INTEGRALS FOR TIME-DEPENDENT COMPLEX DYNAMICAL SYSTEM IN ONE DIMENSION

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Construction of exact integrals for dynamical system on an Extended Complex Phase Space (ECPS) of second order in one-dimension has carried out. To achieve this we use Lie algebraic method to study complex systems on the extended complex phase plane characterized by $x = x_1 + ip_2$, and $p = p_1 + ix_2$. Such integrals play an important role in the analysis of complex trajectories of quantum systems.

Key words: Exact complex integrals, $\mathcal{PT}$-symmetry, complex Hamiltonian.

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1. INTRODUCTION

The role of integrals of motion in physics can hardly be overestimated in the domains of a variety of fields like plasma physics, laser physics, accelerator physics etc [1]. Complex Hamiltonians are expected to admit complex integrals but real Hamiltonians are also found to admit complex integrals, *e.g.* simple harmonic oscillator system possesses a complex integral, namely $u = ln(p + im\omega x) - i\omega t$ [2]. Since the complex Hamiltonians have been in practice for a long time to study many physical systems such as the optical model of nucleus. Also non-hermitian $\mathcal{PT}$-symmetric complex Hamiltonians are used to study delocalization of transitions in condensed matter systems such as vortex flux line depinning in type-II superconductors [3] , to study population biology [4] , study of complex trajectories particularly in laser physics [5] etc. Kaushal *et al.* [6, 7] has investigated the construction of complex integrals of one dimensional complex Hamiltonian systems. Although there are several schemes for complexification of Hamiltonian $H(x,p)$ [8] however, in the present work we follow the approach, used by Xavier and de Aguiar and Kaushal [5, 9] to develop an algorithm for computation of semiclassical coherent-state propagator for a particle going through a simple barrier potential, which is given by the expression as $x = x_1 + ip_2, \quad p = p_1 + ix_2$; for canonical variables x and p in one dimension. In fact the above transformation makes both x and p separately complex by extending each of them to the corresponding complex planes, *i.e.* inserting an imaginary component in each. An important aspect of such a characterization is the manifestation of (x_1, p_1) and (x_2, p_2) as canonical pairs which

in turn provides a link between the complex Hamiltonian and a pair of real Hamiltonians and can be helpful in establishing the integrability of $H(x,p)$. It is worth to mention that the such transformations have been successfully used for solving Schrödinger equation for a large number of one dimensional complex potentials [9]. Also the $\mathcal{PT}$-symmetry of non-Hermitian $H(x,p)$ appears to be a special case of the general transformation under certain limits (in the sense that under $\mathcal{PT}$-symmetry this transformation reduces to a restriction on the variables (x_1,p_1,x_2,p_2), such that $(x_1,p_1,x_2,p_2) \rightarrow (-x_1,p_1,-x_2,p_2; i \rightarrow -i)$ [10, 11].
The organization of the paper is as follows: in next section, the method of complexification and construction of integrals of dynamical systems is described. In section 2 and section 3 we apply the formalism adapted in the section 1 to obtain a complex integral of a dynamical system and finally concluding remarks are given in section 4.

2. CONSTRUCTION OF COMPLEX INTEGRALS

Consider a one-dimensional real phase space (x,p), which can be transform into a complex space (x_1,x_2,p_1,p_2), as

$$x = x_1 + ip_2; \quad p = p_1 + ix_2, \tag{1}$$

Therefore, the Hamiltonian $H(x,p)$ of a one dimensional system in complex space can be expressed, using eq. (2), as $H = H_1(x_1,x_2,p_1,p_2) + iH_2(x_1,x_2,p_1,p_2)$. Clearly, from eq.(1) we get

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_2}; \qquad \frac{\partial}{\partial p} = \frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_2}. \tag{2}$$

Now consider a complex phase space function $I(x,p,t)$ as

$$I = I_1(x_1,x_2,p_1,p_2,t) + iI_2(x_1,x_2,p_1,p_2,t) \tag{3}$$

Thus for function I to be the time dependent dynamical integral of the system in complex phase space, then we write the invariance condition as

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I,H] = 0, \tag{4}$$

where $[.,.]$ is the Poisson bracket (PB), which in view of the definition eq.(1) turns out as

$$[A,B]_{(x,p)} = [A,B]_{(x_1,p_1)} - i[A,B]_{(x_1,x_2)} - i[A,B]_{(p_2,p_1)} - [A,B]_{(p_2,x_2)} \tag{5}$$

which indicates that the computation of Poisson bracket in case of complex Hamiltonian systems becomes a bit tedious. With a view to demonstrate the underlying elegance of the Lie-algebraic approach [12, 13] at the classical level, we briefly describe this in order to construct complex integrals of the dynamical systems. In the

Lie-algebraic approach, one can express the complex Hamiltonian $H(x,p,t)$ of the system as

$$H = \sum_n h_n(t)\Gamma_n(x_1,x_2,p_1,p_2,t), \tag{6}$$

where the set of functions $\{\Gamma_1,....,\Gamma_n\}$ are not explicitly time dependent and $h_n(t)$ are complex coefficient functions of time. The Γ_n's in eq.(6) generate a closed dynamical algebra, implies

$$[\Gamma_n,\Gamma_m] = \sum_l C^l_{nm}\Gamma_l, \tag{7}$$

where C^l_{nm} are the complex structure constants of the algebra. If the Γ_n's in eq.(6) are not sufficient to close the algebra then the set of Γ_n must be extended by adding new Γ_l's, such that $\Gamma_l = [\Gamma_n,\Gamma_m]$, until the closure is obtained along with additional $h_l(t)$'s which are taken to be zero. Since the complex dynamical integral I is also a part of Lie algebra, then one can express this as

$$I(t) = \sum_k \lambda_k(t)\Gamma_k(x_1,x_2,p_1,p_2), \tag{8}$$

where $\lambda_k(t)$'s are time dependent complex coefficients. Thus by using eq.(6) and eq.(7) for H and I respectively in eq.(5), we get a system of linear, first order differential equations, namely

$$\dot{\lambda}_r + \sum_n \left[\sum_m C^r_{nm}h_m(t)\right]\lambda_n = 0, \tag{9}$$

in λ_n's. Therefore, the solutions of these differential equations in turn provide classical complex integrals of a given system from eq.(9). In the next section we will use the prescription given in present section to obtain complex integrals of a classical complex Hamiltonian system.

3. MOMENTUM DEPENDENT HARMONIC OSCILLATOR

With a view to constructing complex integrals for some cases here, in this sections we make the use of methods discussed in previous section. To start with we first consider the case of coupled harmonic oscillator systems with in the frame work of Lie-algebraic method. Consider a momentum dependent coupled harmonic oscillator in one-dimension, whose Hamiltonian is given by

$$H = \frac{1}{2}p^2 + \frac{1}{2}a_0(t)x^2 + a_1(t)xp \tag{10}$$

Where $a_0(t)$ and $a_1(t)$ are time-dependent co-efficient. We demonstrate that the complex version of (10), namely the $\mathcal{PT}$-symmetric one obtained by using (1) in

(10), the above Hamiltonian can be expressed as

$$H = \frac{1}{2}p_1^2 - \frac{1}{2}x_2^2 + ip_1x_2 + a_0(t)[\frac{1}{2}x_1^2 + ip_2x_1 - \frac{1}{2}p_2^2] + a_1(t)[x_1p_1 - p_2x_2 + ip_1p_2 + ix_1x_2] = \sum_{m=1}^{10} h_m(t)\Gamma_m(x_1, x_2, p_1, p_2, t). \quad (11)$$

and the various Γ's and $h(t)'s$ for the above complex H are given as

$$\Gamma_1 = \frac{p_1^2}{2}; \quad \Gamma_2 = \frac{x_2^2}{2}; \quad \Gamma_3 = p_1x_2; \quad \Gamma_4 = \frac{x_1^2}{2}; \quad \Gamma_5 = \frac{p_2^2}{2}; \quad \Gamma_6 = p_2x_1; \quad \Gamma_7 = x_1p_1; \quad \Gamma_8 = p_2x_2; \quad \Gamma_9 = p_1p_2; \quad \Gamma_{10} = x_1x_2.$$

with

$$h_1 = -h_2 = 1;\ h_3 = i;\ h_4 = ih_5 = -h_6 = a_0;\ h_7 = -h_8 = a_1;\ h_9 = h_{10} = ia_1. \quad (12)$$

The dynamical algebra in this case is not closed. To find closure property for the above system, we add four more phase space functions Γ_l's. The additional Γ_l's are as follow

$$\Gamma_{11} = p_1; \qquad \Gamma_{12} = x_2; \qquad \Gamma_{13} = x_1; \qquad \Gamma_{14} = p_2. \quad (13)$$

with corresponding $h_l(t) = 0$. Now in the light of Poisson bracket for complex systems, eq.(5), we get large number of 69 nos nonvanishing Poisson brackets.
Therefore, use of these 69 nos PB in eq.(9) yields the following set of Partial differential equations in λ's as:

$$\dot{\lambda}_1 = 4a_1(\lambda_1 - i\lambda_3) + 4(i\lambda_9 - \lambda_7), \quad (14)$$

$$\dot{\lambda}_2 = 4a_1(\lambda_2 + i\lambda_3) - 4(\lambda_8 + i\lambda_{10}), \quad (15)$$

$$\dot{\lambda}_3 = 2a_1(i\lambda_1 - i\lambda_2 + 2\lambda_3) - 2i\lambda_7 + 2i\lambda_8 - 2\lambda_{10} - 2\lambda_9, \quad (16)$$

$$\dot{\lambda}_4 = 4a_1(-\lambda_4 + i\lambda_6) - 4a_0(i\lambda_{10} - \lambda_7), \quad (17)$$

$$\dot{\lambda}_5 = -4a_1(\lambda_5 + i\lambda_6) + 4a_0(\lambda_8 + i\lambda_9), \quad (18)$$

$$\dot{\lambda}_6 = -2a_1(\lambda_4 - i\lambda_5 + 2i\lambda_6) - 2a_0(\lambda_8 + i\lambda_{10} - i\lambda_9 + i\lambda_7), \quad (19)$$

$$\dot{\lambda}_7 = 2a_0(\lambda_1 - i\lambda_3) - 2a_1(i\lambda_{10} - i\lambda_9 + i\lambda_7) - 2\lambda_4 + 2i\lambda_6, \quad (20)$$

$$\dot{\lambda}_8 = 2a_0(\lambda_2 - i\lambda_3) - 2\lambda_5 + 2i\lambda_6 - 2ia_1(\lambda_8 + \lambda_{10} - \lambda_9), \quad (21)$$

$$\dot{\lambda}_9 = 2a_0(i\lambda_1 + \lambda_3) + 2i\lambda_5 - 2\lambda_6 - 2a_1(i\lambda_7 + \lambda_9 - i\lambda_8), \quad (22)$$

$$\dot{\lambda}_{10} = -2a_0(i\lambda_2 + \lambda_3) - 2i\lambda_4 + 2\lambda_6 + 2ia_1(\lambda_7 + \lambda_8 + i\lambda_{10}), \quad (23)$$

$$\dot{\lambda}_{11} = 2a_1\lambda_{11} - 2\lambda_{13} + 2\lambda_{14}, \quad (24)$$

$$\dot{\lambda}_{12} = 2ia_1\lambda_{11} - 2i\lambda_{13} + 2i\lambda_{14}, \quad (25)$$

$$\dot{\lambda}_{13} = 2a_0(-i\lambda_{12} + \lambda_{11}) - 2a_1(\lambda_{13} - i\lambda_{14}), \tag{26}$$

$$\dot{\lambda}_{14} = 2a_0(\lambda_{12} + i\lambda_{11}) + a_1(\lambda_{14} - i\lambda_{13}). \tag{27}$$

In fact, to solve these fourteen coupled PDEs for complex λ's is very difficult. Thus, here we make certain choices for λ's which facilitate to find solutions of above equations.

From eqs.(14), (15) and (16), we get

$$\dot{\lambda}_3 = i\dot{\lambda}_1 - i\dot{\lambda}_2;$$

and if we $\lambda_3 = c_3$ (a constant, say) consider,then $\dot{\lambda}_3 = 0$, which immediately gives

$$\lambda_1 = \eta(t) + c_1, \quad \lambda_2 = \eta(t) + c_2, \quad \lambda_3 = c_3. \tag{28}$$

where c_1 and c_2 , c_3 are complex integration constants. Again from eqs.(17), (18) and (19) , we obtain

$$2\dot{\lambda}_6 = \dot{\lambda}_4 - \dot{\lambda}_5. \tag{29}$$

and if we $\lambda_6 = c_6$ (a constant, say) consider,then $\dot{\lambda}_6 = 0$, which immediately gives

$$\lambda_4 = \xi(t) + c_4, \quad \lambda_5 = \xi(t) + c_5, \quad \lambda_6 = c_6. \tag{30}$$

where c_4 and c_5 , c_6 are complex integration constants. Now, in order to find solutions for λ_7 , λ_8, λ_9 and λ_{10} subtract eq.(21) from (20),we find

$$\dot{\lambda}_7 - \dot{\lambda}_8 = 2a_0(\lambda_1 - \lambda_2) - 2ia_0(-\lambda_7 + \lambda_8) - 2(\lambda_4 + \lambda_5). \tag{31}$$

and using solutions for λ_1, λ_2, λ_4 and λ_5 in eq.(31) and again if we set $\lambda_7 = \lambda_8 = \phi(t)$ we have

$$\dot{\lambda}_7 - \dot{\lambda}_8 = 0.$$

which gives

$$\lambda_7 = \phi(t) + c_7, \qquad \lambda_8 = \phi(t) + c_8. \tag{32}$$

Here $\phi(t)$ is another arbitrary complex function of time and c_7 and c_8 are complex constants. In the same process, adding eq.(22) and eq.(23),

$$\dot{\lambda}_9 + \dot{\lambda}_{10} = 2ia_0(\lambda_1 - \lambda_2) + 2i(\lambda_5 - \lambda_4) - 2a_1(\lambda_9 + \lambda_{10}). \tag{33}$$

and then with the help of eq.(28) and (30), we get

$$\lambda_9 = \psi(t) + c_9, \qquad \lambda_{10} = -\psi(t) + c_{10}. \tag{34}$$

where, $\phi(t)$ is an arbitrary function of time and c_9 and c_{10} are complex constants.

Now for finding the solutions of λ_{11}, λ_{12}, λ_{13} and λ_{14}, we observe from eq.(24) to eq.(27), that

$$\dot{\lambda}_{11} = i\dot{\lambda}_{12}, \qquad \dot{\lambda}_{13} = i\dot{\lambda}_{14}.$$

which in turns provide

$$\lambda_{11} = \varphi(t) + c_{11}; \;\; \lambda_{12} = \varphi(t) + c_{12}; \; \lambda_{13} = \chi(t) + c_{13}; \;\; \lambda_{14} = \chi(t) + c_{14}. \tag{35}$$

We have solved eqs.[(14) to (27)] in terms of arbitrary functions $\eta, \xi, \phi, \psi, \varphi$ and χ and complex constants, c_i's, $(i = 1, ..., 14)$. If one put back these solutions for λ_i, $(i = 1, ..., 14)$ in eqs.[(14) to (27)], we obtain a number of constraint relations among c_i's, and $\eta, \xi, \phi, \psi, \varphi$ and χ, which limit the choices of these arbitrary complex quantities (for simplicity we set all the c_i's equal to zero). These relations determining arbitrary functions functions $\eta, \xi, \phi, \psi, \varphi$ and χ are written as

$$\ddot{\eta} - 4(a_1\dot{\eta} + i\dot{\psi} - \dot{\phi}) = 0 \ , \ \ddot{\xi} - 4[-a_1\dot{\xi} + a_0(\dot{\phi} - i\dot{\psi})] = 0, \ \ddot{\chi} - 2\dot{\chi} = 0;$$
$$\ddot{\phi} - 2(a_0\dot{\eta} - \dot{\xi} - ia_1\dot{\phi}) = 0 \ , \ \ddot{\psi} - 2i(a_0\dot{\eta} + \dot{\xi} + ia_1\dot{\psi}) = 0 \ , \ \ddot{\varphi} - 2\dot{\varphi} = 0. \quad (36)$$

Therefore, after substituting the solutions of λ_i's in the eq.(8), the complex integral for a two dimensional complex oscillator becomes

$$I = \frac{1}{2}\eta(p_1^2 + x_2^2) + \frac{1}{2}\xi(x_1^2 + p_2^2) + \phi(x_1p_2 + x_2p_2) + \psi(p_1p_2 - x_1x_2) + \varphi(p_1 + ix_2) + \chi(x_1 + ip_2).$$

which conforms to condition eq.(4) in view of the PB of eq.(5).

4. SHIFTED HARMONIC OSCILLATOR

Consider the case of a shifted harmonic oscillator in x-plane, for which the Hamiltonian is written as

$$H = \frac{1}{2}p^2 + \frac{1}{2}k_0(t)x^2 + k_1(t)x. \quad (37)$$

Where k_0 and k_1 are function of t or time dependent. By using Lie-algebra formalism the various $\Gamma's$ and $h(l)'s$ for the above complex H are expressed as

$$\Gamma_1 = \frac{p_1^2}{2}; \Gamma_2 = \frac{x_2^2}{2}; \Gamma_3 = p_1x_2; \Gamma_4 = x_1p_2; \Gamma_5 = \frac{x_1^2}{2}; \Gamma_6 = \frac{p_2^2}{2}; \Gamma_7 = x_1; \Gamma_8 = p_2;$$
$$h_1 = -h_2 = 1; \ \ h_3 = ih_4 = k_0; \ \ h_5 = -h_6 = k_0; \ \ h_7 = ih_8 = k_1. \quad (38)$$

The dynamical algebra in this case is not closed unless one adds six more phase space functions (Γ_l)'s. The additional (Γ_l)'s and their corresponding h_l's, are given as

$$\Gamma_9 = p_1p_2; \ \Gamma_{10} = p_1x_1; \ \ \Gamma_{11} = p_1; \ \Gamma_{12} = x_2; \ \Gamma_{13} = x_1x_2; \ \Gamma_{14} = p_2x_2. \quad (39)$$

with corresponding $h(l)'s = 0$. Now in the light of modified definition of Poisson bracket for complex systems eq.(5), we get 57 number of nonvanishing Poisson brackets.

Therefore using this 57 number of nonvanishing PB's in eq.(9) we obtained a set of 14 partial differential equations. As such solution of these 14 coupled partial differential equations for complex λ's is solved systematically by some ansatze for λ's as

done in pervious example.
Solutions of λ's so obtained are as

$$\lambda_1 = \lambda_2 = \chi(t); \qquad \lambda_3 = \lambda_4 = \text{Constant}; \qquad \lambda_5 = \lambda_6 = \phi(\text{t});$$
$$\lambda_7 = -i\lambda_8 = \beta(t); \qquad \lambda_9 = -\frac{i}{8}(\dot{\xi} - 8\sigma); \qquad \lambda_{10} = -\frac{1}{8}(\dot{\xi} + 8\sigma);$$
$$\lambda_{11} = -i\lambda_{12} = \alpha(t); \qquad \lambda_{13} = \frac{i}{8}(\dot{\xi} - 8\sigma); \qquad \lambda_{14} = -\frac{1}{8}(\dot{\xi} + 8\sigma). \quad (40)$$

If one put back these solutions for λ_i, $(i = 1, ..., 14)$ in PDE's derived, we obtain a number of constraint relations among c_i's, and ξ, σ, α and β, which limit the choices of these arbitrary complex quantities (for simplicity we set all the c_i's equal to zero). So relations determining arbitrary are written as

$$\ddot{\xi} + 16\xi = 0; \; \ddot{\sigma} + 16\sigma = 0; \; \dot{\alpha} - 4\xi = 0; \; \dot{\beta} + 4\sigma = 0. \quad (41)$$

Therefore, after substituting the solutions of λ_i's in the eq.(8), the final form of the integral for a complex shifted harmonic oscillator becomes

$$I = \frac{\chi(p_1^2 + x_2^2)}{2} + \frac{\phi(x_1^2 + p_2^2)}{2} + \beta(x_1 - ip_2) + \frac{i(\dot{\xi} - 8\sigma)}{8}(x_1x_2 - p_1p_2)$$
$$- \frac{(\dot{\xi} + 8\sigma)}{8}(x_1p_1 + x_2p_2) + \alpha(p_1 - ix_2).$$

which conforms to condition eq.(4) in view of the PB in eq.(5).

5. CONCLUSION

In this work, a modest attempt has been made to obtain exact complex second constant of motion (integrals) of a one dimensional complex dynamical systems on an extended complex phase space characterized by eq.(1). The transformations (1) (or eq.(1)) had been a part of many studies [6–8] and can give $\mathcal{PT}$-symmetric Hamiltonians under certain boundary conditions. As the integrals of real Hamiltonian systems have been played a vital role in understanding the underlying dynamics of the systems and in the same spirit, we hope that the complex integrals could also be helpful in exploring some deep insight into features of complex dynamical systems including the real one.

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