TOPOLOGICAL SOLITONS AND OTHER SOLUTIONS OF THE ROSENAU-KdV EQUATION WITH POWER LAW NONLINEARITY

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This paper integrates the Rosenau-KdV equation with power law nonlinearity that appears in applied mathematics. There are several approaches that are applied to study this equation. The ansatz method is applied to obtain the topological soliton solution of this equation. The G'/G method as well as the exp-function method are also applied to extract a few more solutions to this equation. The constraint relations also all naturally fell out during the course of derivation of the solution.

Key words: the Rosenau-KdV equation, power law nonlinearity, the topological soliton solution.

1. INTRODUCTION

There has been an overwhelming amount of solutions to the *Nonlinear Evolution Equations* (NLEEs) obtained in the past few decades using several and newly developed techniques of integration [1-10]. Some of these nonlinear wave solutions are the cnoidal waves, solitons, solitary waves, shock waves, compactons, stumpons, covatons, cuspons, peakons propeller solitons and several many others. These solutions are all indeed very useful in various areas of Applied Mathematics and Theoretical Physics. In fact, these show up sporadically in Plasma Physics, Nonlinear Optics, Nuclear Physics, Fluid Dynamics, Telecommunications Engineering, Mathematical Biology, Mathematical Chemistry, Mathematical Physics just to name a few.

In the past, there was only method of integration that was available to carry out the integration of these several NLEEs. That was known as the *Inverse Scattering Method* (IST). Till today, IST still stands as the most powerful technique of integration, provided the NLEE under study passes the Painleve test of integrability. There are several drawbacks of IST that lead to the invention of these several modern methods of integrability. These latest techniques of integration will however integrate any NLEE even if the Painleve test of integrability will fail and thus IST will no longer be applicable to integrate the equation.

In this paper there will be one such equation that will be integrated to retrieve shock wave solutions, that is also known as topological soliton solution or kinks. This is called the Rosenau-KdV (R-KdV) equation. In a generalized flavor, R-KdV equation is studied in this paper with power law nonlinearity. This will keep things on a generalized perspective. Also, this equation is not integrable by the IST as it will not pass the Painleve test of integrability. Therefore there are three modern methods of integrability that will be applied to integrate this equation. They are the ansatz method for topological solitons, the G'/G-expansion method and the exp-function method integration. These methods will reveal several solutions that will be useful in the literature of NLEEs.

2. TOPOLOGICAL SOLITONS

The generalized R-KdV equation that is going to be studied in this paper is given by

$$u_t + au_x + bu_{xxx} + cu_{xxxxt} + d\left(u^n\right)_x = 0 \tag{1}$$

where a, b, c, and d are real valued constants while for the exponent we assume that $n \neq 0,1$. This equation was studied before [7, 10]. The non-topological 1-soliton solution was already obtained in 2011 [7]. Several other solutions were also obtained in 2009 [7]. In this paper, the search is going to be for shock wave solution or topological 1-soliton solution to the equation (1) by using the solitary wave ansatz. The starting hypothesis is given by

$$u(x,t) = A\tanh^p \tau \tag{2}$$

where

$$\tau = B(x - vt) \tag{3}$$

and

$$p > 0 \tag{4}$$

for solitons to exist. Here, in (2) and (3), A and B are free parameters while v is the velocity of the wave. The unknown exponent p will be determined in term of n during the course of the derivation of the soliton solution to (1). Thus from (2), we have

$$u_t = pvAB\left(\tanh^{p+1}\tau - \tanh^{p-1}\tau\right) \tag{5}$$

$$u_x = pAB\left(\tanh^{p-1}\tau - \tanh^{p+1}\tau\right) \tag{6}$$

$$\begin{aligned} u_{xxx} = pAB^3 \Big[&(p-1)(p-2)\tanh^{p-3}\tau - \left\{2p^2 + (p-1)(p-2)\right\}\tanh^{p-1}\tau \\ &+\left\{2p^2 + (p+1)(p+2)\right\}\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau\Big] \end{aligned} \tag{7}$$

$$\begin{aligned} u_{xxxt} = -pAB^5 v\Big[&(p-1)(p-2)(p-3)(p-4)\tanh^{p-5}\tau \\ &-(p+1)(p+2)(p+3)(p+4)\tanh^{p+5}\tau \\ &-(p-1)(p-2)\left\{2p^2 + 2(p-2)^2 + (p-3)(p-4)\right\}\tanh^{p-3}\tau \\ &+(p+1)(p+2)\left\{2p^2 + 2(p+2)^2 + (p+3)(p+4)\right\}\tanh^{p+3}\tau \\ &+\Big[2(p-1)(p-2)\left\{p^2 + (p-2)^2\right\} + 4p^4 + \\ &+p(p-1)^2(p-2) + p(p+1)^2(p+2)\Big]\tanh^{p-1}\tau \\ &-\Big[2(p+1)(p+2)\left\{p^2 + (p+2)^2\right\} + 4p^4 + \\ &+p(p-1)^2(p-2) + p(p+1)^2(p+2)\Big]\tanh^{p+1}\tau \end{aligned} \tag{8}$$

$$\left(u^n\right)_x = A^n pnB\left(\tanh^{pn-1}\tau - \tanh^{pn+1}\tau\right) \tag{9}$$

Now substituting (5)-(9) into (1) gives

$$\begin{aligned} &p(v-a)AB\left(\tanh^{p+1}\tau - \tanh^{p-1}\tau\right) \\ &+bpAB^3\Big[(p-1)(p-2)\tanh^{p-3}\tau - \left\{2p^2 + (p-1)(p-2)\right\}\tanh^{p-1}\tau \\ &+\left\{2p^2 + (p+1)(p+2)\right\}\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau\Big] \\ &-cpAB^5 v\Big[(p-1)(p-2)(p-3)(p-4)\tanh^{p-5}\tau \end{aligned}$$

$$-(p+1)(p+2)(p+3)(p+4)\tanh^{p+5}\tau$$

$$-(p-1)(p-2)\left\{2p^2+2(p-2)^2+(p-3)(p-4)\right\}\tanh^{p-3}\tau$$

$$+(p+1)(p+2)\left\{2p^2+2(p+2)^2+(p+3)(p+4)\right\}\tanh^{p+3}\tau$$

$$+\Big[2(p-1)(p-2)\left\{p^2+(p-2)^2\right\}+4p^4+$$
$$+p(p-1)^2(p-2)+p(p+1)^2(p+2)\Big]\tanh^{p-1}\tau$$

$$-\Big[2(p+1)(p+2)\left\{p^2+(p+2)^2\right\}+4p^4+$$
$$+p(p-1)^2(p-2)+p(p+1)^2(p+2)\Big]\tanh^{p+1}\tau$$

$$+dA^n pnB\left(\tanh^{pn-1}\tau-\tanh^{pn+1}\tau\right)=0 \tag{10}$$

From (10), equating the exponents $pn+1$ and $p+5$ gives

$$pn+1=p+5 \tag{11}$$

so that

$$p=\frac{4}{n-1} \tag{12}$$

which exist provided that

$$n>1 \tag{13}$$

as seen from (4) and (12). Again this same value of p is obtained on equating the exponents $pn-1$ and $p+3$.

Now from (10) the linearly independent functions are $\tanh^{p+j}\tau$ for $j=\pm1,\pm3,\pm5$. Hence setting their respective coefficients to zero yields the following system of algebraic equations:

$$\begin{aligned}\tanh^{p-1}\tau:&-p\left(v-a\right)AB-bpAB^3\left\{2p^2+(p-1)(p-2)\right\}\\&-cpAB^5v\Big[2(p-1)(p-2)\left\{p^2+(p-2)^2\right\}+\\&+4p^4+p(p-1)^2(p-2)+p(p+1)^2(p+2)\Big]=0\end{aligned} \tag{14}$$

$$\begin{aligned}&\tanh^{p+1}\tau : p\left(v-a\right)AB + bpAB^3\left\{2p^2 + (p+1)(p+2)\right\} + \\ &cpAB^5 v\Big[2(p+1)(p+2)\left\{p^2 + (p+2)^2\right\} + \\ &+4p^4 + p(p-1)^2(p-2) + p(p+1)^2(p+2)\Big] = 0\end{aligned} \tag{15}$$

$$\begin{aligned}&\tanh^{p+3}\tau : -bpAB^3(p+1)(p+2) + dA^n pnB - \\ &cpAB^5 v(p+1)(p+2)\left\{2p^2 + 2(p+2)^2 + (p+3)(p+4)\right\} = 0\end{aligned} \tag{16}$$

$$\tanh^{p+5}\tau : +cpAB^5 v(p+1)(p+2)(p+3)(p+4) - dA^n pnB = 0 \tag{17}$$

$$\begin{aligned}&\tanh^{p-3}\tau : +bpAB^3(p-1)(p-2) + \\ &cpAB^5 v(p-1)(p-2)\times\left\{2p^2 + 2(p-2)^2 + (p-3)(p-4)\right\} = 0\end{aligned} \tag{18}$$

$$\tanh^{p-5}\tau : -cpAB^5 v(p-1)(p-2)(p-3)(p-4) = 0 \tag{19}$$

To solve Eq. (19), we have considered the following two cases:
Case 1: $p-1=0$.

This yields $p=1$ and therefore $n=5$ following to (12). Further substitution of this value into (14)-(18), respectively, gives

$$v = \frac{a-2bB^2}{1+16cB^4} \tag{20}$$

$$v = \frac{a-8bB^2}{1+136cB^4} \tag{21}$$

and

$$A = \left[-\frac{6bB^2}{5d}\right]^{\frac{1}{4}} \tag{22}$$

so that the solitons will exist for

$$bd < 0 \tag{23}$$

Now, equating the two values of the velocity v from (20) and (21) yields

$$B = \frac{1}{2}\left[\frac{5a}{3b} + \frac{1}{3b}\sqrt{\frac{6b^2 + 25a^2c}{c}}\right]^{\frac{1}{2}} \tag{24}$$

which exist provided that

$$c\left(6b^2+25a^2c\right)>0 \tag{25}$$

Case 2: $p-2=0$

This yields $p=2$ and therefore $n=3$ following to (12). By inserting this value into (14)-(18), we obtain

$$v=\frac{a-8bB^2}{1+188cB^4} \tag{26}$$

$$v=\frac{a-20bB^2}{1+184cB^4} \tag{27}$$

and

$$A=\left[-\frac{3bB^2}{d}\right]^{\frac{1}{2}} \tag{28}$$

which shows that solitons will exist for

$$bd<0 \tag{29}$$

Now, equating the two values of the velocity v from (26) and (27) yields

$$B=\frac{1}{2}\left[\frac{a}{286b}+\frac{1}{286b}\sqrt{\frac{a^2c-6864b^2}{c}}\right]^{\frac{1}{2}} \tag{30}$$

which exist provided that

$$c\left(a^2c-6864b\right)>0 \tag{31}$$

Notice that the third case $p-3=0$ and the fourth case $p-4=0$ are not considered here as it does not give a unique value of the free parameter B.

Lastly, we can determine the topological 1-soliton solution of the generalized R-KdV equation given by (1) when we substitute (20) or (21), (22) and (24) in the soliton ansatz (2) with the respective constraints (23) and (25) for the first case of solution or we substitute (26) or (27), (28) and (30) in the soliton ansatz (2) with the respective constraints (29) and (31) for the second case of solution as

$$u(x,t)=A\tanh^{\frac{4}{n-1}}[B(x-vt)] \tag{32}$$

which exist only when $n=3$ and $n=5$.

3. G'/G METHOD

In this section, we describe the main steps of the G'/G-expansion method for finding traveling wave solutions of nonlinear evolution equations.

3.1. DESCRİPTİON OF THE METHOD

Consider a nonlinear evolution equation, say in two independent variables x and t,

$$P\left(u, u_x, u_t, u_{xx}, u_{tx}, u_{tt}, \cdots\right) = 0. \tag{33}$$

In general, the left-hand side of eq.(33) is a polynomial in u and its various partial derivatives. The main steps of the G'/G-expansion method are:

Step 1: By coordinates transformation

$$u(x,t) = U(\xi), \xi = B(x - kt), \tag{34}$$

eq.(33) can be reduced to an ordinary differential equation (ODE) for $U(\xi)$ with

$$Q(U, U', U'', \cdots) = 0. \tag{35}$$

Step 2: We are looking for the solution of ODE (35) assume that it is expressed as

$$U(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'}{G}\right)^i, \tag{36}$$

where $G(\xi)$ satisfies the second order linear ODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{37}$$

where a_i with $a_N \neq 0$, λ and μ are constants to be determined later, $a_N \neq 0$; and the positive integer N can be determined by using the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (35).

Step 3: Substituting (36) into eq.(35), using (37), the left-hand side of (35) can be converted into a polynomial in G'/G. Equating each coefficient of the polynomial to zero yields a system of algebraic equations for a_i, k, B, μ and λ.

Step 4: Solve the algebraic equations in the Step 3 with the aid of Maple. Substituting the values of a_i, k, B, μ and λ obtained into (36), one can obtain the traveling wave solutions expressed by the hyperbolic, trigonometric and rational functions of eq. (35).

3.2. APPLİCATİON TO R-KdV EQUATİON

In this section we will illustrate the G'/G-expansion method in detail by constructing the traveling wave solutions of the generalized R-KdV equation which is given by

$$u_t + au_x + bu_{xxx} + cu_{xxxxt} + d(u^n)_x = 0, \tag{38}$$

where a, b, c, and d are real valued constants while for the exponent we assume that $n \neq 0,1$. Using the wave transformation

$$u(x,t) = U(\xi), \xi = B(x - kt), \tag{39}$$

and integrating once, the eq. (38) can be changed into the following ODE

$$(a-k)U + bB^2U'' - ckB^4U^{(4)} + dU^n = 0. \tag{40}$$

Considering the homogeneous balance between U^n and $U^{(4)}$ in (40), we obtain $N = \dfrac{4}{n-1}$. To obtain a closed form analytic solution we use a transformation formula $U = V^{\frac{4}{n-1}}$ that transforms eq. (40) to

$$\begin{aligned}&(a-k)(n-1)^4V^4 + d(n-1)^4V^8 + 4bB^2(5-n)(n-1)^2V^2(V')^2 \\ &+8ckB^4(n-5)(n-3)(3n-7)(V')^4 + 4bB^2(n-1)^3V^3V'' \\ &+48ckB^4(5-n)(n-1)(n-3)V(V')^2V'' + 12ckB^4(n-5)(n-1)^2V^2(V'')^2 \\ &+16ckB^4(n-5)(n-1)^2V^2V'V''' - 4ckB^4(n-1)^3V^3V^{(4)} = 0\end{aligned} \tag{41}$$

Balancing $V^3V^{(4)}$ with V^8 in eq. (41) yields $N = 1$. Therefore, the solution of eq. (41) is of the form:

$$V(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \tag{42}$$

where a_0, a_1 are constants and $a_1 \neq 0$.

Substituting (42) together with (37) into (41) and collecting all terms with the same power of $\dfrac{G'}{G}$ and setting each coefficient of the resulted polynomial to zero, we obtain a set of algebraic equations for a_0, a_1, B, k, d, b, λ and μ.

Solving the system of algebraic equations with the aid of Maple, we obtain the following solution:

$$a_0 = \frac{1}{2} a_1 \lambda, \;\; a_1 = a_1. \;\; k = a \tag{43}$$

$$B = B, d = \frac{8caB^4 (n+3)(3n+1)(n+1)}{a_1^4 (n-1)^4},$$

$$b = 0, \;\; \mu = \frac{1}{4}\lambda^2.$$

Substituting (43) into (42) gets the following traveling wave solution of the this equation.

Because $\lambda^2 - 4\mu = 0$ then we have the following solution:

$$V(\xi) = -\frac{a_1 c_2}{c_1 + c_2 \xi}, \tag{44}$$

and

$$U(\xi) = \left(-\frac{a_1 c_2}{c_1 + c_2 \xi} \right)^{\frac{1}{m}}, \tag{45}$$

where in (44) and (45) $\xi = B(x - at)$.

4. EXP-FUNCTION METHOD

In this section, we review the expfunction method for finding the exact solutions of nonlinear partial differential equations.

4.1. DESCRİPTİON OF THE METHOD

Suppose that the solution of eq. (35) can be expressed in the following form:

$$U(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}. \tag{46}$$

In order to determine the values of c and p, we balance the linear term of the highest order in (35) with the highest order nonlinearity. Similarly, to determine the values of d and q, we balance the linear term of the lowest order in (35) with the lowest order nonlinear term. Substituting solution (46) into eq. (35) yields a set of algebraic equations for $\exp(\xi)$ then all coefficients of $\exp(\xi)$ have to vanish. After this separated algebraic equation, we can find a_n and b_m constants.

4.2. APPLİCATİON TO R-KdV EQUATİON

As in the aforementioned method using, and the transformation $U = V^{\frac{4}{n-1}}$ eq. (38) becomes

$$(a-k)(n-1)^4 V^4 + d(n-1)^4 V^8 + 4bB^2(5-n)(n-1)^2 V^2 (V')^2 \tag{47}$$
$$+8ckB^4(n-5)(n-3)(3n-7)(V')^4 + 4bB^2(n-1)^3 V^3 V''$$
$$+48ckB^4(5-n)(n-1)(n-3)V(V')^2 V'' + 12ckB^4(n-5)(n-1)^2 V^2 (V'')^2$$
$$+16ckB^4(n-5)(n-1)^2 V^2 V'V''' - 4ckB^4(n-1)^3 V^3 V^{(4)} = 0.$$

Balancing the highest order of expfunction in

$$V^{(4)}V^3 = \frac{c_1 \exp[(4c+15p)\xi] + \cdots}{c_2 \exp[19p\xi] + \cdots}, \tag{48}$$

with that in the nonlinear term

$$V^8 = \frac{c_3 \exp[(8c+11p)\xi] + \cdots}{c_4 \exp[19p\xi] + \cdots}, \tag{49}$$

we have $4c+15p = 8c+11p$, and this gives $p = c$, where c_i are determined coefficients only for simplicity. Now, balancing the lowest order of expfunction in

$$V^{(4)}V^3 = \frac{\cdots + d_1 \exp[-(4d+15q)\xi]}{\cdots + d_2 \exp[19q\xi]}, \tag{50}$$

with that in the nonlinear term

$$V^8 = \frac{\cdots + d_3 \exp[-(8d+11q)\xi]}{\cdots + d_4 \exp[-19q\xi]}, \tag{51}$$

we have $4d+15q = 8d+11q$, and this gives $d = q$, where d_i are determined coefficients only for simplicity. Choosing $p = c = 1$ and $q = d = 1$, eq. (46) becomes

$$V(\xi)=\frac{a_{-1}\exp(-\xi)+a_0+a_1\exp(\xi)}{b_{-1}\exp(-\xi)+b_0+b_1\exp(\xi)}. \tag{52}$$

Substituting the eq. (52) into eq. (47), and equating to zero the coefficients of all powers of $\exp(n\xi)$ yields a set of algebraic equations for $a_0, b_0, a_1, a_{-1}, b_{-1}, b_1$, a, B and k. Solving the system of algebraic equations by the help of Maple, we obtain:

$$a_1=0,\ a_0=a_0,\ a_{-1}=0,\ b_0=0,\ b_1=b_1, \tag{53}$$

$$b_{-1}=b_{-1},\ k=k,\ B=\pm\frac{\sqrt{2}a_0^2(n-1)A}{8b_1b_{-1}b(n+3)(3n+1)(n+1)}$$

$$k=\frac{8b_{-1}^2b_1^2b^2(n+3)(3n+1)(n+1)}{cda_0^2(n^2+2n+5)^2}$$

$$a=\frac{-d^2a_0^2c(n+1)(n^2+2n+5)^2+16b_1^4b_{-1}^4b^2(n+1)(n+3)^2(3n+1)^2}{2cda_0^4b_1b_{-1}^2(n^2+2n+5)(3n^2+10n+3)}.$$

where $A=\sqrt{bd(n+3)(3n+1)(n+1)(n^2+2n+5)}$.

Thus, by substituting the solutions (53) into eq. (52), the exact traveling wave solution to (47) is

$$V(\xi)=\frac{a_0}{b_{-1}e^{-\xi}+b_1e^{\xi}}, \tag{54}$$

and

$$U(\xi)=\left(\frac{a_0}{b_{-1}e^{-\xi}+b_1e^{\xi}}\right)^{\frac{4}{n-1}}, \tag{55}$$

where $\xi=B(x-kt)$ in (54) and (55).

5. CONCLUSIONS

This paper integrates the R-KdV equation by the aid of three methods of integration. They are the solitary wave ansatz method, the G'/G-expansion method as well as the exp-function method of integration. These lead to several solutions to this equation of study. These solutions are going to be extremely useful in further analysis of this equation.

In future, this equation is going to be further analyzed. The Lie symmetry approach is going to be used to obtain several conserved quantities of this equation. Subsequently, the soliton perturbation theory will be used to obtain the adiabatic parameter dynamics of the solitary waves just as this was studied in 2012 [4]. The quasi-stationary solitary waves, in presence of perturbation terms will be obtained. Finally, the stochastic perturbation terms will be taken into consideration. That will also lead to the study of the mean free velocity of the solitary waves. These results will be reported in future publications.

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