calculus became a candidate to solve problems of complex systems that appear in various fields of sciences [5, 6, 7]. There are many open problems in this area both from the theoretical and applied point of view [8, 9, 10], *e.g.* the physical meaning of the fractional derivative. In the fractional modeling area, the fractional partial derivatives appear naturally in dealing with generalization of the existing classical models [8]. Therefore there is a need to improve or to adapt the existing methods and technique from the classical case to the fractional one. At the first sight this process seems simple and direct but in fact this is a complicated process and it requires much attention mainly because the fractional calculus requires some additional conditions in order to be defined correctly [8].

Maravall used the Laplace transform method to obtain the explicit solution of a certain kind of ordinary differential equations with fractional derivatives (see [5] and the references therein). Oldham and Spanier also used the Laplace transform method to solve another type of homogeneous ordinary differential equations of fractional order. The Laplace transform was also used by Dorta, Seitkazieva, Miller and Ross to find solutions of such equations. Many authors like Gerasimov, Fedosov, Yanenko, Biacino, Miseredino and Fujita attempted to solve partial differential equation with fractional orders (see [5] and the references therein). To our knowledge, solving fractional order partial differential equations using the double Laplace transform is still an open problem.

An important application of the heat equation can be seen as the measurement of the thermal diffusivity in polymers [11]. In addition it appears in financial mathematics in the modeling of options. Also we can mention that it appears in describing pressure diffusion in an porous medium. Generally this equation cannot be solve analytically and the numerical techniques are used frequently. The fractional generalization of the heat equation can be related to the nonlocal phenomena and itself presents its own interest.

Taking into account the above mentioned results, in this work, we establish the double Laplace formulas for the partial fractional derivatives and apply these formulas to solve a fractional heat equation with certain initial and boundary conditions.

This manuscript is organized as follows:

In the second section we present the basic definitions of the fractional integrals and derivatives (both ordinary and partial), and the double Laplace transform. In the third section we introduce the double Laplace transforms of the partial fractional integrals and Caputo derivatives. In the fourth section we solve a fractional heat equation with certain initial and boundary conditions. The fifth section is devoted to the conclusion.

2. PRELIMINARIES

In this section we summarize the basic definitions of ordinary and partial fractional integrals and derivatives and the double Laplace transform.

The left Riemann-Liouville fractional integral of order α, on the half axis $\mathbb{R}^+$, of a function f is defined by [5-8]

$$({}_xI_{0^+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)\, \mathrm{d}\xi \tag{1}$$

where $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

The left Riemann-Liouville fractional derivative of order α, on the half axis $\mathbb{R}^+$, of a function f is defined by [5-8]

$$({}_xD_{0^+}^{\alpha} f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\xi)^{\alpha-n-1} f(\xi)\, d\xi \tag{2}$$

where $n-1 < \Re(\alpha) \le n, \quad n \in \mathbb{N}$.

The left Caputo fractional derivative of order α, on the half axis $\mathbb{R}^+$, of a function f is defined by [5, 8]

$$({}_x^cD_{0^+}^{\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\xi)^{\alpha-n-1} \left(\frac{d}{d\xi}\right)^n f(\xi)\, d\xi \tag{3}$$

where $n-1 < \Re(\alpha) \le n, \quad n \in \mathbb{N}$.

Similarly, the partial fractional integrals and Caputo derivatives of a function $f(x,t)$, where $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ are defined [5-8] as follows

$${}_xI_{0^+}^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi,t)\, d\xi, \tag{4}$$

$${}_yI_{0^+}^{\beta} f(x,t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(x,\tau)\, d\tau, \tag{5}$$

$${}_tI_{0^+}^{\beta}\, {}_xI_{0^+}^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^x (t-\tau)^{\beta-1} (x-\xi)^{\alpha-1} f(\xi,\tau)\, d\xi d\tau, \tag{6}$$

$${}_x^cD_{0^+}^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\xi)^{n-\alpha-1} \left(\frac{\partial}{\partial\xi}\right)^n f(\xi,t) d\xi, \tag{7}$$

$${}_t^cD_{0^+}^{\beta} f(x,t) = \frac{1}{\Gamma(m-\beta)} \int_0^t (t-\tau)^{m-\alpha-1} \left(\frac{\partial}{\partial\tau}\right)^m f(x,\tau) d\tau, \tag{8}$$

$${}_t^cD_{0^+}^{\beta}\, {}_x^cD_{0^+}^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)\Gamma(m-\beta)} \int_0^t \int_0^x (t-\tau)^{m-\beta-1} (x-\xi)^{n-\alpha-1} \frac{\partial^{m+n} f(\xi,\tau)}{\partial\tau^m \partial\xi^n} d\xi d\tau, \tag{9}$$

where $n-1 < \Re(\alpha) \le n, \quad m-1 < \Re(\beta) \le m, \quad n,m \in \mathbb{N}$.

The Laplace transforms of the fractional integrals and Caputo fractional derivatives are given in the following lemma.

Lemma 2.1. [5,6]
(a) Let $\Re(\alpha)>0$ and $f\in L_1(0,b)$ for any $b>0$. Also let the estimate $|f(t)|<Ae^{p_0 t}$, $t>b>0$ hold for the constants $A, p_0>0$. Then,

$$L\left\{I_{0^+}^{\alpha} f\right\}(s)=s^{-\alpha}L\left\{f\right\}$$

(b) Let $\alpha>0,\ n-1<\alpha\le n,\ (n\in\mathbb{N})$ be such that $f\in C^n(\mathbb{R}^+),\ f^{(n)}\in L_1(0,b)$ that for any b>0, $|f(t)|<Ae^{p_0 t}$, the Laplace transforms of f and $f^{(n)}$ exist, and $\lim\limits_{t\to\infty} f^{(k)}(t)=0$ for $k=0,1,,...,n$ for. Then

$$L\left\{{}^{C}D_{0^+}^{\alpha} f\right\}(s)=s^{\alpha}L\left\{f\right\}(s)-\sum_{k=0}^{\infty}s^{\alpha-k-1}f^{(k)}(0)\,.$$

Below we present the definition of the double Laplace transform and the formulas for the double Laplace transforms for the first partial derivatives. For more details we refer the readers to references [3, 4] and the references therein.

Defintion 2.2. [3, 4] Let f be a function of 2 variables x and t, where $x,t>0$. The double Laplace transform of f is defined by

$$L_t L_x\left\{f\left(x,t\right)\right\}\left(s_1,s_2\right)=\int_0^{\infty}\mathrm{e}^{-s_2 t}\int_0^{\infty}\mathrm{e}^{-s_1 x}f\left(x,t\right)\mathrm{d}x\mathrm{d}t\,. \tag{10}$$

The following are the formulas of the double Laplace transforms of the first partial derivatives of f with respect to x and t [4].

$$L_t L_x\left\{\frac{\partial f\left(x,t\right)}{\partial x}\right\}(s_1,s_2)=s_1 L_t L_x\left\{f\left(x,t\right)\right\}-L_t\left\{f\left(0,t\right)\right\}(s_2)\,, \tag{11}$$

$$L_t L_x\left\{\frac{\partial f\left(x,t\right)}{\partial t}\right\}(s_1,s_2)=s_2 L_t L_x\left\{f\left(x,t\right)\right\}-L_x\left\{f\left(x,0\right)\right\}(s_1)\,, \tag{12}$$

where L_x and L_t represent the Laplace transforms with respect to the variables x and t, respectively.

3. THE LAPLACE TRANSFORMS OF FRACTIONAL PARTIAL INTEGRALS AND DERIVATIVE

Before we establish the Laplace transforms of the fractional partial integrals and derivatives, we present the Laplace transform formulas for the partial derivatives of an arbitrary integer order in the following theorem.

Theorem 3.1. Let $f \in C^l(\mathbb{R}^+ \times \mathbb{R}^+)$, $l = \max\{m,n\}$ $\exists k, \tau_1, \tau_2 > 0$ such that $\left|\frac{\partial^{i+j} f(x.t)}{\partial x^i \partial t^j}\right| < ke^{x\tau_1 + t\tau_2}$, $i = 0,1,...,n,\ j = 0,1,...,m$. Then the following formulas hold:

$$L_t L_x \left\{ \frac{\partial^n f(x,t)}{\partial x^n} \right\} = s_1^n L_x L_t \{ f(x,t) \} - \sum_{k=0}^{n-1} s_1^{n-1-k} L_t \left\{ \frac{\partial^k f(0,t)}{\partial x^k} \right\}, \tag{13}$$

$$L_t L_x \left\{ \frac{\partial^m f(x,t)}{\partial t^m} \right\} = s_2^m L_x L_t \{ f(x,y) \} - \sum_{j=0}^{m-1} s_2^{m-j-1} L \left\{ \frac{\partial^j f(x,0)}{\partial t^j} \right\}, \tag{14}$$

$$L_t L_x \left\{ \frac{\partial^{m+n} f(x,t)}{\partial t^m \partial x^n} \right\} = s_1^n s_2^m \left[L_t L_x \{ f(x,y) \} - \sum_{j=0}^{m-1} s_2^{-j-1} L_x \left\{ \frac{\partial^j f(x,0)}{\partial t^j} \right\} \right.$$

$$\left. - \sum_{i=0}^{n-1} s_1^{-i-1} L_t \left\{ \frac{\partial^i f(0,t)}{\partial x^i} \right\} + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} s_1^{-i-1} s_2^{-j-1} \frac{\partial^{i+j} f(0,0)}{\partial x^i \partial t^j} \right], \tag{15}$$

where $\frac{\partial^{i+j} f(0,0)}{\partial x^i \partial t^j}$ denotes the value of the mixed derivative at the point $(0,0)$.

Proof. The proof is similar to that of the Laplace transforms of the ordinary derivatives of functions of a single variable.

In the following lemma, we establish the double Laplace transforms of the partial fractional integrals.

Lemma 3.2. Let $\Re(\alpha)$, $\Re(\beta)>0$ and $f \in L_1\left[(0,a)\times(0,b)\right]$ for any $a,b>0$. Also let $|f(x,t)| \le ke^{x\tau_1 + t\tau_2}$, $x > a > 0,\ t > b > 0$ hold for constants $k, \tau_1, \tau_2 > 0$. Then,

$$L_t L_x \{ {}_x I_{0^+}^{\alpha} f(x,t) \}(s_1, s_2) = \frac{1}{s_1^{\alpha}} L_t L_x \{ f(x,t) \}(s_1, s_2), \tag{16}$$

$$L_t L_x \{ {}_t I_{0^+}^{\beta} f(x,t) \}(s_1, s_2) = \frac{1}{s_2^{\beta}} L_t L_x \{ f(x,t) \}(s_1, s_2), \tag{17}$$

and

$$L_t L_x \{ {}_tI^{\beta}_{0^+} \, {}_xI^{\alpha}_{0^+} f(x,t)\}(s_1,s_2) = \frac{1}{s_1^{\alpha} s_2^{\beta}} L_x L_t \{f(x,t)\}(s_1,s_2) \,. \tag{18}$$

Proof. The proof of existence of is analogous of the proof of Lemma 1.1. To derive the formulas (16), (17) and (18) we use the double Laplace transform of the convolution with respect to x for the first, the convolution with respect to t for the second and the Laplace transform of the double convolution for the third. For more information on double convolution, we refer the reader to references [3, 4] and the references therein.

Finally, the double Laplace transforms of the partial fractional Caputo derivatives are given in the following theorem:

Theorem 3.3. Let $\alpha,\beta > 0,\ n-1<\alpha\le n,\ m-1<\beta\le m,\ n,m \in \mathbb{N}$ be such that $f \in C^l(\mathbb{R}^+ \times \mathbb{R}^+),\ l=\max\{m,n\},\ f^{(l)} \in L_1\left[(0,a)\times(0,b)\right]$ for any $a,b>0$, $\left| f(x,t) \right| \le k e^{x\tau_1 + t\tau_2},\ x>a>0,\ t>b>0$, the Laplace transforms of f and $\dfrac{\partial^{i+j} f}{\partial x^i \partial t^j},\ i=0,1,...,n,\ \mathrm{j}=0,1,...,m$ exist. Then,

$$L_t L_x \left\{ {}^c_x D^{\alpha}_{0^+} f\left(x,t\right)\right\} = s_1^{\alpha}\left[L_t L_x \left\{ f\left(x,t\right)\right\} - \sum_{i=0}^{n-1} s_1^{-1-i} L_t \left\{ \frac{\partial^i f\left(0,t\right)}{\partial x^i}\right\}\right], \tag{19}$$

$$L_t L_x \left\{ {}^c_t D^{\beta}_{0^+} f\left(x,t\right)\right\} = s_2^{\beta}\left[L_t L_x \left\{ f\left(x,t\right)\right\} - \sum_{j=0}^{m-1} s_2^{-1-j} L_x \left\{ \frac{\partial^j f\left(x,0\right)}{\partial t^j}\right\}\right] \tag{20}$$

and

$$\begin{aligned} L_t L_x \left\{ {}^c_t D^{\alpha}_{0^+} \, {}^c_x D^{\alpha}_{0^+} f\left(x,t\right)\right\} = s_1^{\alpha} s_2^{\beta} \Bigg[& L_t L_x \left\{ f\left(x,t\right)\right\} - \sum_{i=0}^{n-1} s_1^{-1-i} L_t \left\{ \frac{\partial^i f\left(0,t\right)}{\partial x^i}\right\} - \\ & - \sum_{j=0}^{m-1} s_2^{-1-j} L_x \left\{ \frac{\partial^j f\left(x,0\right)}{\partial t^j}\right\} - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} s_1^{-i-1} \, s_2^{-j-1} \, \frac{\partial^{i+j} f}{\partial y^i \partial x^i}\left(0,0\right)\Bigg]. \end{aligned} \tag{21}$$

Proof. The proof follows from Theorem 3.1 and Lemma 3.2.

Before we discuss the solution of fractional heat equation, we have to mention the Mittag-Leffler function which plays an important role in the theory of fractional differential equations [5-8]. The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)},\ t,\beta\in\mathbb{C},\ \Re(\alpha)>0.$$

The Laplace transform of the function $t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})$ takes the form

$$L\left\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right\}=\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda},\ \text{for}\ |\lambda|<|s^{\alpha}|.$$

4. FRACTIONAL HEAT EQUATION

In this section, we generalize the heat in 1+1 dimension to the fractional case corresponding to the time variable, namely, we will consider the following fractional heat

$$ {}_{t}^{c}D_{0^+}^{\alpha}u(x,t)=\frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}(x,t) \qquad x>0,\quad t>0, \tag{21}$$

with the following initial and boundary values

$$u(0,t)=0,\ \frac{\partial u}{\partial x}(0,t)=\pi\,E_{\alpha}\left(-t^{\alpha}\right) \qquad t>0, \tag{22}$$

$$u(x,0)=\sin\pi x \qquad x>0 \tag{23}$$

where $0<\alpha\le 1$. We have the following

$$L_x\left\{u(x,0)\right\}=L_x\left\{\sin\pi x\right\}=\frac{\pi}{s_1^2+\pi^2},\quad L_t\left\{u(0,t)\right\}=L_t\left\{0\right\}=0,$$

$$L_t\left\{\frac{\partial u(0,t)}{\partial x}\right\}=L_t\left\{\pi\,E_{\alpha}\left(-t^{\alpha}\right)\right\}=\pi\frac{s_2^{\alpha-1}}{1+s_2^{\alpha}}.$$

Applying the double Laplace transforms to both sides of (21), we get

$$s_2^{\alpha}\left[L_tL_x\left\{u(x,t)\right\}-s_2^{-1}\frac{\pi}{s_1^2+\pi^2}\right]=\frac{1}{\pi^2}\left[s_1^2L_tL_x\left\{u(x,t)\right\}\right]-\frac{\pi s_2^{\alpha-1}}{1+s_2^{\alpha}}$$

Thus, $L_tL_x\left\{u(x,t)\right\}=\dfrac{\pi s_2^{\alpha-1}}{\left(s_1^2+\pi^2\right)\left(1+s_2^{\alpha}\right)}=\dfrac{\pi}{s_1^2+\pi^2}\,\dfrac{s_2^{\alpha-1}}{1+s_2^{\alpha}}$. Hence $u(x,t)=\sin\pi x\,E_{\alpha}\left(-t^{\alpha}\right)$.

5. CONCLUSION

In this manuscript, we established the double Laplace transform formulas for partial derivatives of arbitrary positive integer orders. Then we found the double Laplace formulas for the partial fractional integrals and partial fractional derivatives in the sense of Caputo. That is we presented the double Laplace transforms of partial derivatives of arbitrary orders. Finding new methods and techniques to solve partial fractional differential equations is an interesting issue. Because of the importance of the heat equation in physics and other sciences, we applied the double Laplace transform method, to solve a fractional heat equation subject to certain initial and boundary conditions. We believe that this method is applicable also to other kinds of fractional partial differential equations such as the wave equation subject to both initial and boundary conditions.

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REFERENCES

1. D.G. Duff, *Transform Methods for Solving Partial Differential Equations*, Chapman and Hall/CRC, Boca Raton, F.L, 2004.
2. T.A. Estrin, T.J. Higgins, J. Franklin Inst., **252 (2)**, 153–167 (1951).
3. A. Kılıçman, H. E. Gadain, **347**, 848–862 (2010).
4. A.Kılıçman, H. Eltayeb, App. Math. Sc. **4(3)**, 109–118 (2010).
5. A.A. Kilbas, H.H. Srivastava, J.J.Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam. 2006.
6. S.G. Samko, A.A.Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives - Theory and Applications*, Gordon and Breach, Linghorne, P.A. 1993.
7. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.
8. D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus Models and Numerical Methods* (Series on Complexity, Nonlinearity and Chaos. World Scientific), 2012.
9. H. Jafari, A. Kadem, D. Baleanu, T. Yılmaz, Rom. Rep. Phys., **64(2)**, 337–346 (2012).
10. F. Jarad, T. Abdeljawad, E. Gündoğdu, D.Baleanu, Proc. Rom. Acad. A, **12**, 309–314 (2011).
11. D. Unsworth, Am. J. Phys., 47 (11), 891–893 (1979).