

MADELUNG FLUID DESCRIPTION OF A COUPLED SYSTEM OF DERIVATIVE NLS EQUATIONS

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A system of coupled derivative NLS equations is discussed using a Madelung's fluid description. The system is the matrix generalization of the completely integrable derivative NLS equation of Chen-Lee-Liu. For an arbitrary nonlinearity of the form $\beta \left(|\Psi_1|^2 + |\Psi_2|^2 \right)^q \Psi_j$ the bright solitary wave solution is obtained. For the integrable case ($q = 1$) the periodic solutions, expressed through Jacobi elliptic functions, are found. When $k^2 = 1$ (k - the modulus of the elliptic function) various shifted bright, gray and dark solutions are obtained as limit cases of the previous periodic solutions.

Key words: Madelung fluid, NLS equation, PDE coupled system.

1. INTRODUCTION

Eighty five years ago E. Madelung gave a hydrodynamic description of the new born quantum mechanics [1]. For the one-dimensional Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} - U(x)\Psi = 0, \quad (1)$$

seeking a solution of the form

$$\Psi = \sqrt{\rho} e^{i\frac{\theta}{\hbar}},$$

with both ρ, θ depending on x and t the equation (1) is equivalent with the system

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} &= 0, \\ m \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) v &= \frac{\partial}{\partial x} \left[\frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} - U(x) \right], \end{aligned} \quad (2)$$

where

$$v(x, t) = \frac{1}{m} \frac{\partial \theta(x, t)}{\partial x}.$$

The first equation is a continuity equation for the fluid density $\rho(x, t) = |\Psi(x, t)|^2$ and velocity $v(x, t)$, while the second is a Navier-Stokes type equation of motion for the fluid velocity $v(x, t)$. In the *r.h.s.* of the second equation, beside the usual force

term $\left(-\frac{\partial U}{\partial x}\right)$, the derivative of $\left(\frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2}\right)$ appears, known in the literature as Bohm's potential [2], and which contains all the quantum effects. In spite of its flaws (it cannot give a proper solution of the problem of atomic eigenstates and to the quantum description of emission or absorption processes), this approach turns to be fruitful in a number of applications like the stochastic quantum mechanics [3], quantum cosmology [4], description of quantum-like systems [5], the coherent properties of high-energy charged particle beams [6, 7].

Ten years ago, Fedele *et al.* [8–11] have successfully used this approach to study the solitary wave solutions of generalized NLS equation

$$i\alpha \frac{\partial \Psi}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} + U(|\Psi|^2)\Psi = 0, \quad (3)$$

where $U(|\Psi|^2)$ is an arbitrary polynomial in $|\Psi|^2$. A large class of bright, shifted bright, dark and gray solitary wave solutions were obtained for different type of nonlinearities. Quite recently, this Madelung fluid description was used to find periodic solutions of gNLS equation [12], periodic and solitary wave solutions of derivative NLS equations [13, 14] and coupled NLS equations [15]. Also based on this approach an interesting correspondence (at least in the class of traveling wave solutions) between generalized NLS equations and generalized KdV equations was established and discussed [16].

In the present paper this method will be used to study a system of coupled derivative NLS equations of the form of the matrix extension of the completely integrable Chen-Lee-Liu equation [17, 18]. In the next section the basic equations will be presented. In section three, the bright solitary wave solution for a general nonlinearity $\beta\rho_+^q$, $\rho_+ = |\Psi_1|^2 + |\Psi_2|^2$ will be obtained, while in the next one the periodic solutions, expressed through Jacobi elliptic functions, will be found for the integrable case ($q = 1$). A short comment on the results will be given in the last section.

2. BASIC EQUATIONS

One considers the following system of coupled derivative NLS equations

$$\begin{aligned} i\frac{\partial \Psi_1}{\partial t} + \frac{1}{2} \frac{\partial^2 \Psi_1}{\partial x^2} + iU(|\Psi_1|^2 + |\Psi_2|^2) \frac{\partial \Psi_1}{\partial x} &= 0, \\ i\frac{\partial \Psi_2}{\partial t} + \frac{1}{2} \frac{\partial^2 \Psi_2}{\partial x^2} + iU(|\Psi_1|^2 + |\Psi_2|^2) \frac{\partial \Psi_2}{\partial x} &= 0, \end{aligned} \quad (4)$$

where U can be an arbitrary power function of $\rho_+ = |\Psi_1|^2 + |\Psi_2|^2$, $U = \beta\rho_+^q$. The case $q = 1$ is the two-component case of the completely integrable matrix extension of the type-1 of Chen-Lee-Liu system [18]. Seeking solutions of the form

$$\Psi_j = \sqrt{\rho_j} e^{i\theta_j}, \quad j = 1, 2,$$

where ρ_j, θ_j are real functions of (x, t) , the system (4) is equivalent with

$$\frac{\partial \rho_j}{\partial t} + \frac{\partial(v_j \rho_j)}{\partial x} + U(\rho_+) \frac{\partial \rho_j}{\partial x} = 0 \quad (5)$$

and

$$\frac{\partial \theta_j}{\partial t} + \frac{1}{2} v_j^2 = \frac{1}{2\sqrt{\rho_j}} \frac{\partial^2 \sqrt{\rho_j}}{\partial x^2} - v_j U(\rho_+), \quad (6)$$

where

$$v_j(x, t) = \frac{\partial \theta_j(x, t)}{\partial x}.$$

By differentiating the relation (6) with respect to x we obtain

$$\left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x} \right) v_j = \frac{\partial}{\partial x} \left[\frac{1}{2\sqrt{\rho_j}} \frac{\partial^2 \sqrt{\rho_j}}{\partial x^2} - v_j U(\rho_+) \right]. \quad (7)$$

Following Fedele [8], [9] the equation (7) is transformed into

$$\begin{aligned} -\rho_j \frac{\partial v_j}{\partial t} + v_j \frac{\partial \rho_j}{\partial t} + 2 \left[c_j(t) - \int \frac{\partial v_j}{\partial t} dx \right] \frac{\partial \rho_j}{\partial x} + \\ \frac{1}{4} \frac{\partial^3 \rho_j}{\partial x^3} - \rho_j \frac{\partial}{\partial x} (v_j U(\rho_+)) - (v_j U(\rho_+)) \frac{\partial \rho_j}{\partial x} = 0. \end{aligned} \quad (8)$$

In spite of the complicated form of equation (8) we shall obtain solutions in two simplifying situations, namely

A) equal, constant velocities

$$v_1 = v_2 = v_0 = \text{const.}$$

B) equal velocities and constant profile solution, when $v_1(x, t) = v_2(x, t) = v(\xi)$,
 $\rho_1(x, t) = \rho_1(\xi)$, $\rho_2(x, t) = \rho_2(\xi)$ with $\xi = x - u_0 t$.

For equal and constant velocities, case (A), from (5) we obtain

$$\frac{\partial \rho_+}{\partial t} + (v_0 + U(\rho_+)) \frac{\partial \rho_+}{\partial x} = 0,$$

which has the solution

$$\rho_+(x, t) = f[x - (v_0 + U(\rho_+))t], \quad (9)$$

where $f(x) = \rho_+(x, t=0)$ is the initial condition. In the same way from (8) we get ($c_1 = c_2 = c_0 = \text{const.}$)

$$v_0 \frac{\partial \rho_+}{\partial t} + \frac{1}{4} \frac{\partial^3 \rho_+}{\partial x^3} + 2c_0 \frac{\partial \rho_+}{\partial x} - v_0 \rho_+ \frac{\partial U}{\partial x} - v_0 U \frac{\partial \rho_+}{\partial x} = 0. \quad (10)$$

The solution (9) is incompatible with the dispersive equation (10) and consequently a solution with equal and constant velocities for our problem doesn't exist. The same conclusion was obtained in the case of derivative NLS equations [13, 14].

In the case (B) of constant profile solutions, from equations (5) we obtain

$$\frac{d}{d\xi}(-u_0\rho_+ + v\rho_+ + G) = 0, \quad (11)$$

where we denoted

$$U(\rho_+) = \frac{dG(\rho_+)}{d\rho_+}. \quad (12)$$

Integrating (11) we get

$$v(\xi) = u_0 - \frac{G}{\rho_+} + \frac{A_0}{\rho_+}, \quad (13)$$

where A_0 is an integration constant, which for vanishing solutions at infinity has to be taken equal to zero. Introducing (13) in (5) we find

$$\frac{1}{\rho_j} \frac{d\rho_j}{d\xi} = \frac{1}{\rho_+} \frac{d\rho_+}{d\xi}$$

and consequently any ρ_j is proportional with ρ_+ ,

$$\rho_j = p_j^2 \rho_+, \quad p_1^2 + p_2^2 = 1. \quad (14)$$

Therefore the problem is reduced to finding the solution of the equation satisfied by ρ_+ , which for a nonlinearity $U(\rho_+) = \beta\rho_+^q$ writes

$$\begin{aligned} \frac{1}{4} \frac{d^3\rho_+}{d\xi^3} + (2c_0 + u_0^2) \frac{d\rho_+}{d\xi} - (2+q)u_0\beta\rho_+^q \frac{d\rho_+}{d\xi} \\ + \frac{2q+1}{q+1} \beta^2 \rho_+^{2q} \frac{d\rho_+}{d\xi} - A_0\beta q \rho_+^{q-1} \frac{d\rho_+}{d\xi} = 0. \end{aligned} \quad (15)$$

The above considerations are straightforwardly extended to a system of n -components Ψ_j ($j = 1, \dots, n$) with $\rho_+ = \rho_1 + \dots + \rho_n$. All the previous conclusions remain valid, namely in the case of equal velocities and for constant profile current (all the quantities depending only on $\xi = x - u_0t$) each density ρ_j is proportional to ρ_+ , $\rho_j = p_j^2 \rho_+$, $p_1^2 + \dots + p_n^2 = 1$. Thus, the problem is reduced to solving the equation (15) satisfied by ρ_+ .

A physical interpretation of this result comes by observing that ρ_+ is the first conserved density of the system (4). Indeed, it is easily shown that ρ_+ satisfies the equation

$$\frac{\partial\rho_+}{\partial t} + \frac{\partial}{\partial x}(J + G) = 0, \quad (16)$$

where G is defined by (12) and

$$J = \sum_{j=1}^n J_j, \quad (17)$$

where each current density J_j is given by

$$J_j = \frac{1}{2i} \left(\Psi_j^* \frac{\partial \Psi_j}{\partial x} - \Psi_j \frac{\partial \Psi_j^*}{\partial x} \right) = v_j \rho_j.$$

For equal velocities $v_j(x, t) = v(\xi)$ any current density J_j is proportional to ρ_j , $J_j = v(\xi)\rho_j(\xi)$ and the total current density J is proportional to the total density ρ_+ , $J = v(\xi)\rho_+(\xi)$. Then this approach can be considered a "mean field approximation" since all the quantities are depending only on the mean density ρ_+ .

3. SOLITARY WAVE SOLUTION

Solitary wave solutions, vanishing at infinity (bright solutions), of the equation (15) are easily obtained for arbitrary q . Since, in this case $A_0 \equiv 0$, after integrating twice the equation we get

$$\frac{1}{4} \left(\frac{d\rho_+}{d\xi} \right)^2 = E\rho_+^2 + 2\frac{\beta u_0}{q+1}\rho_+^{q+2} - \frac{\beta^2}{(q+1)^2}\rho_+^{2q+2}, \quad (18)$$

where we denoted $E = -(2c_0 + u_0^2)$. With the change of variable $z = \rho^{-q}$ the equation (18) becomes

$$\frac{1}{4q^2} \left(\frac{dz}{d\xi} \right)^2 = Ez^2 + 2\frac{\beta u_0}{(q+1)}z - \frac{\beta^2}{(q+1)^2}. \quad (19)$$

The physical solutions of (19) have to be positive, satisfying the boundary condition $\lim_{\xi \rightarrow \infty} z = \infty$, and for which the *r.h.s.* of (19) is positive also.

Let us assume $E > 0$. Then the second order polynomial has two real solutions, one negative z_1 and the other positive z_2 , and the physical domain is $z \in [z_2, \infty)$. Denoting $A = 2q\sqrt{E}$, the solution is

$$z(\xi) = z_m + z_M \cosh A\xi, \quad (20)$$

where

$$z_m = \frac{z_2 + z_1}{2} = -\frac{u_0|\beta|}{(q+1)E} \operatorname{sgn} \beta, \quad z_M = \frac{z_2 - z_1}{2} = \frac{u_0|\beta|}{(q+1)E} \sqrt{1 + \frac{E}{u_0^2}}.$$

Then the amplitude of the bright solitary wave is given by

$$\rho_1(\xi) = \frac{1}{(z_m + z_M \cosh A\xi)^{\frac{1}{q}}}. \quad (21)$$

In Fig. 1 we represent $\rho(\xi)$ for three different values of q ($q = 1, 2, 3$) and the set of parameters $\beta > 0$, $\frac{u_0|\beta|}{E} = 1$, $\frac{E}{u_0^2} = 3$, $x = A\xi$.

It is obvious that for $E < 0$ no physical acceptable solutions exist.

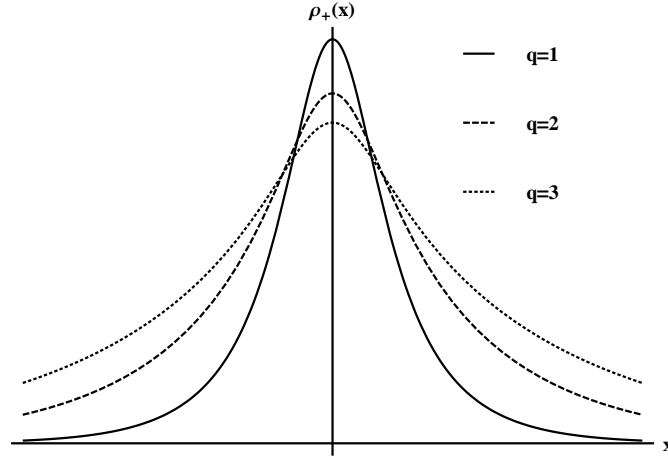


Fig. 1 – Graphic of $\rho_+(\xi)$ for three values of the nonlinearity exponent q ($q = \overline{1, 2, 3}$, $\beta > 0$, $x = A\xi$, $\frac{u_0\beta}{E} = 1$, $\frac{E}{u_0^2} = 3$).

The phase $\theta(x, t)$ is easily calculated from (13) using the expression (21) for $\rho_+(\xi)$, namely

$$\theta(\xi) = u_0\xi - \frac{\beta}{(q-1)Az_M} \int_0^{A\xi} \frac{dx}{a + \cosh x}$$

where $a = \frac{z_m}{z_M} < 1$. The integral is straightforward giving

$$\theta(\xi) = u_0\xi - \frac{\text{sgn}\beta}{q} \arctan\left(\sqrt{\frac{1-a}{1+a}} \tanh q\sqrt{E}\xi\right) \quad (22)$$

But, $\theta_j(x, t)$ is given by $\theta(\xi)$ up to an arbitrary function of t , $\theta_j(x, t) = \theta(\xi) + \phi(t)$. Then using (6) we get

$$\frac{d\phi}{dt} = \frac{1}{2} \frac{1}{\sqrt{\rho_+}} \frac{d^2\sqrt{\rho_+}}{d\xi^2} + u_0v - \frac{1}{2}v^2 + U(\rho_+)v$$

and the *r.h.s.* has to be a constant. Indeed using the expression (13) of $v(\xi)$ and the equation satisfied by ρ_+ we obtain

$$\frac{d\phi}{dt} = -c_0, \quad \phi(t) = -c_0t + \theta_0$$

and the final expression of the phase $\theta_j(x, t)$ is

$$\theta_j(x, t) = u_0(x - u_0t) - \frac{\text{sgn}\beta}{q} \arctan\left(\sqrt{\frac{1-a}{1+a}} \tanh q\sqrt{E}(x - u_0t)\right) - c_0t + \theta_{0j}. \quad (23)$$

As $E = -(2c_0 + u_0^2) < 0$ the constant $2c_0 < -u_0^2$. These results are identical to those

obtained by us for derivative NLS equations [13]. This is not surprising because in this approximation the problem is reduced to a single component ρ_+ .

4. PERIODIC SOLUTIONS

Analytical expressions for periodic solutions can be obtained only for the integrable case $q = 1$. For this case the equation (15), integrated twice, gives

$$\frac{1}{\beta^2} \left(\frac{d\rho_+}{d\xi} \right)^2 = P_4(\rho_+), \quad (24)$$

where $P_4(\rho_+)$ is a polynomial of fourth power in ρ_+

$$P_4(\rho_+) = -\rho_+^4 + \frac{4u_0}{\beta} \rho_+^3 + \frac{4E}{\beta^2} \rho_+^2 + M\rho_+ + N.$$

Here E , M , N are arbitrary constants. Different situations can be envisaged, depending on the roots of the polynomial $P_4(\rho_+)$.

- a) The polynomial $P_4(\rho_+)$ has four distinct real roots with two of them positive, $\rho_1 > \rho_2 > 0$. The physical situation is encountered for $\rho_+ \in [\rho_2, \rho_1]$ where $P_4(\rho_+) > 0$. Then the solution of (24) is given by the elliptic integral (see [19] pg. 124)

$$\int_{\rho_2}^{\rho_1} \frac{dt}{\sqrt{(\rho_1 - t)(t - \rho_2)(t - \rho_3)(t - \rho_4)}} = |\beta|\xi = gu \quad (25)$$

and can be expressed in terms of Jacobi elliptic functions in the following way

$$\rho_+ = \frac{\rho_1 + \rho_4 \operatorname{sn}^2 u}{1 + \mu_a^2 \operatorname{sn}^2 u}. \quad (26)$$

Here $u = \frac{|\beta|\xi}{g}$ and the modulus k of the elliptic function and the quantities μ_a and g are given by

$$k^2 = \frac{(\rho_1 - \rho_2)(\rho_3 - \rho_4)}{(\rho_1 - \rho_3)(\rho_2 - \rho_4)}, \quad \mu_a^2 = \frac{\rho_1 - \rho_2}{\rho_2 - \rho_4}, \quad g = \frac{2}{\sqrt{(\rho_1 - \rho_3)(\rho_2 - \rho_4)}}. \quad (27)$$

It is a periodic function of u , of period $2K(k)$ and $\rho_+(u=0) = \rho_1$, while $\rho_+(u=K) = \rho_2$. If $\rho_2 = \rho_3$, $k^2 = 1$ and the periodic solution degenerates in a solitary wave solution ($\operatorname{sn} u \rightarrow \tanh u$)

$$\rho_+ \rightarrow \frac{\rho_1 + \rho_4 \mu_a^2 \tanh^2 u}{1 + \mu_a^2 \tanh^2 u}, \quad (28)$$

which describes a shifted bright soliton (a bright soliton with nonvanishing value at infinity; $\rho_+(0) = \rho_1$, $\rho_+(\infty) = \rho_2$). If $\rho_2 = 0$ it becomes a true bright soliton.

- b) The polynomial $P_4(\rho_+)$ has four real distinct roots, all of them positive. Then besides the case a) another physical solution is possible, namely $\rho \in [\rho_4, \rho_3]$. it is given by (see [19] pg. 103)

$$\rho_+ = \frac{\rho_4 + \rho_1 \mu_b^2 \operatorname{sn}^2 u}{1 + \mu_b^2 \operatorname{sn}^2 u}, \quad (29)$$

with k^2 and g having the same expressions as in the case (a) and

$$\mu_b^2 = \frac{\rho_3 - \rho_4}{\rho_1 - \rho_3}.$$

For $u = 0$, $\rho_+ = \rho_4$ and for $u = K$, $\rho_+ = \rho_3$. In the same limit case $\rho_2 = \rho_3$, $k = 1$ and the equation (29) transforms into

$$\rho_+ \rightarrow \frac{\rho_4 + \rho_1 \mu_b^2 \tanh^2 u}{1 + \mu_b^2 \tanh^2 u} \quad (30)$$

which describes a gray soliton ($\rho_+(0) = \rho_4$, $\rho_+(\infty) = \rho_3$). In the limit case $\rho_2 = \rho_3$, $\mu_b^2 = \frac{1}{\mu_a^2}$ and both solitary waves have the same asymptotic value ρ_2 . A graphic representation of both solutions (28) and (30) is given in Fig. 2. Both a shifted bright and gray solitary wave can be present.

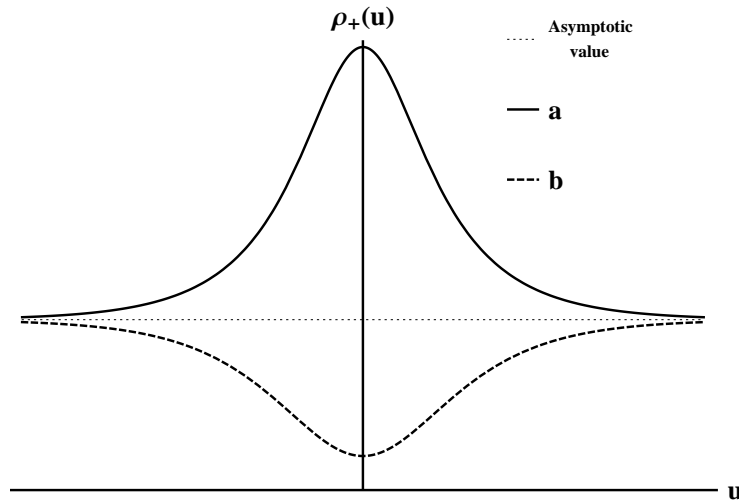


Fig. 2 – Solitary wave solutions when $P_4(\rho_+)$ has four real positive roots $\rho_1 > \rho_2 = \rho_3 > \rho_4$ ($\mu_a^2 = 2$, $\mu_b^2 = 1/2$).

- c) The last situation of physical interest is when the polynomial $P_4(\rho_+)$ has two real positive roots $\rho_1 > \rho_2 > 0$ and two complex conjugated ones $\rho_3 = \rho_4^* = a + ib$.

Then the solution is given by (see [19], pg. 133)

$$\rho_+ = \frac{(A\rho_2 + B\rho_1) + (A\rho_2 - B\rho_1) \operatorname{cn} u}{(A+B) + (A-B) \operatorname{cn} u}, \quad (31)$$

where

$$\begin{aligned} A^2 &= (\rho_1 - a) + b^2, & B^2 &= (\rho_2 - a)^2 + b^2 \\ k^2 &= \frac{(\rho_1 - \rho_2)^2 - (A - B)^2}{4AB}, & g &= \frac{1}{\sqrt{AB}}, & u &= \frac{|\beta|\xi}{g}. \end{aligned} \quad (32)$$

It is a periodic function of period $4K(k)$ and for $u = 0$, $\rho_+ = \rho_2$, while for $u = 2K$, $\rho_+ = \rho_1$. The limit case $k^2 = 1$ is obtained when $\rho_1 - \rho_2 = A + B$. As $A^2 - B^2 = (\rho_1 - \rho_2)(\rho_1 + \rho_2 - 2a)$ it results $A - B = \rho_1 + \rho_2 - 2a$ and therefore $A = \rho_1 - a$, $B = -(\rho_2 - a)$, $b = 0$, and the two roots ρ_3, ρ_4 become equal and real, $\rho_3 = \rho_4 = a$. As in this case $\operatorname{cn} u \rightarrow \operatorname{sech} u$ the solution (31) transforms into the solitary wave

$$\rho_+ \rightarrow \frac{a(\rho_1 - \rho_2) + [2\rho_1\rho_2 - a(\rho_1 + \rho_2)] \operatorname{sech} u}{(\rho_1 - \rho_2) + (\rho_1 + \rho_2 - 2a) \operatorname{sech} u}. \quad (33)$$

For $u = 0$, $\rho_+ \rightarrow \rho_2$ and for $u \rightarrow \pm\infty$, $\rho_+ \rightarrow a$. Since ρ_+ has to be positive for any value of u we have the additional requirement $a > 0$. The solution (33) describes a shifted bright solitary wave if $a < \rho_2$, or a gray solitary wave if $a > \rho_1$.

The phase determination $\theta_j(x, t)$ in the case of periodic solutions involves integrals of expression of Jacobi elliptic functions. As an example for the periodic solutions (26) we have

$$\begin{aligned} \theta(\xi) &= \left(u_0 + \frac{\beta|\rho_4|}{2} - \frac{A_0}{|\rho_4|} \right) \xi - \operatorname{sgn} \beta \frac{g(\rho_1 + |\rho_4|)}{2} \int_0^u \frac{du'}{1 + \mu^2 \operatorname{sn}^2 u'} \\ &\quad + \frac{A}{|\beta|} \frac{g(\rho_1 + |\rho_4|)}{\rho_1 |\rho_4|} \int_0^u \frac{du'}{1 - \frac{|\rho_4|\mu^2}{\rho_1} \operatorname{sn}^2 u'}. \end{aligned} \quad (34)$$

The integrals appearing in (34) are incomplete elliptic integrals of third kind (see [19], pg. 223)

$$\Pi(u, \alpha^2) = \int_0^u \frac{du'}{1 - \alpha^2 \operatorname{sn}^2 u'}, \quad (35)$$

where $\alpha^2 = -\mu^2$ for the first one and $\alpha^2 = \frac{|\rho_4|\mu^2}{\rho_1}$ for the second one. Then

$$\begin{aligned} \theta(\xi) &= \left(u_0 + \frac{\beta|\rho_4|}{2} - \frac{A_0}{|\rho_4|} \right) \xi - \operatorname{sgn} \beta \frac{g(\rho_1 + |\rho_4|)}{2} \Pi(u, -\mu^2) \\ &\quad + \frac{A}{|\beta|} \frac{g(\rho_1 + |\rho_4|)}{\rho_1 |\rho_4|} \Pi\left(u, \mu^2 \frac{|\rho_4|}{\rho_1}\right). \end{aligned} \quad (36)$$

As it is well known, the various cases of the incomplete elliptic integral of third kind, $\Pi(u, \alpha^2)$ can be classified according to the value of α^2 , namely

$$\begin{aligned} \text{circular case if } & \begin{cases} 0 < -\alpha^2 < \infty \\ k^2 < \alpha^2 < 1 \end{cases} \\ \text{hyperbolic case if } & \begin{cases} 0 < \alpha^2 < k^2 \\ 1 < \alpha^2 < \infty \end{cases} \end{aligned}$$

Then $\Pi(u, -\mu^2)$ is circular, while $\Pi\left(u, \mu^2 \frac{|\rho_4|}{\rho_1}\right)$ can be circular if $k^2 < \mu^2 \frac{|\rho_4|}{\rho_1} < 1$ and hyperbolic otherwise. As in the case of solitary wave solution $\theta_j(x, t) = \theta(\xi) - c_0 t + \theta_{0j}$.

5. CONCLUSIONS

Few comments impose themselves. In the present paper the Madelung fluid description was used to discuss a system of coupled nonlinear Schrödinger equations of form (4) with the nonlinearity $U(\rho_1, \dots, \rho_n) = U(\rho_+) = \beta \rho_+^q$, $\rho_+ = \rho_1 + \dots + \rho_n$. If $q = 1$ the above system is completely integrable (type-1 of the complete integrable matrix generalization of the Chen-Lee-Liu model [18]). A complete solution was obtained only if all the the velocities are equal. For traveling wave solutions, when all the quantities are depending only on $\xi = x - u_0 t$, we get that $\rho_j = p_j^2 \rho_+$, $p_1^2 + \dots + p_n^2 = 1$ and the problem is reduced to solve the equation satisfied by ρ_+ , therefore a one-component problem. This approximation can be called "a mean field approximation" as all the quantities are depending on the total (mean) density ρ_+ . The solutions are similar to those obtained by us for the derivative NLS equations [13].

In the scientific literature, two well known, nonlinear derivative Schrödinger equations have been mainly discussed, namely the Kaup-Newell equation [20]

$$i \frac{\partial \Psi}{\partial t} + \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial(|\Psi|^2 \Psi)}{\partial x} = 0$$

and the Chen-Lee-Liu [17]

$$i \frac{\partial \Psi}{\partial t} + \frac{\partial^2 \Psi}{\partial x^2} + |\Psi|^2 \frac{\partial \Psi}{\partial x} = 0.$$

Both are completely integrable by IST method and moreover they are related by gauge transformations to the well known cubic NLS equation [21, 22]. Therefore it is not surprising that many equations in the NLS class have comparable solutions. Especially during the last years, solutions of the derivative NLS equations and their multicomponent extensions were investigated by different authors (see [23–28] and

references therein). In agreement with previous results [26], the phase of the complex envelope was expressed in terms of incomplete elliptic integrals of the third kind.

Although the Madelung fluid approach (called "polar representation" in [26]) has given useful results in very simplifying conditions (equal velocities), this method is easily managed and can be applied to a large class of equations and even to systems of coupled equations.

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