Dedicated to Academician Aureliu Sandulescu's 80th Anniversary

CONFORMAL MAPS AND GROUP CONTRACTIONS IN NUCLEAR STRUCTURE MODELS

DENNIS BONATSOS

Institute of Nuclear Physics,
N.C.S.R. "Demokritos", GR-15310 Aghia Paraskevi, Attiki, Greece
Email: bonat@inp.demokritos.gr

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In mathematics, a conformal map is a function which preserves angles. We show how this procedure can be used in the framework of the Bohr Hamiltonian, leading to a Hamiltonian in a curved space, in which the mass depends on the nuclear deformation β, while it remains independent of the collective variable γ and the three Euler angles. This Hamiltonian is proved to be equivalent to that obtained using techniques of Supersymmetric Quantum Mechanics.

Group contraction is a procedure in which a symmetry group is reduced into a group of lower symmetry in a certain limiting case. Examples are provided in the large boson number limit of the Interacting Boson Approximation (IBA) model by a) the contraction of the SU(3) algebra into the [R^5]SO(3) algebra of the rigid rotator, consisting of the angular momentum operators forming SO(3), plus 5 mutually commuting quantities, the quadrupole operators, b) the contraction of the O(6) algebra into the [R^5]SO(5) algebra of the γ-unstable rotator. We show how contractions can be used for constructing symmetry lines in the interior of the symmetry triangle of the IBA model.

Key words: Bohr collective model, Interacting Boson Approximation model, supersymmetric quantum mechanics, group contraction, Davidson potential, Alhassid–Whelan arc of regularity.

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It is a pleasure and a great honour for the author to dedicate this work to Professor Aurel Sandulescu on the occasion of his 80th birthday. In addition to his many seminal contributions to nuclear physics, I would like to point out the tireless encouragement he has provided over the years to many young students through far reaching comments, always characterized by the broadness and clarity of his ideas, of which I also have profited in many cases.

1. INTRODUCTION

The Interacting Boson Approximation (IBA) model [1] and the collective model of Bohr [2] provide complementary descriptions of the properties of medium-mass and heavy nuclei.

In the present work we first show how the Bohr Hamiltonian can be generalized into a curved space [3]. This generalization turns out to be equivalent to the use of a nuclear mass which is not a constant, as in the usual case, but depends on the nuclear deformation. This modification solves [3] a long-standing problem of nuclear structure, namely the too rapid increase of the moment of inertia with deformation [4].

Furthermore in this work we are going to show [5] how the mathematical concept of group contraction [6] can be used for determining approximate symmetries in the framework of the IBA model [1]. In particular, an approximate SU(3) symmetry will be determined within the symmetry triangle [7] of the IBA model, at the vertices of which the dynamical symmetries of the model appear.

2. CONFORMAL MAPS

2.1. A FORMULATION OF THE BOHR HAMILTONIAN

The original Bohr Hamiltonian [2] is

$$H_B = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right] + V(\beta,\gamma), \quad (1)$$

where β and γ are the usual collective coordinates (β being a deformation coordinate measuring departure from spherical shape, and γ being an angle measuring departure from axial symmetry), while Q_k ($k = 1, 2, 3$) are the components of angular momentum in the intrinsic frame, and B is the mass parameter, which is usually considered constant.

In Ref. [8] it has been proved that the position-dependent effective mass formalism can be equivalently expressed in a curved space. We shall prove here that this connection is possible also in the case of the Bohr Hamiltonian.

Ordering the coordinates as

$$q_1 = \Phi, \quad q_2 = \Theta, \quad q_3 = \psi, \quad q_4 = \beta, \quad q_5 = \gamma, \quad (2)$$

the kinetic energy in the standard Bohr Hamiltonian [2] can be represented as

$$T = \frac{B}{2}\left(\frac{ds}{dt}\right)^2, \quad (3)$$

where

$$ds^2 = g_{ij}dq_idq_j, \quad (4)$$

the symmetric matrix g_{ij} having the form

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & 0 & g_{33} & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 \\ 0 & 0 & 0 & 0 & g_{55} \end{pmatrix}, \tag{5}$$

with

$$g_{11} = \frac{J_1}{B}\sin^2\Theta\cos^2\psi + \frac{J_2}{B}\sin^2\Theta\sin^2\psi + \frac{J_3}{B}\cos^2\Theta, \tag{6}$$

$$g_{12} = \frac{1}{B}(J_2 - J_1)\sin\Theta\sin\psi\cos\psi, \qquad g_{13} = \frac{J_3}{B}\cos\Theta, \tag{7}$$

$$g_{22} = \frac{J_1}{B}\sin^2\psi + \frac{J_2}{B}\cos^2\psi, \quad g_{33} = \frac{J_3}{B}, \quad g_{44} = 1, \quad g_{55} = \beta^2, \tag{8}$$

where the moments of inertia are

$$J_k = 4B\beta^2\sin^2\left(\gamma - k\frac{2\pi}{3}\right). \tag{9}$$

The determinant of the matrix is

$$g = \frac{J_1J_2J_3}{B^3}\beta^2\sin^2\Theta = 4\beta^8\sin^2 3\gamma\sin^2\Theta. \tag{10}$$

The relevant volume element is then

$$dV = 2\beta^4\sin 3\gamma\sin\Theta d\Phi d\Theta d\psi d\beta d\gamma. \tag{11}$$

The inverse matrix is found to be

$$(g^{-1}_{ij}) = \begin{pmatrix} g^{-1}_{11} & g^{-1}_{12} & g^{-1}_{13} & 0 & 0 \\ g^{-1}_{21} & g^{-1}_{22} & g^{-1}_{23} & 0 & 0 \\ g^{-1}_{31} & g^{-1}_{32} & g^{-1}_{33} & 0 & 0 \\ 0 & 0 & 0 & g^{-1}_{44} & 0 \\ 0 & 0 & 0 & 0 & g^{-1}_{55} \end{pmatrix}, \tag{12}$$

with

$$g^{-1}_{11} = \frac{B}{\sin^2\Theta}\left(\frac{\cos^2\psi}{J_1} + \frac{\sin^2\psi}{J_2}\right), \tag{13}$$

$$g^{-1}_{12} = -B\left(\frac{1}{J_1} - \frac{1}{J_2}\right)\frac{\sin\psi\cos\psi}{\sin\Theta}, \tag{14}$$

$$g^{-1}_{13} = -B\left(\frac{\cos^2\psi}{J_1} + \frac{\sin^2\psi}{J_2}\right)\frac{\cot\Theta}{\sin\Theta}, \quad g^{-1}_{22} = B\left(\frac{\sin^2\psi}{J_1} + \frac{\cos^2\psi}{J_2}\right), \tag{15}$$

$$g^{-1}_{23} = B\left(\frac{1}{J_1} - \frac{1}{J_2}\right)\cot\Theta\sin\psi\cos\psi, \tag{16}$$

$$g_{33}^{-1} = B\left(\frac{\cos^2\psi}{J_1} + \frac{\sin^2\psi}{J_2}\right)\cot^2\Theta + \frac{B}{J_3}, \quad g_{44}^{-1} = 1, \quad g_{55}^{-1} = \frac{1}{\beta^2}. \tag{17}$$

Using these matrix elements and the value of the determinant from Eq. (10) in the usual Pauli–Podolsky prescription [9]

$$(\nabla\Phi)^i = g^{ij}\frac{\partial\Phi}{\partial x^j}, \qquad \nabla^2\Phi = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j\Phi, \tag{18}$$

we obtain

$$T = -\frac{\hbar^2}{2B}\nabla^2 = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\right.$$

$$\left. -\frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right], \tag{19}$$

where Q_k are the components of the angular momentum in the intrinsic frame

$$Q_x = -i\left(-\frac{\cos\psi}{\sin\Theta}\frac{\partial}{\partial\Phi} + \sin\psi\frac{\partial}{\partial\Theta} + \cot\Theta\cos\psi\frac{\partial}{\partial\psi}\right), \tag{20}$$

$$Q_y = -i\left(-\frac{\sin\psi}{\sin\Theta}\frac{\partial}{\partial\Phi} + \cos\psi\frac{\partial}{\partial\Theta} - \cot\Theta\sin\psi\frac{\partial}{\partial\psi}\right), \quad Q_z = -i\frac{\partial}{\partial\psi}. \tag{21}$$

2.2. TRANSITION TO A CURVED SPACE

The connection between position-dependent effective mass and curved spaces has been considered in Ref. [8]. According to the findings of Ref. [8], one expects in the present case all elements of the matrix (5) to be divided by f^2

$$g'_{ij} = \frac{g_{ij}}{f^2}, \tag{22}$$

where f is a function of β, which will appear in subsection 2.4. As a result, the determinant of the matrix will be

$$g' = \frac{g}{f^{10}}, \tag{23}$$

and the volume element will be

$$dV' = \frac{dV}{f^5}. \tag{24}$$

The elements of the inverse matrix will be

$$g'^{-1}_{ij} = f^2 g_{ij}^{-1}. \tag{25}$$

2.3. FORMALISM OF POSITION-DEPENDENT EFFECTIVE MASSES

In order to proceed to the Bohr Hamiltonian with mass dependent on the deformation, we need the general formalism for handling effective masses depending in general on the position. The main problem encountered is the generalization of the kinetic energy term. We show how this can be solved in an unambiguous way.

When the mass $m(\mathbf{x})$ is position dependent [8], it does not commute with the momentum $\mathbf{p} = -i\hbar\nabla$. Therefore, there are many ways to generalize the usual form of the kinetic energy, $\mathbf{p}^2/(2m_0)$, where m_0 is a constant mass, in order to obtain a Hermitian operator. In order to avoid any specific choices, one can use the general two-parameter form proposed by von Roos [10], with a Hamiltonian

$$H = -\frac{\hbar^2}{4}[m^{\delta'}(\mathbf{x})\nabla m^{\kappa'}(\mathbf{x})\nabla m^{\lambda'}(\mathbf{x}) + m^{\lambda'}(\mathbf{x})\nabla m^{\kappa'}(\mathbf{x})\nabla m^{\delta'}(\mathbf{x})] + V(\mathbf{x}), \quad (26)$$

where V is the relevant potential and the parameters δ', κ', λ' are constrained by the condition $\delta' + \kappa' + \lambda' = -1$. Assuming a position dependent mass of the form

$$m(\mathbf{x}) = m_0 M(\mathbf{x}), \quad M(\mathbf{x}) = \frac{1}{(f(\mathbf{x}))^2}, \quad f(\mathbf{x}) = 1 + g(\mathbf{x}), \quad (27)$$

where m_0 is a constant mass and $M(\mathbf{x})$ is a dimensionless position-dependent mass, the Hamiltonian becomes

$$H = -\frac{\hbar^2}{4m_0}[f^{\delta}(\mathbf{x})\nabla f^{\kappa}(\mathbf{x})\nabla f^{\lambda}(\mathbf{x}) + f^{\lambda}(\mathbf{x})\nabla f^{\kappa}(\mathbf{x})\nabla f^{\delta}(\mathbf{x})] + V(\mathbf{x}), \quad (28)$$

with $\delta + \kappa + \lambda = 2$. It is known [8] that this Hamiltonian can be put into the form

$$H = -\frac{\hbar^2}{2m_0}\sqrt{f(\mathbf{x})}\nabla f(\mathbf{x})\nabla\sqrt{f(\mathbf{x})} + V_{eff}(\mathbf{x}), \quad (29)$$

with

$$V_{eff}(\mathbf{x}) = V(\mathbf{x}) + \frac{\hbar^2}{2m_0}\left[\frac{1}{2}(1-\delta-\lambda)f(\mathbf{x})\nabla^2 f(\mathbf{x}) + \left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f(\mathbf{x}))^2\right], \quad (30)$$

where δ and λ are free parameters.

In the final part of the project, in which comparison to experiment is carried out by fitting the theoretical predictions to the experimental data, it is seen that the predictions for the theoretical spectra turn out to be independent of the choice made for δ and λ [3].

2.4. EQUIVALENCE TO THE DEFORMATION-DEPENDENT MASS CASE

Using the formalism of the previous subsection, in Ref. [3] the Bohr equation with a mass depending on the deformation

$$B(\beta) = \frac{B_0}{(f(\beta))^2}, \tag{31}$$

where B_0 is a constant, has been considered, leading to a modified Bohr equation of the form

$$H\Psi = \left[-\frac{1}{2}\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f \frac{\partial}{\partial\beta}\sqrt{f} - \frac{f^2}{2\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma \frac{\partial}{\partial\gamma} \right.$$
$$\left. + \frac{f^2}{8\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)} + v_{eff} \right] \Psi = \epsilon\Psi, \tag{32}$$

where reduced energies $\epsilon = B_0 E/\hbar^2$ and reduced potentials $v = B_0 V/\hbar^2$ have been used, with

$$v_{eff} = v(\beta,\gamma) + \frac{1}{4}(1-\delta-\lambda) f \nabla^2 f + \frac{1}{2}\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f)^2. \tag{33}$$

According to Ref. [8], in order to obtain the Schrödinger equation in the form of Eq. (32), one has to start with the equation

$$H_g\tilde{\Psi} = \left[-\frac{1}{2}\nabla^2 + u_g\right]\tilde{\Psi} = \left[-\frac{1}{2}\frac{1}{\sqrt{g'}}\partial_i\sqrt{g'}{g'}^{-1}_{ij}\partial_j + u_g\right]\tilde{\Psi}, \tag{34}$$

where

$$\tilde{\Psi} = f^{5/2}\Psi, \tag{35}$$

while reduced energies and reduced potentials are used, as in Eq. (32). The exponent in the last equation is related to the dimensionality of the space.

Substituting the g' matrix elements and determinant in Eq. (34), and performing the relevant calculation (which closely resembles the pure Bohr case, except for the 44-term), we see that Eqs. (34) and (32) do coincide with

$$u_g = u_{eff} + f\ddot{f} - 2(\dot{f})^2 + 4\frac{f\dot{f}}{\beta}, \quad \dot{f} = \frac{df}{d\beta}, \quad \ddot{f} = \frac{d^2 f}{d\beta^2}. \tag{36}$$

2.5. DISCUSSION

This result has several important consequences.

1) It becomes clear that solving the Schrödinger equation (32) with deformation dependent mass is equivalent to solving a modified Bohr equation (34) with different metric matrix g' and another effective potential, u_g. Between the two equivalent

schemes, one chooses to solve Eq. (32) instead of Eq. (34), just because the former can be solved analytically through the use of SUSYQM techniques [11].

2) The wave functions $\tilde{\Psi} = f^{5/2}\Psi$ are accompanied by the volume element $dV' = dV/f^5$. As a result

$$\int \tilde{\Psi}^* \tilde{\Psi} dV' = \int (f^{5/2}\Psi^*)(f^{5/2}\Psi)\frac{dV}{f^5} = \int \Psi^* \Psi dV, \tag{37}$$

i.e., the wave functions Ψ of the deformation dependent mass problem correspond to the usual Bohr volume element dV.

3) The simple relation between $\tilde{\Psi}$ and Ψ also shows that the wave functions Ψ satisfy the well-known 24 symmetries of Bohr wave functions [2], which the wave functions $\tilde{\Psi}$ satisfy by construction. If these symmetries were not satisfied, the solutions could not have been used for the description of nuclei.

2.6. THE DEFORMATION DEPENDENT MASS (DDM) DAVIDSON MODEL

The solution of Eq. (32) using SUSYQM techniques is described in Ref. [3] for the particular case of the Davidson potential [12]

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}, \tag{38}$$

where the parameter β_0 indicates the position of the minimum of the potential. Using SUSYQM techniques it turns out that in this case the Bohr Hamiltonian with mass dependent on the deformation is exactly soluble if the deformation function is of the form

$$f(\beta) = 1 + a\beta^2, \qquad a \ll 1. \tag{39}$$

In other words, the functional dependence of the mass on the deformation is dictated by SUSYQM, being different for different potentials.

The final results for the spectra, which are used for comparison to experiment, read

$$\begin{aligned} \epsilon_0 = {} & \frac{19}{4}a + \frac{5}{2}(1-\delta-\lambda)a + \frac{1}{2}\sqrt{a^2+4k_1} \\ & + \frac{a}{2}\sqrt{1+4k_{-1}} + \frac{1}{4}\sqrt{(a^2+4k_1)(1+4k_{-1})} + a\Lambda, \end{aligned} \tag{40}$$

$$\epsilon_1 = \epsilon_0 + 4a + \sqrt{a^2+4k_1} + a\sqrt{1+4k_{-1}}, \tag{41}$$

$$\epsilon_2 = \epsilon_0 + 12a + 2\sqrt{a^2+4k_1} + 2a\sqrt{1+4k_{-1}}, \tag{42}$$

where

$$\begin{aligned} k_1 &= 2 + a^2[5(1-\delta-\lambda) + (1-2\delta)(1-2\lambda) + 6 + \Lambda]. \\ k_{-1} &= 2 + \Lambda + 2\beta_0^4. \end{aligned} \tag{43}$$

Λ has a different form in different physical situations.

1) In the case of γ-unstable nuclei, the potential is independent of γ, *i.e.*, $v(\beta) = u(\beta)$ [13]. Then $\Lambda = \tau(\tau+3)$ represents the eigenvalues of the second order Casimir operator of SO(5), while τ is the seniority quantum number, characterizing the irreducible representations of SO(5). The values of angular momentum L occurring for each τ are provided by a well known algorithm and are listed in [1, 13]. Within the ground state band (gsb) one has $L = 2\tau$. The $L = 2$ member of the quasi-γ_1 band is degenerate with the $L = 4$ member of the gsb, the $L = 3$, 4 members of the quasi-γ_1 band are degenerate to the $L = 6$ member of the gsb, the $L = 5$, 6 members of the quasi-γ_1 band are degenerate to the $L = 8$ member of the gsb, and so on.

2) In the case of axially symmetric prolate deformed nuclei, in order to achieve exact separation of variables, one assumes a potential of the form [13–17]

$$v(\beta,\gamma) = u(\beta) + \frac{f^2}{\beta^2} w(\gamma), \tag{44}$$

with $w(\gamma)$ having a deep minimum at $\gamma = 0$,

$$w(\gamma) = \frac{1}{2}(3c)^2\gamma^2, \tag{45}$$

and seeks solutions for $\gamma \approx 0$ [17, 18]. Then the allowed bands are characterized by

$$n_\gamma = 0, \quad K = 0; \qquad n_\gamma = 1, \quad K = \pm 2; \qquad n_\gamma = 2, \quad K = 0, \pm 4; \quad \ldots, \tag{46}$$

where $n_\gamma = 0, 1, 2, \ldots$ is the quantum number related to γ-oscillations, while K is the quantum number of the projection of angular momentum on the body-fixed z-axis. Then Λ is given by

$$\Lambda = \frac{L(L+1) - K^2}{3} + (6c)(n_\gamma + 1). \tag{47}$$

3) In the case of triaxial nuclei with $\gamma = \pi/6$, a potential of the form of Eq. (44) is assumed again, but with $w(\gamma)$ having a deep minimum at $\gamma = \pi/6$ [19],

$$w(\gamma) = \frac{1}{4} c \left(\gamma - \frac{\pi}{6}\right)^2. \tag{48}$$

In this case K, the angular momentum projection on the body-fixed z-axis, is not a good quantum number any more, but α, the angular momentum projection on the body-fixed x-axis, is a good quantum number, as found [20] in the study of the triaxial rotator [21, 22]. In the literature on triaxial nuclei it is customary, instead of the projection α of the angular momentum on the x-axis, to introduce the wobbling quantum number [20, 23] $n_w = L - \alpha$. Seeking solutions in the case $\gamma \approx \pi/6$ [19], one obtains

$$\Lambda = \frac{L(L+4) + 3n_w(2L - n_w)}{4} + \sqrt{2c}\left(n_\gamma + \frac{1}{2}\right). \tag{49}$$

The ground state band is obtained from Eq. (40), while the quasi-β_1 band is obtained from Eq. (41), and the quasi-β_2 band is obtained from Eq. (42).

In the special case of $a = 0$ (no dependence of the mass on the deformation) one easily obtains

$$\epsilon_1 = \epsilon_0 + 2\sqrt{2}, \qquad \epsilon_2 = \epsilon_0 + 4\sqrt{2}, \tag{50}$$

i.e. the β-bandheads become equidistant.

The details for the calculation of wave functions and $B(E2)$ transition rates are given in Ref. [3]. Numerical results for spectra and $B(E2)$ transition rates for about 50 γ-unstable nuclei and 50 deformed nuclei have been obtained [3] and compared to experimental data, with positive results.

2.7. CONNECTION TO EARLIER WORK

It is instructive to examine the relation between the present approach and earlier numerical work.

1) The formalism of subsections 2.1-2.4 clarifies the relation between the present approach and the numerical solution of Kumar and Baranger [24], who used a matrix of the form (5) with g_{ij}, $i,j = 1,2,3$ the same as in Eq. (6), but with

$$g_{44} = B_{00}, \quad g_{55} = B_{2'2'}, \quad g_{45} = g_{54} = B_{02'}, \tag{51}$$

where B_{00}, $B_{2'2'}$, $B_{02'}$, as well as the moments of inertia $\mathcal{J}_i$ ($i = 1,2,3$) and the potential V have been treated as seven arbitrary functions of the variables $\beta_0 = \beta\cos\gamma$ and $\beta_{2'} = \beta\sin\gamma$ [while in the Bohr formulation [2] $a_0 = \beta\cos\gamma$ and $a_2 = \beta\sin\gamma/\sqrt{2}$ are used]. On one hand, the present solution is a special case of Ref. [24], since it contains no non-diagonal terms $g_{45} = g_{54}$. On the other hand, in the present approach the above mentioned quantities are interrelated by the overall symmetry in a specific way, greatly reducing the number of free parameters (down to two or three in total). It should be pointed out that the functional dependence of the mass on the deformation for the potential used is dictated by SUSYQM. Therefore, the successful prediction of the behavior of the moments of inertia, for example, provides credit for the present approach. What is seen in Ref. [3], independently of the parameter values, is that the increase of the moments of inertia as a function of deformation is moderated by the f^2 factor, which can be seen as a result of the dependence of the mass on the deformation, or, alternatively, as seen in subsection 2.4, as a result of using a curved space.

2) It should be pointed out that in Ref. [8] the equivalence between the position dependent mass case and the curved space approach has been established in the special case of $\kappa = 2$ and $\delta = \lambda = 0$ (see Eq. (28) for the meaning of the symbols), which represents the Ben Daniel and Duke Hamiltonian [25]

$$H_{BD} = -\frac{\hbar^2}{2}\nabla f^2 \nabla + V_{BD}. \tag{52}$$

This resembles the collective Hamiltonian

$$H_{coll} = -\frac{\hbar^2}{2}\Sigma_{i,j}\frac{\partial}{\partial q_i}[\mathcal{M}(q)_{ij}]^{-1}\frac{\partial}{\partial q_j} + V(q) \tag{53}$$

used by Libert *et al.* [26] in mean field calculations, in which a tensor mass appears.

2.8. OUTLOOK

The present work opens the way for several investigations, listed here.

1) The method should be applied to other potentials, like the Kratzer potential [27, 28], since the way in which the mass depends on the deformation , dictated by a deformed shape invariance condition [3], is in general different for each potential. Spectra and electromagnetic transition rates obtained [29,30] with a constant mass for the Morse potential through the use of the Asymptotic Iteration Method will also be useful for comparison, given the similarities in the shapes of the Kratzer and Morse potentials, which become flat in the right hand side, while the Davidson potential rises to infinity.

2) It has been recently suggested [31, 32], based on experimental evidence, that a mass tensor is required in the Bohr Hamiltonian instead of a constant mass. A connection between this approach and the present method has to be established.

3) Using a variational procedure [33, 34], one can determine how critical point symmetries [35] are modified by the deformation dependence of the mass.

4) The present analytical method offers a test ground both for the recently developed algebraic collective model [36–39], as well as for microscopic approaches in the relativistic mean field framework [40].

3. GROUP CONTRACTIONS

3.1. THE SU(3)→ [R^5]SO(3) CONTRACTION

The SU(3)→ [R^5]SO(3) contraction [41, 42] is a procedure in which the full SU(3) algebra, consisting of 8 noncommuting generators, is shrunk into an SO(3) algebra (consisting of 3 noncommuting generators), accompanied by 5 mutually commuting operators (the quadrupole operators). This simplification occurs in the limit of large boson number in which, in SU(3), all intrinsic excitations rise in energy, isolating the ground state band so that SU(3) goes over, approximately, into a simple rigid rotator. The resulting algebraic structure is, indeed, known [43] to be the algebra of the rigid rotator.

The SU(3) commutation relations read

$$[\hat{L}_\xi, \hat{L}_\nu] = -\sqrt{2}(1\xi 1\nu|1\xi+\nu)\hat{L}_{\xi+\nu}, \tag{54}$$

$$[\hat{L}_\xi, \hat{Q}^{(2)}_{SU(3),\nu}] = -\sqrt{6}(1\xi 2\nu|2\xi+\nu)\hat{Q}^{(2)}_{SU(3),\xi+\nu}, \tag{55}$$

$$[\hat{Q}^{(2)}_{SU(3),\xi}, \hat{Q}^{(2)}_{SU(3),\nu}] = \frac{3}{4}\sqrt{\frac{5}{2}}(2\xi 2\nu|1\xi+\nu)\hat{L}_{\xi+\nu}. \tag{56}$$

The second order Casimir operator is

$$\hat{C}_2[SU(3)] = \frac{2}{3}\left[2\hat{Q}^{(2)}_{SU(3)}\cdot\hat{Q}^{(2)}_{SU(3)} + \frac{3}{4}\hat{L}\cdot\hat{L}\right], \tag{57}$$

while its eigenvalues in the Elliott basis, (λ,μ), are

$$C_2(\lambda,\mu) = \frac{2}{3}(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu). \tag{58}$$

If we consider SU(3) irreducible representations (irreps) with large values of $C_2(\lambda,\mu)$, that is for large boson numbers, we can rescale the quadrupole operator as

$$\hat{q}^{(2)}_{SU(3),\xi} = \frac{\hat{Q}^{(2)}_{SU(3),\xi}}{\sqrt{C_2(\lambda,\mu)}}. \tag{59}$$

The first two commutation relations remain unchanged by the rescaling, while the last one becomes

$$[\hat{q}^{(2)}_{SU(3),\xi}, \hat{q}^{(2)}_{SU(3),\nu}] = \frac{3}{4}\sqrt{\frac{5}{2}}(2\xi 2\nu|1\xi+\nu)\frac{\hat{L}_{\xi+\nu}}{C_2(\lambda,\mu)}. \tag{60}$$

Then in the limit of large values of $C_2(\lambda,\mu)$ one gets

$$[\hat{q}^{(2)}_{SU(3),\xi}, \hat{q}^{(2)}_{SU(3),\nu}] = 0. \tag{61}$$

This result, which is obtained for large boson number, is called the contraction of SU(3) to [R^5]SO(3), where [R^5]SO(3) is the algebra of the rigid rotator [43], generated by the angular momentum operators of SO(3) and the five commuting operators $\hat{q}^{(2)}_{SU(3),\xi}$, $\xi = -2,-1,0,1,2$.

An immediate consequence of Eqs. (60) and (61) is that, in the contraction limit, terms proportional to the angular momentum $\hat{L}$ can be ignored. In the IBA framework, in which $\hat{L}$ is proportional to $(d^\dagger\tilde{d})^{(1)}$, this implies that $(d^\dagger\tilde{d})^{(1)}$ terms can be ignored.

In the limit of large values of $C_2(\lambda,\mu)$ and $\lambda \geq \mu$ the intrinsic quadrupole moments become [42]

$$q_0 = \frac{1}{2\sqrt{2}}(2\lambda+\mu+3), \qquad q_2 = \frac{1}{4}\sqrt{3(\mu-K)(\mu+K+2)}, \tag{62}$$

where K is the eigenvalue of the angular momentum projection on the body-fixed z-axis, for which $K \leq L$ is valid, as one can see from the algorithm of the SU(3)⊃SO(3) reduction [1]. For states with $\lambda >> L$ (therefore also $\lambda >> K$) and $\lambda >> \mu$ one then obtains [41]

$$q_0 = \frac{\lambda}{\sqrt{2}}, \tag{63}$$

while q_2 becomes negligible. Since the ground state band belongs to the $(2N,0)$ irreducible representation (irrep) of SU(3), while other low-lying bands belong to irreps $(2N-4i-6j,2i)$, $i=0,1,2,\ldots$, $j=0,1,2,\ldots$ with relatively low i, j, the contraction occurs in the large N limit. Thus in the case of interest the intrinsic quadrupole moment becomes

$$q_0 = N\sqrt{2}. \tag{64}$$

An equivalent statement is that one can approximately replace the operator $\hat{Q}^{(2)}_{SU(3)}$ by the scalar $\lambda/\sqrt{2}$, as one can see from Eqs. (57) and (58), since the terms containing $\hat{L}$ and μ are negligible in this limit, having as a consequence that only the first term in the r.h.s. of these equations survives. A formal justification for this replacement is given in Ref. [5], where matrix elements of the commutators of the relevant parts of the Hamiltonian with the quadrupole operator are properly considered, resulting in the appearance of the intrinsic quadrupole moment.

It should be noticed that the above results have been obtained in irreps with $\lambda >> L$, thus they regard the low lying part of the spectrum.

3.2. $\overline{\mathrm{SU(3)}}$

In SU(3) the irreps are built out of the (2,0) irrep, while in the case of $\overline{\mathrm{SU(3)}}$ the irreps are built out of the (0,2) irrep [1]. As a result, in the $\overline{\mathrm{SU(3)}}$ framework one is interested in states with large values of $C_2(\lambda,\mu)$ and $\lambda < \mu$, in which the intrinsic quadrupole moments become [42,44]

$$q_0 = -\frac{1}{2\sqrt{2}}(\lambda+2\mu+3), \qquad q_2 = -\frac{1}{4}\sqrt{3(\lambda-K)(\lambda+K+2)}. \tag{65}$$

For states with $\mu >> L$ and $\mu >> \lambda$ one then obtains

$$q_0 = -\frac{\mu}{\sqrt{2}}, \tag{66}$$

while q_2 becomes negligible. Since the ground state band belongs to the $(0,2N)$ irrep of $\overline{\mathrm{SU(3)}}$, while other low-lying bands belong to irreps $(2i,2N-4i-6j)$, $i=0,1,2,\ldots$, $j=0,1,2,\ldots$ with relatively low i, j, the contraction does occur in the large N limit, the intrinsic quadrupole moment becoming

$$q_0 = -N\sqrt{2}. \tag{67}$$

Since SU(3) is associated to prolate shapes, while $\overline{\mathrm{SU(3)}}$ is related to oblate shapes, the signs in Eqs. (64) and (67) are consistent with the fact that intrinsic quadrupole moments are known to be positive for prolate nuclei and negative for oblate nuclei [7].

The first contraction has been used in Ref. [5] for determining a line of approximate SU(3) symmetry inside the symmetry triangle of the IBA, as described in subsection 3.5.

3.3. THE O(6)→ [R^5]SO(5) CONTRACTION

A procedure similar to that of subsection 3.1 is followed in the contraction of O(6) to [R^5]SO(5) [45, 46]. This is a procedure in which the full O(6) algebra, consisting of 15 noncommuting generators, is shrunk into an SO(5) algebra (consisting of 10 noncommuting generators), accompanied by 5 mutually commuting operators (the quadrupole operators). The resulting algebraic structure is known [46] to be the algebra of the γ-unstable rotator.

The commutation relation for the quadrupole operators reads

$$[\hat{Q}^{(2)}_{O(6),\xi}, \hat{Q}^{(2)}_{O(6),\nu}] = 2 \sum_{k=1,3} (2\xi 2\nu | k\xi+\nu)(d^\dagger \tilde{d})^{(k)}_{\xi+\nu}. \tag{68}$$

The second order Casimir operator is [1]

$$\hat{C}_2[O(6)] = 2\hat{Q}^{(2)}_{O(6)} \cdot \hat{Q}^{(2)}_{O(6)} + 4 \sum_{k=1,3} (d^\dagger \tilde{d})^{(k)} \cdot (d^\dagger \tilde{d})^{(k)}. \tag{69}$$

Its eigenvalues are

$$C_2(\sigma) = 2\sigma(\sigma+4), \tag{70}$$

where σ is the quantum number characterizing the irreps of O(6).

If we consider O(6) irreps with large σ, we can rescale the quadrupole operator as

$$\hat{q}^{(2)}_{O(6),\xi} = \frac{\hat{Q}^{(2)}_{O(6)),\xi}}{\sqrt{C_2(\sigma)}}. \tag{71}$$

Then the commutation relation for the quadrupole operators becomes

$$[\hat{q}^{(2)}_{O(6),\xi}, \hat{q}^{(2)}_{O(6),\nu}] = 2 \sum_{k=1,3} (2\xi 2\nu | k\xi+\nu) \frac{(d^\dagger \tilde{d})^{(k)}_{\xi+\nu}}{C_2(\sigma)}. \tag{72}$$

Then in the limit of large σ (and small τ, where τ is the quantum number characterizing the irreps of O(5)) [46] one gets

$$[\hat{q}^{(2)}_{O(6),\xi}, \hat{q}^{(2)}_{O(6),\nu}] = 0. \tag{73}$$

This procedure is called the contraction of O(6) to [R^5]SO(5), where [R^5]SO(5) is the algebra of the γ-unstable rotator, generated by the operators of SO(5) and the five commuting operators $q^{(2)}_{O(6),\xi}$, $\xi = -2, -1, 0, 1, 2$, which are the coordinates [46].

An immediate consequence of Eqs. (72) and (73) is that, in the contraction limit, terms proportional to $(d^\dagger \tilde{d})^{(k)}$ can be ignored.

The most leading O(6) irrep, to which the ground state band belongs, is (N). Thus in the large boson number limit it is appropriate to use this contraction. The intrinsic quadrupole moment will then be

$$q_0' = \sigma, \tag{74}$$

as can be seen from Eqs. (69) and (70). Thus in the case of interest the intrinsic quadrupole moment becomes

$$q_0' = N. \tag{75}$$

It should be noticed that the above results have been obtained in irreps with $\sigma >> \tau$, thus they regard the low lying part of the spectrum (since $L \leq 2\tau$, as seen from the algorithm of the SO(5)⊃SO(3) reduction [1]).

3.4. THE ALHASSID–WHELAN ARC OF REGULARITY

The study of chaotic properties of the Interacting Boson Approximation (IBA) model [1], both classically and quantum mechanically, led to the discovery [47, 48] of a narrow strip of nearly regular behaviour inside the symmetry triangle [7] of the IBA, connecting the U(5) and SU(3) limiting symmetries, in addition to the regular region along the U(5)-O(6) leg of the triangle. While the existence of the latter is known to be due to the underlying SO(5) symmetry, a common subalgebra of both U(5) and O(6) present throughout the U(5)-O(6) leg [49], the origin of the former, called the Alhassid–Whelan (AW) arc of regularity, has remained an open question.

The IBA Hamiltonian used by Alhassid and Whelan [47, 48] reads

$$\hat{H}(\eta,\chi) = \hat{H}_1 + \hat{H}_2 = c\left[\eta \hat{n}_d + \frac{\eta-1}{N}\hat{Q}^{(2)}_\chi \cdot \hat{Q}^{(2)}_\chi\right], \tag{76}$$

where N is the number of valence bosons, c is a scaling factor, $\hat{H}_1$ and $\hat{H}_2$ denote the first and the second term respectively,

$$\hat{n}_d = d^\dagger \cdot \tilde{d} = \sqrt{5}(d^\dagger \tilde{d})^{(0)}, \tag{77}$$

$$\hat{Q}^{(2)}_{\chi,\xi} = (s^\dagger \tilde{d} + d^\dagger s)^{(2)}_\xi + \chi (d^\dagger \tilde{d})^{(2)}_\xi. \tag{78}$$

The above Hamiltonian contains two parameters, η and χ, with the parameter η ranging from 0 to 1, and the parameter χ ranging from 0 to $-\sqrt{7}/2 = -1.323$. The U(5) symmetry limit corresponds to $\eta = 1$, the SU(3) limit to $\eta = 0$, $\chi = -\sqrt{7}/2$, and the O(6) limit to $\eta = 0$, $\chi = 0$. These symmetries are placed at the vertices of the symmetry triangle [7] of the IBA, shown in Fig. 1(a). In the symmetry triangle, the narrow coexistence region [50] surrounding the critical line [51] separating the spherical phase from the prolate deformed phase, corresponding to a first-order shape/phase transition [52], is also shown. It corresponds to $\eta \sim 0.8$. The point at which the critical line reaches the U(5)-O(6) side of the triangle is known to correspond to a second-order shape/phase transition [52].

3.5. AN SU(3) SYMMETRY UNDERLYING THE ARC

The SU(3) [1] algebra is generated by the angular momentum operators

$$\hat{L}_\xi = \sqrt{10}(d^\dagger \tilde{d})^1_\xi, \tag{79}$$

and the quadrupole operators

$$\hat{Q}^{(2)}_{SU(3),\xi} = (s^\dagger \tilde{d} + d^\dagger s)^{(2)}_\xi - \frac{\sqrt{7}}{2}(d^\dagger \tilde{d})^{(2)}_\xi. \tag{80}$$

In order to have an underlying SU(3) symmetry, the Hamiltonian of Eq. (76) has to commute with the generators of SU(3). It does commute with the angular momentum operators $\hat{L}_\xi$ by construction, since it is a scalar quantity. We will examine the special conditions under which the Hamiltonian also commutes (approximately) with the quadrupole operators. The commutation relations needed for this task can be found in Ref. [5].

The first term of the Hamiltonian gives

$$[\hat{H}_1, \hat{Q}^{(2)}_{SU(3),\nu}] = c\eta[\hat{n}_d, \hat{Q}^{(2)}_{SU(3),\nu}] = c\eta(d^\dagger s - s^\dagger \tilde{d})^{(2)}_\nu. \tag{81}$$

Using

$$\hat{Q}^{(2)}_{\chi,\xi} = \hat{Q}^{(2)}_{SU(3),\xi} + \left(\chi + \frac{\sqrt{7}}{2}\right)(d^\dagger \tilde{d})^{(2)}_\xi, \tag{82}$$

in the second term of the Hamiltonian one gets the intermediate result

$$\begin{aligned}
&[\hat{Q}^{(2)}_\chi \cdot \hat{Q}^{(2)}_\chi, \hat{Q}^{(2)}_{SU(3),\nu}] = \\
&\sum_\xi (-1)^\xi \Big\{ [\hat{Q}^{(2)}_{SU(3),\xi}, \hat{Q}^{(2)}_{SU(3),\nu}]\hat{Q}^{(2)}_{\chi,-\xi} + \hat{Q}^{(2)}_{\chi,\xi}[\hat{Q}^{(2)}_{SU(3),-\xi}, \hat{Q}^{(2)}_{SU(3),\nu}] \\
&+ \left(\chi + \frac{\sqrt{7}}{2}\right)\Big\{ [(d^\dagger \tilde{d})^{(2)}_\xi, \hat{Q}^{(2)}_{SU(3),\nu}]\hat{Q}^{(2)}_{\chi,-\xi} + \hat{Q}^{(2)}_{\chi,\xi}[(d^\dagger \tilde{d})^{(2)}_{-\xi}, \hat{Q}^{(2)}_{SU(3),\nu}] \Big\} \Big\}.
\end{aligned} \tag{83}$$

In order to obtain the conditions for which the Hamiltonian of Eq. (76) commutes with the generators of SU(3), we exploit a simplification of Eq. (83) that occurs in the large N limit. In this limit, the eigenvalue expression for the second order Casimir of SU(3) [see Eq. (58)] reduces to just the λ^2 term for SU(3) irreducible representations (irreps) (λ, μ) with $\lambda >> \mu$ and hence the ground state band [which belongs to the (2N,0) irrep] becomes energetically isolated from all other excitations. That is, SU(3) effectively reduces to a simple rigid rotator. This situation is formally known as the contraction of SU(3) to R^5[SO(3)] [41,42] and occurs when the $Q^{(2)}_{SU(3)}$ operators can be replaced by mutually commuting quantities [For a detailed explanation, see in subsection 3.1 the discussion leading to Eq. (61)]. If the $Q^{(2)}_{SU(3)}$ operators can be approximated by mutually commuting quantities, Eq. (83) greatly simplifies.

In the large N limit, where contraction occurs, the commutators in the first two terms in Eq. (83) will vanish. Furthermore, the vanishing of the commutator

$$[\hat{Q}^{(2)}_{SU(3),\xi}, \hat{Q}^{(2)}_{SU(3),\nu}] = \frac{15}{4}(2\xi 2\nu|1\xi+\nu)(d^\dagger \tilde{d})^{(1)}_{\xi+\nu} \tag{84}$$

in this limit, implies that terms containing $(d^\dagger \tilde{d})^{(1)}$ can be ignored. This fact can be understood qualitatively as a consequence of the relevant dominance of s bosons over d bosons within the ground state band, especially for relatively low-lying states in the large boson number limit.

Eq. (83) can be rewritten, without using any approximations yet, as

$$\begin{aligned}
[\hat{Q}^{(2)}_\chi \cdot \hat{Q}^{(2)}_\chi, \hat{Q}^{(2)}_{SU(3),\nu}] &= \frac{3\sqrt{15}}{4}[((d^\dagger \tilde{d})^{(1)}\hat{Q}^{(2)}_\chi)^{(2)}_\nu - (\hat{Q}^{(2)}_\chi (d^\dagger \tilde{d})^{(1)})^{(2)}_\nu] \\
&+ \left(\chi + \frac{\sqrt{7}}{2}\right)\Bigg\{[((d^\dagger s - s^\dagger \tilde{d})^{(2)}\hat{Q}^{(2)}_\chi)^{(2)}_\nu + (\hat{Q}^{(2)}_\chi (d^\dagger s - s^\dagger \tilde{d})^{(2)})^{(2)}_\nu] \\
&+ \sum_{k=1,3}\sqrt{35(2k+1)}\begin{Bmatrix} 2 & 2 & k \\ 2 & 2 & 2 \end{Bmatrix}[((d^\dagger \tilde{d})^{(k)}\hat{Q}^{(2)}_\chi)^{(2)}_\nu - (\hat{Q}^{(2)}_\chi (d^\dagger \tilde{d})^{(k)})^{(2)}_\nu]\Bigg\}.
\end{aligned} \tag{85}$$

In the large N limit the terms containing $(d^\dagger \tilde{d})^{(k)}$ (in the first line and in the third line) can be omitted. Furthermore, in the second line, $\hat{Q}^{(2)}_\chi$ can be replaced by $\hat{Q}^{(2)}_{SU(3)}$, since, as seen from Eq. (82), they differ by terms $(d^\dagger \tilde{d})^{(2)}$, which are small. In addition, in this limit $\hat{Q}^{(2)}_{SU(3)}$ can be replaced by the intrinsic quadrupole moment (a scalar), which is $N\sqrt{2}$ in the present case (see subsection 3.1). This replacement is justified in detail in Ref. [5]. This result is perhaps familiar in the context of the well-known property of SU(3) that $B(E2: 2^+_1 \to 0^+_1)$ goes as N^2 [1], that is, the collectivity of yrast transition strengths increases quadratically with boson number. Then in the large N limit one is left with

$$[\hat{Q}^{(2)}_\chi \cdot \hat{Q}^{(2)}_\chi, \hat{Q}^{(2)}_{SU(3),\nu}] = 2\sqrt{2}N\left(\chi + \frac{\sqrt{7}}{2}\right)(d^\dagger s - s^\dagger \tilde{d})^{(2)}_\nu. \tag{86}$$

Then in the large N limit the commutator for the second part of the Hamiltonian reads

$$[\hat{H}_2, \hat{Q}^{(2)}_{SU(3),\nu}] = c(\eta - 1)2\sqrt{2}\left(\chi + \frac{\sqrt{7}}{2}\right)(d^\dagger s - s^\dagger \tilde{d})^{(2)}_\nu. \tag{87}$$

In order to get a vanishing commutator, the coefficients of $(d^\dagger s - s^\dagger \tilde{d})^{(2)}_\nu$ in Eqs. (81) and (87) should cancel, leading in the large N limit to the condition

$$\chi(\eta) = \frac{1}{2\sqrt{2}} \frac{\eta}{(1-\eta)} - \frac{\sqrt{7}}{2}. \tag{88}$$

When χ is taking values between $-\sqrt{7}/2$ and 0, the parameter η takes values between 1 and 0.789 . From the formulae reported in Refs. [53] and [54] it is clear that the critical line in the large N limit corresponds to $\eta_{crit} = 0.8$ for $\chi = 0$ and to $\eta_{crit} = 9/11 = 0.818$ for $\chi = -\sqrt{7}/2$. Thus the line described by Eq. (88) cannot reach the critical line, confined in the region between the critical line and the SU(3) vertex.

It should be noticed that the arc of regularity found in Refs. [47, 48] has been approximately described by [55]

$$\chi(\eta) = \frac{\sqrt{7}-1}{2}\eta - \frac{\sqrt{7}}{2}. \tag{89}$$

The similarity between the lines described by Eqs. (88) and (89) can be seen numerically. Indeed, the two equations give very similar predictions for values of η between 0 and 0.6, i.e., from the SU(3) vertex until quite close to the critical line.

The symmetry triangle of IBA in the Alhassid–Whelan parametrization is shown in Fig. 1(a), together with the arc corresponding to Eq. (89) and the line of Eq. (88). The degeneracy line corresponding to $E(2^+_\beta) = E(2^+_\gamma)$, found in Ref. [56], is also shown (on the right of the critical line) for comparison.

We see that the present line remains very close to both the $E(2^+_\beta) = E(2^+_\gamma)$ degeneracy line and the original arc line from the SU(3) vertex until quite close to the critical region, where both the $E(2^+_\beta) = E(2^+_\gamma)$ degeneracy line and the present line turn upwards, avoiding to meet the critical line.

In Figs. 1(b) and 1(c), the ν and $\bar{\lambda}$ diagrams are reproduced from Ref. [48], with the lines of Fig. 1(a) plotted on them. We see that the present line remains within the valley corresponding to the arc of regularity for most of the way from the SU(3) vertex towards the critical line, turning upwards a little before reaching the critical line.

It should be noted that the present study is greatly facilitated by the fact that the position of the arc of regularity appears to be practically independent of the number of bosons, as already remarked in Refs. [47, 48] and corroborated in Ref. [56].

3.6. OUTLOOK

In summary, we have achieved by now two goals.

1) To prove analytically the existence of a line in the parameter space of the IBA, along which the Hamiltonian approximately commutes with the SU(3) genera-

tors in the large N limit.

2) To prove that this line closely follows the Alhassid–Whelan arc of regularity in the region between the SU(3) vertex and the critical line of first order shape/phase transition.

Further work is needed in several directions.

1) The present efforts have been limited to the lower part of the spectrum. The whole spectrum has to be considered [57], using measures of chaos, like the nearest neighbour level spacings distribution [58].

2) The branch of the arc between the U(5) vertex and the first order critical line has to be considered by similar appropriate approaches.

4. CONCLUSION

We have shown how conformal maps can be used for introducing a deformation-dependent mass in the Bohr collective model, as well as how group contractions can be used for determining approximate symmetries within the symmetry triangle of the Interacting Boson Approximation model.

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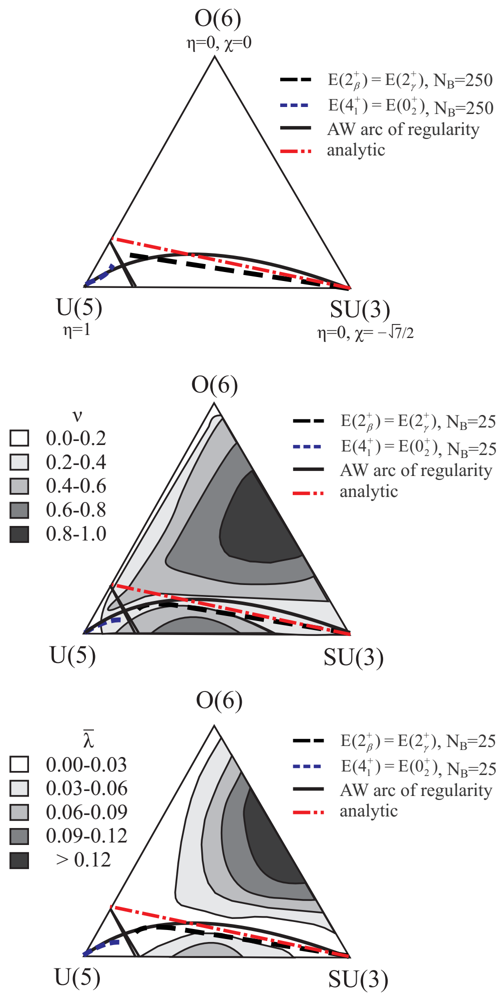

Fig. 1 – IBA symmetry triangle in the parametrization of Eq. (76) with the three dynamical symmetries, the Alhassid–Whelan arc of regularity [Eq. (89)], and the present line of Eq. (88) (labeled as analytic). The shape coexistence region [50] between spherical and deformed phases is shown by slanted lines near the U(5) vertex. In addition, the loci of the degeneracies $E(2^+_\beta)$=$E(2^+_\gamma)$ (dashed line on the right, corresponding to the SU(3) quasi-dynamical symmetry (QDS) discussed in Ref. [56]) and $E(4^+_1)$=$E(0^+_2)$ (dotted line on the left, also discussed in Ref. [56]) are shown for N_B=250 (top) and N_B = 25 (bottom). In the middle and bottom parts, the ν-diagram and the $\bar{\lambda}$-diagram, based on Ref. [48], are shown. See subsection 3.5 for further discussion.