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GRAVITATIONAL INTERACTION OF YANG-MILLS FIELDS FROM FREE-FIELD COHOMOLOGY

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Using cohomological methods we determine the most general form of the interaction between the gravitational field and an arbitrary system of Yang-Mills fields (massless and massive). We solve the corresponding descent equations and obtain the first order chronological product (interaction Lagrangian). Surprisingly enough we find that gravitational ghost and Yang-Mills anti-ghost fields appear in the coupling in the massive Yang-Mills case.

Key words: Perturbation theory, standard model, quantum gravity.

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1. INTRODUCTION

It is standard knowledge that matter fields couple to gravity by their energy - momentum tensor which must be conserved. This statement has a well-defined meaning in classical field theory, but in quantum theory the situation is more subtle. The reason is that the naive expression of the energy - momentum tensor $\mathcal{T}^{\mu\nu} \equiv F^{\mu\rho}\ F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}\ \ F_{\rho\sigma}\ F^{\rho\sigma}$ is not conserved in general. This is due to the fact that gauge conditions like $\partial_\mu A^\mu(x) = 0$ are true on the physical subspace only (Gupta - Bleuler method), not as an operator equation. In this situation we better abstain from classical arguments. Instead we shall construct the mixed spin-1 (for Yang - Mills) and spin-2 (for gravity) quantum gauge theory from scratch without using any classical Lagrangian. Our method is the proper definition of gauge invariance by means of free-field cohomology. It is a pity that this powerful method is not so widely known, because it allows to solve "*the* biggest open problem of theoretical physics" [15], namely the unification of general relativity with quantum theory - at least in flat background. The power-counting non - renormalizability is a necessary consequence of spin-2 [16] and not a sign of failure.

Without having a classical Lagrangian we start from a collection of free fields which are the asymptotic fields of a S-matrix. In a quantum gauge theory a gauge variation d_Q of these free fields is defined by a graded commutator with a nilpotent gauge charge Q

$$d_Q A = [Q, A], \quad Q^2 = 0. \tag{1}$$

Here A is an arbitrary free field operator or any Wick polynomial of them. A coupling $T(x)$ of the free fields is called gauge invariant if it is a Wick polynomial satisfying

$$d_Q T = [Q, T] = i\partial_\mu T^\mu \tag{2}$$

for some other Wick polynomials T^μ. Although the gauge variation (1) is quite similar to the BRS transformation which is defined for interacting fields, we notice the following differences in (2): (i) in BRS the r.h.s. vanishes but we have a non-trivial divergence, (ii) d_{BRS} is non-linear [2] and makes sense only for classical field theories; on the contrary the gauge charge Q is a linear operator in the Hilbert - Fock space generated by the free fields. We should mention however that one can define the non-linear BRST operator in the perturbative sense [14].

The construction of gauge theories now is the problem to find non-trivial solutions T, T^μ of (2). Technically this is a cohomology problem. In all known models one finds out that there exists a chain of Wick polynomials T^μ, $T^{\mu\nu}$, $T^{\mu\nu\rho}, \ldots$ such that:

$$[Q, T] = i\partial_\mu T^\mu, \quad [Q, T^\mu] = i\partial_\nu T^{\mu\nu}, \quad [Q, T^{\mu\nu}] = i\partial_\rho T^{\mu\nu\rho}, \ldots \tag{3}$$

In all cases $T^{\mu\nu}$, $T^{\mu\nu\rho}, \ldots$ are completely antisymmetric in all indices; it follows that the chain of relation stops at the step 4 (if we work in four dimensions). For concrete models the equations (3) can stop earlier: for instance in the case of gravity $T^{\mu\nu\rho\sigma} = 0$.

The paper is organized as follows. In the next Section we introduce the various free fields and their gauge structure. This is well-known (apart from massive gravity which we also consider), we must summarize this material to fix our notation. We also describe the cohomology of the operator d_Q for Yang-Mills models and gravity (the proofs can be found in some previous papers [7, 8].) Using this cohomology we solve in Section 3 the descent equations (3) and we determine the interaction between gravity and massless and massive Yang-Mills fields in the most general case. In this way we generalize the result from [8] where only the case of massless Yang-Mills fields was considered.

We stress the fact that our analysis is of quantum nature. Indeed, relation (2) expresses the fact that the physical states from the Hilbert space are left invariant by the interaction Lagrangian, at least in the adiabatic limit: using a test function f we have

$$[Q, T(f)] = -i\, T^\mu(\partial_\mu f) \tag{4}$$

so, if the test function becomes flatter and flatter we have with a better and better approximation:

$$T(f)\,\mathcal{H}_{\rm phys} \subset \mathcal{H}_{\rm phys}. \tag{5}$$

Also the consistency of the definition of the gauge charge involves the canonical (anti)commutation relations.The result of this paper is the interaction Lagrangian $T(x)$. In terms of Feynman graphs this means only tree contributions which corresponds to the classical theory. However, if higher orders are suitably constructed by the inductive method of Epstein and Glaser $[1, 4, 17]$ then causal gauge invariance like (2) holds for loop graphs, too. This clearly shows that our subject is the quantum theory.

One can also prove that the energy-momentum tensor is conserved when averaged between physical states. Indeed the divergence of this tensor is not null (as said before) but it is a sum between a coboundary and a total divergence. As above we can argue that such an expression does not contribute in the adiabatic limit. Alternatively it is possible to reformulate the coupling so that only physical degrees of freedom are involved. But then manifest renormalizability is lost.

2. FREE FIELDS OF SPIN 1 AND 2

We remind here some results and notations from [7] and [8]. The Hilbert space $\mathcal{H}$ we use is of Fock type generated by some free fields (in the sense of Borchers' theorem) and it should describe particles of spin 1 and 2 with null or positive mass; we will denote by Ω the vacuum state in $\mathcal{H}$. The Pauli-Jordan distribution of mass m is denoted by D_m and $D_m^{(+)}$ is its positive frequency part. The Minkowski metrics (with diagonal $1, -1, -1, -1$) is denoted by $\eta_{\mu\nu}$. We will always mean by $[\cdot, \cdot]$ the graded commutator. We only need to give: the set of fields, the statistics and the 2-point distributions. Because we assume that the fields are free the n-point distributions are generated from the 2-point distributions using Wick theorem for the appropriate statistics. In a more familiar language, it is only necessary to give the non-trivial canonical (anti)commutation relations.

2.1. MASSLESS VECTOR FIELD

We consider a vector space $\mathcal{H}$ of Fock type generated by the vector field A_μ (with Bose statistics) and the ghost fields u, $\tilde{u}$ which are scalars with Fermi statistics. We suppose that all these (quantum) fields are of null mass. The non-trivial (anti)commutators are by definition:

$$[A_\mu(x), A_\mu(y)] = i\,\eta_{\mu\nu}\,D_0(x-y), \qquad \{u(x), \tilde{u}(y)\} = -i\,D_0(x-y) \tag{6}$$

and we also suppose that

$$A_\mu^\dagger = A_\mu, \qquad u^\dagger = u, \qquad \tilde{u}^\dagger = -\tilde{u}. \tag{7}$$

Now we can introduce the operator Q according to the following formulas:

$$[Q, A_\mu] = i\, \partial_\mu u, \qquad [Q, u] = 0, \qquad [Q, \tilde{u}] = -i\, \partial_\mu A^\mu, \qquad Q\Omega = 0 \tag{8}$$

where by $[\cdot,\cdot]$ we mean the graded commutator. One can prove that Q is well defined. Indeed, the operator Q should leave invariant the canonical (anti)commutation relations, in particular

$$[Q, [A_\mu(x_1), \tilde{u}(x_2)]] + \text{cyclic permutations} = 0 \tag{9}$$

which is true according to the previous definition. The usefulness of this construction follows from:

Theorem 2.1 *The operator Q verifies $Q^2 = 0$. The factor space $Ker(Q)/Im(Q)$ is isomorphic to the Fock space of particles of zero mass and helicity 1 (photons, gluons).*

2.2. MASSIVE VECTOR FIELDS

We consider a vector space $\mathcal{H}$ of Fock type generated by the vector field A_μ, the scalar field Φ (with Bose statistics) and the scalar fields u, $\tilde{u}$ (with Fermi statistics). We suppose that all these (quantum) fields are of mass $m > 0$. The non-trivial (anti)commutators are by definition:

$$\begin{gathered}[A_\mu(x), A_\mu(y)] = i\, \eta_{\mu\nu}\, D_m(x-y), \quad [\Phi(x), \Phi(y)] = -i\, D_m(x-y), \\ \{u(x), \tilde{u}(y)\} = -i\, D_m(x-y),\end{gathered} \tag{10}$$

and we also suppose that:

$$A_\mu^\dagger = A_\mu, \qquad u^\dagger = u, \qquad \tilde{u}^\dagger = -\tilde{u}, \qquad \Phi^\dagger = \Phi. \tag{11}$$

Now we introduce $\mathcal{H}$ the operator Q according to the following formulas:

$$\begin{gathered}[Q, A_\mu] = i\partial_\mu u, \qquad [Q, u] = 0, \qquad [Q, \tilde{u}] = -i\, (\partial_\mu A^\mu + m\, \Phi), \\ [Q, \Phi] = i\, m\, u, \qquad Q\Omega = 0.\end{gathered} \tag{12}$$

One can prove that Q is well defined. We then have:

Theorem 2.2 *The operator Q verifies $Q^2 = 0$. The factor space $Ker(Q)/Im(Q)$ is isomorphic to the Fock space of particles of mass m and spin 1 (massive photons, vector bosons).*

2.3. MASSLESS GRAVITONS

We consider the vector space $\mathcal{H}$ of Fock type generated by the symmetric tensor field $h_{\mu\nu}$ (with Bose statistics) and the (ghost) vector fields u^ρ, $\tilde{u}^\sigma$ (with Fermi statistics). We suppose that all these (quantum) fields are of null mass. The non-trivial (anti)commutators are by definition:

$$[h_{\mu\nu}(x), h_{\rho\sigma}(y)] = -\frac{i}{2}(\eta_{\mu\rho}\,\eta_{\nu\sigma} + \eta_{\nu\rho}\,\eta_{\mu\sigma} - \eta_{\mu\nu}\,\eta_{\rho\sigma}) D_0(x-y),$$
$$\{u_\mu(x), \tilde{u}_\nu(y)\} = i\,\eta_{\mu\nu}\,D_0(x-y), \tag{13}$$

and we define the conjugation by

$$h_{\mu\nu}^\dagger = h_{\mu\nu}, \qquad u_\rho^\dagger = u_\rho, \qquad \tilde{u}_\sigma^\dagger = -\tilde{u}_\sigma. \tag{14}$$

Now we can introduce the operator Q according to the following formulas:

$$[Q, h_{\mu\nu}] = -\frac{i}{2}\,(\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu}\partial_\rho u^\rho), \qquad [Q, u_\mu] = 0,$$
$$[Q, \tilde{u}_\mu] = i\,\partial^\nu h_{\mu\nu}, \qquad Q\Omega = 0, \tag{15}$$

where by $[\cdot,\cdot]$ we mean the graded commutator. One can prove that Q is well defined. The usefulness of this construction follows from the following result [6]:

Theorem 2.3 *The operator Q verifies $Q^2 = 0$. The factor space $Ker(Q)/Im(Q)$ is isomorphic to the Fock space of particles of zero mass and helicity 2 (gravitons).*

2.4. MASSIVE GRAVITONS

We consider a vector space $\mathcal{H}$ of Fock type generated by the tensor field $h_{\mu\nu}$, the vector field v_μ (with Bose statistics) and the vector fields u_μ, $\tilde{u}_\mu$ (with Fermi statistics). We suppose that all these (quantum) fields are of mass $m > 0$.

The non-trivial commutators are

$$[h_{\mu\nu}(x), h_{\rho\sigma}(y)] = -\frac{i}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma}) D_m(x-y),$$
$$\{u(x), \tilde{u}(y)\} = i\eta_{\mu\nu} D_m(x-y), \qquad [v_\mu(x) v_\mu(y)] = \frac{i}{2}\eta_{\mu\nu} D_m(x-y) \tag{16}$$

We define the operator Q according to the following formulas [16]:

$$[Q, h_{\mu\nu}] = -\frac{i}{2}\,(\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu}\partial_\rho u^\rho),$$
$$[Q, u_\mu] = 0,\ [Q, \tilde{u}_\mu] = i\,(\partial^\nu h_{\mu\nu} - m v_\mu), \quad [Q, v_\mu] = -\frac{i\,m}{2}\,u_\mu,\ Q\Omega = 0. \tag{17}$$

We have the result [11]:

Theorem 2.4 *The operator Q verifies $Q^2 = 0$. The factor space $Ker(Q)/Im(Q)$ is isomorphic to the Fock space of particles of mass m and spin 2 (massive gravitons) plus a spin 0 particle of mass m.*

The gauge invariant first-order coupling between the Lorentz vector field v_μ and the tensor field comes out to be $\sim h^{\mu\nu}\,\partial_\mu v^\lambda\,\partial_\nu v_\lambda$ (see [11]) with partial derivatives instead of covariant derivatives $\nabla_\mu v^\lambda \equiv \partial_\mu v^\lambda + \Gamma^\lambda_{\mu\nu}\,v^\nu$ as we might expect if

v_μ would be a vector field with respect to general coordinates transformation. If that would be the case the additional terms depending on the Christoffel symbols Γ from $h^{\mu\nu}\ \nabla_\mu v^\lambda\ \nabla_\nu v_\lambda$ should appear in the second order of the perturbation theory as finite renormalizations. However the second order finite renormalizations can be chosen independent of v_μ [11] so the last hope would be that the additional terms depending on the Christoffel symbols Γ can be rewritten as a divergence plus a coboundary. However, one can prove that this is not possible (see [16] for details) and that means that v_μ must be considered as scalars with respect to general coordinates transformations.

2.5. THE GENERAL CASE

The situations described above are susceptible to the following generalizations. First we consider the Yang-Mills case. We take a system of r_1 species of particles of null mass and helicity 1, that means we use from the first part of this Section r_1 triplets $(A_a^\mu, u_a, \tilde{u}_a), a \in I_1$ of massless fields; here I_1 is a set of indices of cardinal r_1. All the relations have to be modified by appending an index a to all these fields. In the massive case we have to consider r_2 quadruples $(A_a^\mu, u_a, \tilde{u}_a, \Phi_a), a \in I_2$ of fields of mass m_a ; here I_2 is a set of indices of cardinal r_2. We want to include some arbitrary scalar fields with indices $a \in I_3$. Then we take $I = I_1 \cup I_2 \cup I_3$ a set of indices and for any index we take a quadruple $(A_a^\mu, u_a, \tilde{u}_a, \Phi_a), a \in I$ of fields with the following conventions: *(a)* the first entry are vector fields and the last three ones are scalar fields; *(b)* the fields $A_a^\mu,\ \Phi_a$ are obeying Bose statistics and the fields $u_a,\ \tilde{u}_a$ are obeying Fermi statistics; *(c)* For $a \in I_1$ we impose $\Phi_a = 0$ and we take the masses to be null $m_a = 0$; *(d)* For $a \in I_2$ we take all the masses strictly positive: $m_a > 0$; *(e)* For $a \in I_3$ we take $A_a^\mu, u_a, \tilde{u}_a$ to be null and the fields $\Phi_a \equiv \phi_a^H$ of mass $m_a^H \geq 0$. The fields $u_a,\ \tilde{u}_a,\ \ a \in I_1 \cup I_2$ and $\Phi_a,\ a \in I_2$ are called *ghost fields* and the fields $\phi_a^H,\ a \in I_3$ are called *Higgs fields*; *(f)* we include spinorial matter fields also *i.e.* some set of Dirac fields with Fermi statistics: $\Psi_A,\ A \in I_4$; *(g)* we consider that the Hilbert space is generated by all these fields applied on the vacuum and define in $\mathcal{H}$ the gauge charge operator Q according to the following formulas for all indices $a \in I$:

$$\begin{aligned} [Q, A_a^\mu] = i\ \partial^\mu u_a, \qquad [Q, u_a] = 0, \\ [Q, \tilde{u}_a] = -i\ (\partial_\mu A_a^\mu + m_a\ \Phi_a), \qquad [Q, \Phi_a] = i\ m_a\ u_a, \end{aligned} \tag{18}$$

$$[Q, \Psi_A] = 0 \quad \text{and} \quad Q\Omega = 0. \tag{19}$$

If we want to include gravitons also then we extend the Fock space including the corresponding free fields $h_{\mu\nu}, u^\rho, \tilde{u}^\sigma$ and we extend the definition of the gauge charge Q in a natural way using (15). In this way the Fock space will describe a system of Yang-Mills particles together with gravitons.

2.6. THE COHOMOLOGY OF THE d_Q OPERATOR

We can reduce the analysis of the descent equations (3) to the determination of the cohomology of the operator d_Q in the space of Wick polynomials. One can solve this problem in a quite general setting using the jet bundle formalism [7] and [8]. However, for practical purposes we only need to know that every cocycle of this operator is cohomologous to a polynomial depending only on gauge-invariant variables.

For the case of massless or massive spin 1 fields, beside the the gauge invariant field u we define the *field strength* according to

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \tag{20}$$

and observe that it is also gauge invariant: $d_Q F^{\mu\nu} = 0$. The derivatives of $F^{\mu\nu}$ are also gauge invariants.

In the case of a massive vector field A_μ with mass m we define:

$$\phi_\mu \equiv \partial_\mu \Phi - m\ A_\mu \tag{21}$$

and we observe that $d_Q \phi_\mu = 0$. For the general Yang-Mills system we have to append an index a and we have the invariants $u_a,\ F_{a\mu\nu},\ (a \in I_1 \cup I_2)$ and $\phi_{a\mu}\ (a \in I_2)$.

In the case of the gravitational field it is convenient to introduce some other notations: first

$$h \equiv \eta^{\mu\nu} h_{\mu\nu} \qquad \hat{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\ \eta_{\mu\nu}\ h \tag{22}$$

and the we define the *Christoffel symbols* according to:

$$\Gamma_{\mu;\nu\rho} \equiv \partial_\rho \hat{h}_{\mu\nu} + \partial_\nu \hat{h}_{\mu\rho} - \partial_\mu \hat{h}_{\nu\rho}. \tag{23}$$

The expression

$$R_{\mu\nu;\rho\sigma} \equiv \partial_\rho \Gamma_{\mu;\nu\sigma} - (\rho \leftrightarrow \sigma) \tag{24}$$

is called the *Riemann tensor*; we can easily prove that it is gauge invariant $d_Q R_{\mu\nu;\rho\sigma} = 0$. The derivatives of the Riemann tensor are also gauge invariant. We also define

$$u_{\mu\nu} = u_{[\mu\nu]} \equiv \frac{1}{2}\ (\partial_\mu u_\nu - \partial_\nu u_\mu). \tag{25}$$

The results from [7] and [8] describe the cohomology of the operator d_Q in the space of polynomials depending on all these variables (and their derivatives). Essentially, it says that every cocycle *i.e.* expression verifying $d_Q p = 0$ is cohomologous to a polynomial in the gauge invariants defined above. Some of these invariant expressions can be coboundary: for instance the (non-null) traces of the derivatives of the field strength $\partial_{\lambda_1} \dots \partial_{\lambda_n} F_{a\mu\nu}$, of the Riemann tensor $\partial_{\lambda_1} \dots \partial_{\lambda_n} R_{\mu\nu;\rho\sigma}$ and $\partial_{\lambda_1} \dots \partial_{\lambda_n} u_{\mu\nu}$. However, we do not need an explicit description of the space $Ker(d_Q)/Im(d_Q)$ the previous result being sufficient for computations.

We mention in closing that for polynomials at least tri-linear in the fields, the Poincaré lemma holds *i.e.* if $\partial_\mu T^\mu = 0$ holds for such a polynomial expression then $T^\mu = \partial_\nu T^{\mu\nu}$ where the expression $T^{\mu\nu}$ is antisymmetric [7].

3. THE INTERACTION OF GRAVITY WITH OTHER QUANTUM FIELDS

We here consider a system of massive and massless Yang-Mills fields: the set of fields is $(A_a^\mu, u_a, \tilde{u}_a, \Phi_a), a \in I$ and we determine the coupling with the massless gravitational field $h_{\mu\nu}$, u^ρ, $\tilde{u}^\sigma$. By definition the ghost number is the sum of the ghost numbers of the YM and gravity sectors. As in [8] we consider that the interaction has null ghost number and its canonical dimension is bounded by 5. We will get an expression of the form

$$T_{\rm int} = T_{\rm int}^{\rm YM} + T_{\rm int}^{\rm scalar} + T_{\rm int}^{\rm Fermi} \tag{26}$$

with the scalar and Fermi contributions being the same as in [8]. We concentrate only on the Yang-Mills contribution and we have our main result. We use the following definitions. A Wick polynomial T is called a *relatively co-cycle* iff it verifies the relation $d_Q T = \partial_\mu T^\mu$; two Wick polynomials are *relatively cohomologous* iff they differ by an expression of the type $d_Q B + \partial_\mu B^\mu$. As it was already seen in previous papers, one can reduce the problem of determining the relative cohomology groups to the cohomology of the operator d_Q using the descent procedure. We determine here only the first term of (26) *i.e.* we take: $T_{\rm int} = T_{\rm int}^{\rm YM}$.

Theorem 3.1 *(i) The expression $T_{\rm int}^{\rm YM}$ is relatively cohomologous to*

$$\begin{aligned} t_{\rm int} \equiv \sum_{a\in I_1} f_a\ (4 h_{\mu\nu}\ F_a^{\mu\rho}\ F^\nu_{a\ \rho} - h\ F_{a\rho\sigma}\ F_a^{\rho\sigma} + 4\ u_\mu\ \partial_\nu \tilde{u}_a\ F_a^{\mu\nu}) \\ + \sum_{a\in I_2} f_a\ (4 h_{\mu\nu}\ F_a^{\mu\rho}\ F^\nu_{a\ \rho} - h\ F_{a\rho\sigma}\ F_a^{\rho\sigma} \\ + 4\ u_\mu\ \partial_\nu \tilde{u}_a\ F_a^{\mu\nu} - 4\ h_{\mu\nu}\ \phi_a^\mu\ \phi_a^\nu - 4\ m_a\ u_\mu\ \tilde{u}_a\ \phi_a^\mu) \end{aligned} \tag{27}$$

with real constants f_a.
(ii) The relation $d_Q t_{\rm int} = i\ \partial_\mu t_{\rm int}^\mu$ is verified by:

$$\begin{aligned} t_{\rm int}^\mu \equiv \sum_{a\in I_1} f_a\ (u^\mu\ F_a^{\rho\sigma}\ F_{a\rho\sigma} + 4\ u^\rho\ F_a^{\mu\nu}\ F_{a\nu\rho}) \\ + \sum_{a\in I_2} f_a (u^\mu\ F_a^{\rho\sigma} F_{a\rho\sigma} + 4\ u^\rho\ F_a^{\mu\nu} F_{a\nu\rho} - 2\ u^\mu\ \phi_{a\nu}\ \phi_a^\nu + 4\ u_\nu\ \phi_a^\mu \phi_a^\nu) \end{aligned} \tag{28}$$

and we also have

$$d_Q t_{\rm int}^\mu = 0. \tag{29}$$

Proof: *(i)* By hypothesis we have

$$d_Q T_{\rm int} = i\,\partial_\mu T^\mu_{\rm int} \tag{30}$$

and the descent procedure based on Poincaré lemma (see [7,8,12]) leads to

$$\begin{array}{cc} d_Q T^\mu_{\rm int} = i\,\partial_\nu T^{\mu\nu}_{\rm int}, & d_Q T^{\mu\nu}_{\rm int} = i\,\partial_\rho T^{\mu\nu\rho}_{\rm int}, \\ d_Q T^{\mu\nu\rho}_{\rm int} = i\,\partial_\sigma T^{\mu\nu\rho\sigma}_{\rm int}, & d_Q T^{\mu\nu\rho\sigma}_{\rm int} = 0 \end{array} \tag{31}$$

and can choose the expressions $T^I_{\rm int}$ to be Lorentz covariant; we also have

$$gh(T^I_{\rm int}) = |I|, \quad \omega(T^I_{\rm int}) \leq 5. \tag{32}$$

From the last relation we find that

$$T^{\mu\nu\rho\sigma}_{\rm int} = d_Q B^{\mu\nu\rho\sigma} + T^{\mu\nu\rho\sigma}_{\rm int,0} \tag{33}$$

with $T^{\mu\nu\rho\sigma}_{\rm int,0}$ a polynomial in the invariants described in the preceding Section and we can choose the expressions $B^{\mu\nu\rho\sigma}$ and $T^{\mu\nu\rho\sigma}_{\rm int,0}$ completely antisymmetric. The generic form of $T^{\mu\nu\rho\sigma}_{\rm int,0}$ can be easily obtained. If we substitute the expression of $T^{\mu\nu\rho\sigma}_{\rm int}$ in the third relation (31) we find out

$$d_Q(T^{\mu\nu\rho}_{\rm int} - i\,\partial_\sigma B^{\mu\nu\rho\sigma}) = i\,\partial_\sigma T^{\mu\nu\rho\sigma}_{\rm int,0} \tag{34}$$

so the expression in the right hand side must be a co-boundary and we immediately obtain $T^{\mu\nu\rho\sigma}_{\rm int,0} = 0$ so:

$$T^{\mu\nu\rho\sigma}_{\rm int} = d_Q B^{\mu\nu\rho\sigma} \tag{35}$$

and

$$d_Q(T^{\mu\nu\rho}_{\rm int} - i\,\partial_\sigma B^{\mu\nu\rho\sigma}) = 0. \tag{36}$$

We continue in the same way and obtain:

$$T^{\mu\nu\rho}_{\rm int} = d_Q B^{\mu\nu\rho} + i\,\partial_\sigma B^{\mu\nu\rho\sigma} \tag{37}$$

and

$$T^{\mu\nu}_{\rm int} = d_Q B^{\mu\nu} + i\,\partial_\rho B^{\mu\nu\rho}. \tag{38}$$

(ii) We substitute the expression of $T^{\mu\nu}_{\rm int}$ in the first relation (31) and get:

$$d_Q(T^\mu_{\rm int} - i\,\partial_\nu B^{\mu\nu}) = 0. \tag{39}$$

As above we obtain

$$T^\mu_{\rm int} = d_Q B^\mu + i\,\partial_\nu B^{\mu\nu} + T^\mu_{\rm int,0}, \tag{40}$$

where $T^\mu_{\rm int,0}$ is a polynomial in the invariants. We get from the first relation (30)

$$d_Q(T_{\rm int} - i\,\partial_\mu B^\mu) = i\,\partial_\mu T^\mu_{\rm int,0}, \tag{41}$$

so the right hand side must be a co-boundary. At this stage of the computation some non-trivial co-cycles do appear in $T^{\mu}_{\rm int,0}$ namely

$$T^{\mu}_{\rm int,0} = f^{(1)}_{ab}\ u^{\mu}\ F^{\rho\sigma}_{a}\ F_{b\rho\sigma} + f^{(2)}_{ab}\ u^{\rho}\ F^{\mu\nu}_{a}\ F_{b\nu\rho} + g^{(1)}_{ab}\ u^{\mu}\ \phi_{a\nu}\ \phi^{\nu}_{b} + g^{(2)}_{ab}\ u_{\nu}\ \phi^{\mu}_{a}\ \phi^{\nu}_{b} + \cdots, \tag{42}$$

where by $\cdots$ we mean parity violating terms with the ϵ - tensor; we can impose the symmetry conditions

$$f^{(1)}_{ab} = a \leftrightarrow b, \qquad g^{(1)}_{ab} = a \leftrightarrow b. \tag{43}$$

If one computes the divergence $\partial_{\mu} T^{\rm YM,\mu}_{\rm int,0}$ and imposes the condition that it is a co-boundary, then one gets:

$$\begin{gathered} f^{(2)}_{ab} = 4\ f^{(1)}_{ab}, \qquad g^{(2)}_{ab} = -2\ g^{(1)}_{ab}, \\ 2m_b\ f^{(1)}_{ab} + m_a\ g^{(1)}_{ab} = 0\ (\forall a, b \in I_2), \qquad f^{(1)}_{ab} = 0\ (\forall a \in I_1, b \in I_2) \end{gathered} \tag{44}$$

and also the parity violating terms $\cdots$ from the expression of $T^{\mu}_{\rm int,0}$ are null. The first equality above comes from terms without a mass factor and the second one from terms with a mass factor. It follows that $f^{(1)}_{ab} = 0$ if $m_a \neq m_b$. In the sector of fields with equal mass $m_a = m$ the real symmetric matrix $f^{(1)}_{ab}$ can be diagonalized by an orthogonal transformation. Such a transformation of the fields is always possible without changing commutation relations, gauge structure etc. With this choice of the basic fields it follows

$$T^{\mu}_{int,0} = t^{\mu}_{\rm int} \tag{45}$$

with $t^{\mu}_{\rm int}$ the expression from the statement of the theorem. Because we have by direct computation $d_Q t_{\rm int} = i\ \partial_{\mu} t^{\mu}_{\rm int}$ we get

$$d_Q (T_{\rm int} - t_{\rm int} - i\ \partial_{\mu} B^{\mu}) = 0 \tag{46}$$

so known results lead to

$$T_{\rm int} = t_{\rm int} + d_Q B + i\ \partial_{\mu} B^{\mu} + T_{\rm int,0} \tag{47}$$

where $T_{\rm int,0}$ is a polynomial in the invariants. But there are no such expression *i.e.* $T_{\rm int,0} = 0$ and we have

$$T_{\rm int} = t_{\rm int} + d_Q B + i\ \partial_{\mu} B^{\mu} \tag{48}$$

which is the final result.

We point out that in all these computations one should consider all polynomials in the invariants, even or odd with respect to parity invariance. Indeed, because the Yang-Mills interaction is not parity invariant, there are no reasons to suppose that the interaction with gravity is parity invariant. Fortunately, the odd sectors do not produce non-trivial obstructions to the descent procedure.

We end up with a number of comments. First, let us note that the result of the theorem stays true if we replace massless gravity by massive gravity. The interaction of massive gravity with scalar fields does produce some new coupling; this case is studied in detail in [13].

Second, let us clear up in what sense the energy-momentum tensor is conserved in our setting. From the statement of the theorem we extract the following expression for this tensor;

$$\mathcal{T}^{\mu\nu} \equiv f_a\,(4\,F_a^{\mu\rho}\,F^{\nu}_{a\;\rho} - \eta^{\mu\nu}\,F_{a\rho\sigma}\,F_a^{\rho\sigma} - 4\,\phi_a^{\mu}\,\phi_a^{\nu}) \tag{49}$$

where we use the summation convention over $a \in I_1 \cup I_2$. This expression is the coefficient of $h_{\mu\nu}$ from formula (27). Then one can easily prove that we have

$$\partial_\nu\,\mathcal{T}^{\mu\nu} \equiv 4i\,f_a\,d_Q(F_a^{\mu\nu}\,\partial_\nu\tilde{u}_a - m_b\,\phi^{\mu}\,\tilde{u}_a) - 2\partial^{\mu}(f_a\,\phi_a^{\nu}\,\phi_{a\nu}) \tag{50}$$

and this shows that this tensor is conserved in the physical subspace, at least in the adiabatic limit (see the Introduction).

Finally we mention that the interaction between massive Yang-Mills fields and gravity (*i.e.* the second line in the formula (27)) can be put into a simpler form. For simplicity we first consider one massive vector field *i.e.* $|I_2| = 1$. In this case we can skip the index $a = 1$ and we define the *physical part of the vector field* A_μ according to the formula [16]:

$$A_\mu^{\rm phys} \equiv A_\mu + \frac{1}{m^2}\,\partial^{\mu}\,\partial_\nu\,A_\nu. \tag{51}$$

This field has the following properties:

$$d_Q A_\mu^{\rm phys} = 0, \qquad \partial^{\mu} A_\mu^{\rm phys} = 0. \tag{52}$$

Then one can prove by some computations the following formula:

$$h_{\mu\nu}\phi_a^{\mu}\phi_b^{\nu} + m_a u_\mu \tilde{u}_a \phi_b^{\mu} - u_\mu \partial_\nu \tilde{u}_a F_b^{\mu\nu} = m^2 h^{\mu\nu} A_\mu^{\rm phys} A_\nu^{\rm phys} + d_Q B + \partial_\mu B^{\mu}, \tag{53}$$

i.e. we can express the interaction between the massive vector field A_μ and gravity in terms of the physical part of A_μ in a standard form:

$$t_{\rm int} = \hat{h}^{\mu\nu}\,\mathcal{T}_{\mu\nu}^{\rm phys}, \tag{54}$$

where we have defined h and $\hat{h}_{\mu\nu}$ in formula (22) and

$$\mathcal{T}_{\mu\nu}^{\rm phys} \equiv F_{\mu\rho}\,F_\nu{}^{\rho} - \frac{1}{4}\,\eta_{\mu\nu}\,F_{\rho\sigma}\,F^{\rho\sigma} - m^2\,A_\mu^{\rm phys}\,A_\nu^{\rm phys} + \frac{m^2}{2}\,\eta_{\mu\nu}\,A^{{\rm phys},\rho}\,A_\rho^{\rm phys} \tag{55}$$

is the *energy-momentum tensor*; one can prove directly that it is conserved:

$$\partial^{\nu}\,\mathcal{T}_{\mu\nu}^{\rm phys} = 0. \tag{56}$$

In this form gravity couples to the physical degrees of freedom only, because there is no coupling to the ghost and $A_\mu^{\rm phys}$ contains the three transverse physical modes only [11]. However, the new expression of the interaction Lagrangian has canonical dimension 7 so for the purpose of perturbation theory it is better to work with the expression appearing in the theorem.

The energy-momentum tensor in (27) is additive with respect to the vector fields, but the coupling constants f_a may be still different. It follows from gauge invariance in the second order of the perturbation theory that they are all equal to $f_a = \sqrt{8\pi G}$ where G is Newton's constant. The proof is similar to the one for scalar fields in [13].

4. CONCLUSIONS

We have obtained the Feynman rules for the interaction between the particles of the standard model and (quantum) gravity. Indeed, by Feynman rules one understands usually the Feynman propagators and the Feynman vertices. From the expressions (6), (10) and (13) we obtain, using causal splitting, the Feynman propagators. In the first two cases the distributions from the right hand side has negative order of singularity this splitting is unique. For instance we have from (6) the corresponding advanced and retarded products:

$$\begin{aligned} A(A_\mu(x), A_\mu(y)) &= i\, \eta_{\mu\nu}\, D_0^{\rm adv}(x-y), \\ R(A_\mu(x), A_\mu(y)) &= i\, \eta_{\mu\nu}\, D_0^{\rm ret}(x-y). \end{aligned} \tag{57}$$

Then we use the general definition of the chronological product (see [4], sect. 3):

$$\begin{aligned} T(A(x), B(y)) = A(A(x), B(y)) + :B(y)A(x): \\ = R(A(x), B(y)) + :A(x)B(y): \end{aligned} \tag{58}$$

and obtain the (well-known) corresponding chronological product:

$$T(A_\mu(x), A_\mu(y)) = i\, \eta_{\mu\nu}\, D_0^F(x-y). \tag{59}$$

If we consider the first relation (13) we have in the right hand side a distribution with null order of singularity. We obtain in the same way the chronological product for the gravitational field:

$$T(h_{\mu\nu}(x), h_{\rho\sigma}(y)) = -\frac{i}{2}\, (\eta_{\mu\rho}\, \eta_{\nu\sigma} + \eta_{\nu\rho}\, \eta_{\mu\sigma} - \eta_{\mu\nu}\, \eta_{\rho\sigma})\, D_0^F(x-y) \tag{60}$$

but we have the freedom to add in the right hand side terms proportional to $\delta(x-y)$. However, it is convenient to choose the preceding expression as the Feynman propagator for the gravitational field. The (tri-linear) Feynman vertices follow from (27) if we go in the momentum space, as it is usually done. So, we see that, in the Epstein-Glaser formalism, we obtain an unique expression for the interaction between quantum gravity and the particles of the standard model. We are not aware if such an expression was previously derived (and in an unique way) using the more popular functional formalism.

Further developments are possible. For instance, one can go to the second order of perturbation theory using the off-shell formalism [9, 10].

In classical general relativity one says that gravity couples to everything which carries energy and momentum. That means one must have an energy-momentum tensor which is conserved $\partial_\nu\ \mathcal{T}^{\mu\nu} = 0$. In quantum theory it is harder to find the correct gravitational couplings by classical Lagrangian arguments.

Fortunately, the requirement of gauge invariance $d_Q T = i\ \partial_\mu T^\mu$ is so strong that it determines all couplings uniquely, if some natural additional properties are assumed. It is a surprise that the resulting gravitational couplings contain ghost fields because they do not "carry energy and momentum". The paradox is resolved by observing that the ghost coupling terms do not contribute to S-matrix elements between physical states.

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