VARIATIONAL CALCULUS WITH FRACTIONAL AND CLASSICAL DERIVATIVES

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This paper presents necessary and sufficient optimality conditions of Euler-Lagrange type for fractional variational problems with functionals containing classical derivatives and right or left fractional derivatives in both Riemann-Liouville and Caputo senses. We use, as variational functionals, right and left fractional integrals instead of the classical integral.

Key words: Fractional integral, fractional derivative, fractional calculus of variations.

1. INTRODUCTION

Fractional calculus is one of the generalizations of the classical calculus. Several fields of application of fractional differentiation and fractional integration are already well established, some others have just started. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, etc. (see [12, 14, 15, 19, 20, 22–24, 33]).

Integer variational calculus play a significant role in many areas of science, engineering and applied mathematics. In recent years, there has been a growing interest in the area of fractional variational calculus and its applications which include classical and quantum mechanics, field theory, and optimal control (see [1-11,13-18,21,25-32,34-36]).

In the papers cited above, the problems have been formulated mostly in terms of two types of fractional derivative, namely Riemann-Liouville (RL) and Caputo derivatives.

The necessary optimality conditions for problems of calculus of variations with a Lagrangian containing both classical and fractional Riemann derivative is given in [30].

In [16] the fractional variational problems with fractional integral and fractional derivative in the sense of Riemann-Liouville and the Caputo derivatives were discussed and the fractional Euler-Lagrange equations with the natural boundary conditions were given.

In almost all papers the classical theory is not given as a special case but as a limit, when $\alpha \to 1$, which is not clear as a special case because, as known, Riemann derivative is put as y' in definition but not equal y' as a limit. In [30, 31] the authors give the classical theory as a special case not as a limit.

Here we develop the theory of fractional variational calculus further by proving the necessary and sufficient optimality conditions, for more general problems of the fractional calculus of variations with a fractional integral and a Lagrangian that contains classical and fractional derivatives in both Riemann–Liouville and Caputo senses.

We consider for $\alpha \in (0,1)$ and $\gamma \in \mathbb{R}^+$ the following four types of fractional variational calculus

$$J(y) = {}_aI_x^\gamma L(t, y(t), {}_a^CD_t^\alpha y(t), y'(t)), \quad y(a) = y_a, \tag{1}$$

$$J(y) = {}_xI_b^\gamma L(t, y(t), {}_t^CD_b^\alpha y(t), y'(t)), \quad y(b) = y_b, \tag{2}$$

$$J(y) = {}_aI_x^\gamma L(t, y(t), {}_a^RD_t^\alpha y(t), y'(t)), \quad y(a) = y_a, \tag{3}$$

$$J(y) = {}_xI_b^\gamma L(t, y(t), {}_t^RD_b^\alpha y(t), y'(t)), \quad y(b) = y_b. \tag{4}$$

we find in each of these the necessary and sufficient optimality conditions at any arbitrary fixed $x \in [a,b]$. This type of Lagrangian is used to describe the systems involving dissipation.

This paper is organized as follows: In section 2, we give the principal definitions used in this paper. In section 3 Necessary optimality conditions are proved for the problems (1) - (4) at any fixed x with giving some special cases which prove the generalization of our problems. Sufficient conditions are given in section 4 and an example is given in section 5 to illustrate our main results. Finally our conclusion is given in section 6.

2. PRELIMINARIES

Here we give the standard definitions of left and right Riemann-Liouville fractional integral, Riemann-Liouville fractional derivatives and Caputo fractional derivatives (see [20, 24, 33, 37]).

Definition 1: If $f(t) \in L^1(a,b)$, the set of all integrable functions, and $\alpha > 0$ then the left and right Riemann-Liouville fractional integral of order α, denoted respectively by ${}_aI_t^{\alpha}$ and ${}_tI_b^{\alpha}$, are defined by

$$ {}_aI_t^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau, $$
$$ {}_tI_b^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}f(\tau)d\tau. $$

Definition 2: For $\alpha > 0$ the left and right Riemann-Liouville fractional derivative of order α, denoted respectively by ${}_a^RD_t^{\alpha}$ and ${}_t^RD_b^{\alpha}$, are defined by

$$ {}_a^RD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}D^n\int_a^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau, $$
$$ {}_t^RD_b^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}(-D)^n\int_t^b (\tau-t)^{n-\alpha-1}f(\tau)d\tau, $$

where n is such that $n-1<\alpha<n$ and $D=\frac{d}{dt}$
If α is an integer, these derivatives are defined in the usual sense

$$ {}_a^RD_t^{\alpha} := D^{\alpha}, \ \ {}_t^RD_b^{\alpha} := (-D)^{\alpha}, \ \ \alpha = 1,2,3,... $$

Definition 3: For $\alpha > 0$ the left and right Caputo fractional derivative of order α, denoted respectively by ${}_a^CD_t^{\alpha}$ and ${}_t^CD_b^{\alpha}$, are defined by

$$ {}_a^CD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}D^nf(\tau)d\tau, $$
$$ {}_t^CD_b^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}(-D)^nf(\tau)d\tau, $$

where n is such that $n-1<\alpha<n$ and $D=\frac{d}{d\tau}$
If α is an integer, then these derivatives takes the ordinary derivatives

$$ {}_a^CD_t^{\alpha} = D^{\alpha}, \quad {}_t^CD_b^{\alpha} = (-D)^{\alpha}, \quad \alpha = 1,2,3,\ldots $$

3. NECESSARY OPTIMALITY CONDITIONS

To develop the necessary conditions for the extremum for (1), assume that $y^*(t)$ is the desired function, which makes the value of the given functional a minimum or maximum at any arbitrary fixed $x\in[a,b]$, let $\epsilon\in\mathbb{R}$, and define a family of curves $y(t)=y^*(t)+\epsilon\eta(t)$, where $\eta\in C^1[a,b]$, since ${}_a^CD_t^{\alpha}$ is a linear operator then we get

$$ J(\epsilon)=\int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}L(t,y^*(t)+\epsilon\eta(t),{}_a^CD_t^{\alpha}y^*(t)+\epsilon\,{}_a^CD_t^{\alpha}\eta(t),y^{*\prime}(t)+\epsilon\eta'(t))dt, \quad (5) $$

which has an extremum at $\epsilon = 0$. Thus, by differentiating both sides with respect to ϵ and set $\frac{dJ}{d\epsilon} = 0$ we get

$$\int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \left[\frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial\, {}_a^C D_t^\alpha y}\, {}_a^C D_t^\alpha \eta + \frac{\partial L}{\partial y'}\eta' \right] dt = 0. \tag{6}$$

But we have (by integration by parts) that

$$\int_a^x \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_a^C D_t^\alpha y}\, {}_a^C D_t^\alpha \eta \right) dt = \left(\eta\, {}_t I_x^{1-\alpha} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_a^C D_t^\alpha y} \right) \right) \Bigg|_a^x + \int_a^x \eta\, {}_t^R D_x^\alpha \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_a^C D_t^\alpha y} \right) dt \tag{7}$$

and

$$\int_a^x \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'}\eta' \right) dt = \left(\eta \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right) \right) \Bigg|_a^x - \int_a^x \eta \frac{d}{dt} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right) dt. \tag{8}$$

Substituting in (6) we get

$$\begin{aligned} &\int_a^x \eta(t) \left[\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y} + {}_t^R D_x^\alpha \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_a^C D_t^\alpha y} \right) - \frac{d}{dt} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right) \right] dt \\ &+\eta(x) \left[\left({}_t I_x^{1-\alpha} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_a^C D_t^\alpha y} \right) \right) \Bigg|_{t=x} + \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right) \Bigg|_{t=x} \right] \\ &-\eta(a) \left[\left({}_t I_x^{1-\alpha} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_a^C D_t^\alpha y} \right) \right) \Bigg|_{t=a} + \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right) \Bigg|_{t=a} \right] = 0 \end{aligned} \tag{9}$$

Since $y(a) = y_a$ we get that $\eta(a) = 0$ and because $\eta(t)$ is arbitrary, it follows that the Euler-Lagrange equation is

$$\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y} + {}_t^R D_x^\alpha \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_a^C D_t^\alpha y} \right) - \frac{d}{dt} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right) = 0 \tag{10}$$

with the natural boundary condition (transversality conditions)

$$\left({}_t I_x^{1-\alpha} \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_a^C D_t^\alpha y} \right) \right) \Bigg|_{t=x} + \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right) \Bigg|_{t=x} = 0 \tag{11}$$

Thus we prove that:

Theorem 1: Let $J(y)$ be a functional of the form (1) defined on the set of functions $y \in C^1[a,b]$ and satisfy the boundary condition $y(a) = y_a$. Then a necessary condition for $J(y)$ to have an extremum for a given function $y(t)$ is that $y(t)$ satisfies the fractional Euler-Lagrange equation (10) with the natural boundary condition (transversality condition) (11).

Similar to the proof of Theorem 1 we can prove the following theorems:

Theorem 2: Let $J(y)$ be a functional of the form (2) defined on the set of functions $y \in C^1[a,b]$ and satisfy the boundary condition $y(b) = y_b$. Then a necessary condition for $J(y)$ to have an extremum for a given function $y(t)$ is that $y(t)$ satisfies the fractional Euler-Lagrange equation

$$\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L}{\partial y} + {}_x^R D_t^\alpha \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_t^C D_b^\alpha y} \right) - \frac{d}{dt}\left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right) = 0 \quad (12)$$

with the natural boundary condition (transversality condition)

$$\left({}_x I_t^{1-\alpha} \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_t^C D_b^\alpha y} \right) \right)\Bigg|_{t=x} + \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right)\Bigg|_{t=x} = 0. \quad (13)$$

Theorem 3: Let $J(y)$ be a functional of the form (3) defined on the set of functions $y \in C^1[a,b]$ and satisfy the boundary condition $y(a) = y_a$. Then a necessary condition for $J(y)$ to have an extremum for a given function $y(t)$ is that $y(t)$ satisfies the fractional Euler-Lagrange equation

$$\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L}{\partial y} + {}_t^C D_x^\alpha \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_a^R D_t^\alpha y} \right) - \frac{d}{dt}\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right) = 0 \quad (14)$$

with the natural boundary condition (transversality conditions

$$\left(\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_a^R D_t^\alpha y} \right) \right)\Bigg|_{t=x} + \left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right)\Bigg|_{t=x} = 0 \quad (15)$$

Theorem 4: Let $J(y)$ be a functional of the form (4) defined on the set of functions $y \in C^1[a,b]$ and satisfy the boundary condition $y(b) = y_b$. Then a necessary condition for $J(y)$ to have an extremum for a given function $y(t)$ is that $y(t)$ satisfies the fractional Euler-Lagrange equation

$$\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L}{\partial y} + {}_x^C D_t^\alpha \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_t^R D_b^\alpha y} \right) - \frac{d}{dt}\left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right) = 0 \quad (16)$$

with the natural boundary condition (transversality condition)

$$\left(\left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial\, {}_t^R D_b^\alpha y} \right) \right)\Bigg|_{t=x} + \left(\frac{(t-x)^{\gamma-1}}{\Gamma(\gamma)} \frac{\partial L}{\partial y'} \right)\Bigg|_{t=x} = 0. \quad (17)$$

Remarks:

1. We have different cases for the boundary condition (11) and similar cases to (13), (15) and (17), these cases are:
   (a) For $\gamma \in (0,1)$ we have no solution satisfying the boundary conditions, except the very special case when the Lagrangian L contains the term $(x-t)^{1-\gamma}$;

(b) For $\gamma=1$ then (11) takes the form $\left({}_tI_x^{1-\alpha}\frac{\partial L}{\partial\, {}_a^CD_t^\alpha y}\right)\Big|_{t=x}+\left(\frac{\partial L}{\partial y'}\right)\Big|_{t=x}=0;$

(c) For $\gamma>1$ we have no boundary conditions where (11) is trivially satisfied.

2. Putting $L=L(t,y(t),\, {}_a^CD_t^\alpha y)$ in (1) and $L=L(t,y(t),\, {}_a^RD_t^\alpha y)$ in (3) gives the results as given in [16].
3. Putting $\gamma=1$ and $L=L(t,y(t),\, {}_a^CD_t^\alpha y)$ in (1) gives the results as given in [2].
4. Putting $\gamma=1$ and $L=L(t,y(t),\, {}_a^RD_t^\alpha y)$ in (3) gives the results as given in [1].
5. Putting $L=L(t,y(t),\, {}_a^RD_t^\alpha y)$ in (3) gives the results as given in [13].
6. Putting $\gamma=1$ and $L=L(t,y(t),y'(t))$ in each one of our problems we get the classical Euler-Lagrange equation.

4. SUFFICIENT CONDITIONS

In this section we prove the sufficient conditions that ensure the existence of a minimum (maximum) of our fractional variational problems. Some conditions of convexity (concavity) are in order.

Definition 4: Given a function $L=L(t,y,z,u)$, we say that L is jointly convex (concave) in (y,z,u) if $\frac{\partial L}{\partial y},\frac{\partial L}{\partial z},\frac{\partial L}{\partial u}$ exist and are continuous and verify the following condition:

$$L(t,y+y_1,z+z_1,u+u_1)-L(t,y,z,u)\geq(\leq)\frac{\partial L}{\partial y}y_1+\frac{\partial L}{\partial z}z_1+\frac{\partial L}{\partial u}u_1 \tag{18}$$

for all $(t,y,z,u),(t,y+y_1,z+z_1,u+u_1)\in[a,b]\times\mathbb{R}^3$.

Theorem 5: Let $L(t,y,z,u)$ be jointly convex (concave) in(y,z,u). If y_0 satisfies conditions (10)-(11), then y_0 is a global minimizer (maximizer) to problem (1).

Proof. We shall give the proof for only the convex case (and similarly we can prove it for the concave case). Since L is jointly convex in (y,z,u) for any admissible function y_0+h, we have

$$\begin{aligned}
&J(y_0+h)-J(y_0)=\\
&\quad\int_a^x\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\left[L(t,y_0(t)+h(t),\, {}_a^CD_t^\alpha(y_0(t)+h(t)),y_0'(t)+h'(t))\right.\\
&\qquad\qquad\left.-L(t,y_0(t),\, {}_a^CD_t^\alpha y_0(t),y_0'(t))\right]dt\\
&\qquad\geq\int_a^x\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\left[\frac{\partial L}{\partial y_0}h+\frac{\partial L}{\partial\, {}_a^CD_t^\alpha y_0}\, {}_a^CD_t^\alpha h+\frac{\partial L}{\partial y_0'(a)}h'(t)\right]dt
\end{aligned}$$

By using integration by parts (as in proving (10)-(11) we get)

$$
\begin{aligned}
&J(y_0+h)-J(y_0)\geq\\
&\int_a^x h(t)\left[\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L}{\partial y}+{}_t^RD_x^\alpha\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L}{\partial\, {}_a^CD_t^\alpha y}\right)-\frac{d}{dt}\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L}{\partial y'}\right)\right]dt\\
&\quad+h(x)\left[\left({}_tI_x^{1-\alpha}\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L}{\partial\, {}_a^CD_t^\alpha y}\right)\right)\Big|_{t=x}+\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L}{\partial y'}\right)\Big|_{t=x}\right]\\
&\quad-h(a)\left[\left({}_tI_x^{1-\alpha}\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L}{\partial\, {}_a^CD_t^\alpha y}\right)\right)\Big|_{t=a}+\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L}{\partial y'}\right)\Big|_{t=a}\right]=0.
\end{aligned}
$$

Since y_0 satisfies conditions (10)-(11) with given $h(a)=0$ then $J(y_0+h)-J(y_0)\geq 0$ which completes the proof.

Similar to proving the previous theorem we can prove similar theorems to (2), (3) and (4).

5. EXAMPLE

We shall provide in this section an example in order to illustrate our main results.

Consider the following problem:

$$\min J(y)=\frac{1}{2}\,{}_0I_x^\gamma[y^2(t)+({}_0^CD_t^\alpha y(t))^2+\delta y'^2(t)],\quad x\in[0,1],\delta\geq 0, y(0)=y_0 \tag{19}$$

For this problem, we get the fractional Euler-Lagrange equation and the natural boundary conditions respectively in the form:

$$\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}y(t)+{}_t^RD_x^\alpha\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\,{}_0^CD_t^\alpha y(t)\right)-\frac{d}{dt}\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\delta y'(t)\right)=0 \tag{20}$$

$$\left({}_tI_x^{1-\alpha}\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\,{}_0^CD_t^\alpha y\right)\right)\Big|_{t=x}+\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\delta y'\right)\Big|_{t=x}=0, \tag{21}$$

and where $L(y,z,u)=\frac{1}{2}(y^2+z^2+\delta u^2)$ is a jointly convex then the solution of (20)-(21) is a global minimizer to problem (19). Note that it is difficult to solve the above fractional equations, for $0<\alpha<1$, a numerical method should be used to get an approximation to the solution. When α and γ go to 1, problem (19) tends to

$$J(y)=\frac{1}{2}\int_0^x(y^2(t)+(\delta+1)y'^2)dt,\quad x\in[0,1],\delta\geq 0,\ y(0)=y_0 \tag{22}$$

and equations(20) and (21) take the form

$$y''(t)-\frac{1}{\delta+1}y(t)=0, \tag{23}$$

$$y'(x)=0,\qquad y(0)=y_0. \tag{24}$$

Solving this differential equation we get the solution in the form

$$\overline{y}(t) = \frac{y_0 e^{\frac{1}{\sqrt{1+\delta}}x}}{1+e^{\frac{2}{\sqrt{1+\delta}}x}} \left(e^{\frac{1}{\sqrt{1+\delta}}(x-t)} + e^{-\frac{1}{\sqrt{1+\delta}}(x-t)} \right) \tag{25}$$

which is a candidate for minimizer at arbitrary fixed $x \in [0,1]$. Problem (19) satisfies the assumptions of Theorem 5 which gives that $\overline{y}$ is a global minimizer to problem (22) for any arbitrary fixed $x \in [0,1]$.

6. CONCLUSION

In this paper we consider a new class of fractional functionals of the calculus of variations containing classical and fractional derivatives in each Riemann-Liouville and Caputo senses. Moreover, instead of the classical integral we use fractional integration. Necessary and sufficient optimality conditions for such variational functionals are obtained.

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