

THE PSEUDO-NEWTONIAN FORCE AND POTENTIAL OF THE STRINGY BLACK HOLES

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Received January 5, 2011

This paper is devoted to investigate the structure of the pseudo-Newtonian force and potential of the stringy black holes. We discuss conditions for the force character from an attractive to repulsive. It is also found that the force will reach a maximum under certain conditions. Also, the ratio of mass and charge is evaluated for the maximum force.

Key words: Force and Potential; Stringy Black Holes.

PACS: 04.20.Cv

General Relativity replaces the bending of paths by the curvature of spacetime rather than as caused by a force. While retaining the path predicted by general relativity, we can flatten the background spacetime and ask what force gives the required curvature. This relativistic analogue of the gravitational force has been called the pseudo-Newtonian (ψN) force. For a point gravitating source, it just gives the usual Newtonian gravitational force. In the ψN approach [1-3], the curvature of the spacetime is *straightened out* to yield a relativistic force which bends the path, so as to again supply the guidance of the earlier, force-based, intuition.

The quantity whose gradient gives the ψN force is the ψN potential. Since the ψN force depends on the choice of geodesic the procedure becomes unwieldy for more complicated cases. However, there is a well-defined class of observers for whom the ψN force can be shown to be the gradient of a scalar ψN potential [4],

$$V = \frac{1}{2}(\mathbf{k} \cdot \mathbf{k} - 1), \quad (1)$$

where \mathbf{k} is the Killing vector corresponding to the timelike isometry. This V is exactly the usual conjecture for the gravitational potential [5]. The ψN force is the generalisation of the force which gives the usual Newtonian force for the Schwarzschild metric and a " $\frac{Q^2}{r^3}$ " correction to it in the Riessner-Nordstrom metric. The ψN force may be regarded as the *Newtonian fiction* which *explains* the same motion (geodesic)

as the *Einsteinian reality* of the curved spacetime does. We can, thus, translate back to Newtonian terms and concepts where our intuition may be able to lead us to ask, and answer, questions that may not have occurred to us in relativistic terms. The structure of the ψN force and potential for a highly charged and rotating black hole is very interesting. Infact the ψN force itself reaches a maximum and then starts to decrease outside the black hole. Thus an observer armed with an accelerometer could deduce the presence of the black hole and its nature while still able to communicate with the outside.

Some insights have already been obtained [1,4] by expressing the consequences of general relativity in terms of forces by applying it to Kerr and Kerr-Newmann metrics. Ivanov and Prodanov [6] have studied the pseudo-Newtonian potential for charged particle in Kerr-Newman geometry. In recent papers, we have investigated the structure of the pseudo-Newtonian force and potential about a five [7] and n -dimensional [8] rotating black holes. Here we analyse the structure of force and potential of the stringy black holes.

In the free fall rest-frame, the ψN force is given [2,3] by

$$F_i = -M(\ln \sqrt{g_{00}})_{,i}, \quad (i = 1, 2, 3). \quad (2)$$

This can be written as

$$F_i = -V_{,i}, \quad (3)$$

where

$$V = M(\ln \sqrt{g_{00}}). \quad (4)$$

It is clear that V is the generalisation of the classical gravitational potential and, for small variations from Minkowski space

$$V \approx \frac{1}{2}M(g_{00} - 1) \quad (5)$$

which is the pseudo-Newtonian potential. We shall investigate the behaviour of these quantities for the stringy black holes.

The non-trivial geometry of different types of black holes is of crucial interest in general relativity. We shall consider some typical four dimensional stringy black hole spacetimes. In four dimensions, the simplest eternal black hole geometry is, of course, the Schwarzschild. We consider the variations of the Schwarzschild which have arisen in the context of string theory [9-11]. All these four dimensional geometries have a Schwarzschild limit (obtainable by setting a parameter in the line element to zero). We also have asymptotically flat solutions representing black holes in dilaton-Maxwell gravity. Such solutions, due to Garfinkle, Horowitz and Strominger [9], represent electric and dual magnetic black holes [10]. The spacetime geometry of these line elements are causally similar to the Schwarzschild geometry.

In the respective limit of $\alpha = 0$ or $Q = 0$, we have Schwarzschild geometry in both the cases.

The metric for the black hole with electric charge is given by

$$ds^2 = \frac{(1 - \frac{2m}{r})}{(1 + \frac{2m \sinh^2 \alpha}{r})^2} dt^2 + \frac{dr^2}{(1 - \frac{2m}{r})} + r^2 d\Omega^2, \quad (6)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric on a 2-dimensional unit sphere and α is a parameter related to the electric charge of the black hole such that $\tanh^2 \alpha = \frac{Q^2}{M^2}$, m is a parameter related to the physical mass of the black hole. For $\alpha = 0$, the spacetime reduces to the Schwarzschild spacetime. Also, it reduces to Minkowski spacetime for $m = 0$.

The structure of the ψN force (per unit mass of the test particle) for the black hole with electric charge takes the following form

$$F_r = -\frac{\frac{m}{r^2}(1 + \sinh^2 \alpha - \frac{2m}{r} \sinh^2 \alpha)}{(1 - \frac{2m}{r})(1 + \frac{2m}{r} \sinh^2 \alpha)}, \quad F_\theta = 0. \quad (7)$$

If we take $\alpha = 0$, the results exactly coincide with those $a = 0 = Q$ in [4], i.e. for the Schwarzschild black hole. Equation (7) shows that the radial component cannot be zero outside the horizon. Consequently, the attractive force cannot change to a repulsive force outside the black hole. It is obvious that naked singularities can give repulsive as well as attractive forces [5]. The force structure would provide interesting features if it reaches a maximum and then drops as we reduce r provided the turnover lies outside the horizon. Since our observers are seeing *force* in a flat space, the metric to be used is the plane polar one. Thus the magnitude of the force is

$$F = \frac{\frac{m}{r^2}(1 + \sinh^2 \alpha - \frac{2m}{r} \sinh^2 \alpha)}{(1 - \frac{2m}{r})(1 + \frac{2m}{r} \sinh^2 \alpha)}. \quad (8)$$

The equation for the turnover along radial direction is

$$(1 + 2 \sinh^2 \alpha)r^3 - m(1 + 4 \sinh^2 \alpha + 2 \sinh^4 \alpha)r^2 + 4m^2 \sinh^2 \alpha(1 - \sinh^2 \alpha)r + 4m^3 \sinh^3 \alpha = 0.$$

This is a cubic equation. In general a cubic equation can be solved and we can have the following three different possibilities:

- (i) All the three roots are real and distinct.
- (ii) All the three roots are real, two of them being equal.
- (iii) One root is real and the other two are a conjugate pair. When we solve the above

cubic equation, we obtain one real and two imaginary roots. The real root is given by

$$\begin{aligned}
 r = & \frac{1}{3 \cosh 2\alpha} [\{m + m(3 + \cosh 2\alpha) \sinh^2 \alpha + (m^2(1 - 4 \sinh^2 \alpha \\
 & + 8 \sinh^4 \alpha + 40 \sinh^6 \alpha + 4 \sinh^8 \alpha))\} / \{6\sqrt{3}\{(-m^6 \cosh^2 2\alpha \sinh^3 \alpha \\
 & \times (1 + \sinh \alpha(1 + \sinh \alpha(-6 + \sinh \alpha(-37 + \sinh \alpha(-36 + \sinh \alpha(-87 \\
 & + 4 \sinh \alpha(10 + \sinh \alpha(-19 + \sinh \alpha(54 + \sinh \alpha(-22 + \sinh \alpha(30 \\
 & + \sinh \alpha(10 + \sinh \alpha(2 + \sinh \alpha)))))))))\}^{\frac{1}{2}} + m^3(1 + 2 \sinh^2 \alpha(-3 \\
 & + \frac{1}{64} \sinh \alpha(-864 - 864 \cosh 4\alpha + 234 \sinh \alpha - 664 \sinh 3\alpha + 48 \sinh 5\alpha \\
 & + 51 \sinh 7\alpha + \sinh 9\alpha))\}^{\frac{1}{3}}] + \{6\sqrt{3}\{(-m^6 \cosh^2 2\alpha \sinh^3 \alpha(1 + \sinh \alpha \\
 & \times (1 + \sinh \alpha(-6 + \sinh \alpha(-37 + \sinh \alpha(-36 + \sinh \alpha(-87 + 4 \sinh \alpha(10 \\
 & + \sinh \alpha(-19 + \sinh \alpha(54 + \sinh \alpha(-22 + \sinh \alpha(30 + \sinh \alpha(10 \\
 & + \sinh \alpha(2 + \sinh \alpha)))))))))\}^{\frac{1}{2}} + m^3(1 + 2 \sinh^2 \alpha(-3 \\
 & + \frac{1}{64} \sinh \alpha(-864 - 864 \cosh 4\alpha + 234 \sinh \alpha \\
 & - 664 \sinh 3\alpha + 48 \sinh 5\alpha + 51 \sinh 7\alpha + \sinh 9\alpha))\}^{\frac{1}{3}}]. \quad (9)
 \end{aligned}$$

This equation is not easy to analyse analytically. The general comments can be given as follows. We see that a maximum of the magnitude does occur at this value of r . The maximum value of the force can be obtained by replacing this value of r in Eq.(8) which turns out to be a complicated equation. However, this will obviously yield that the maximum value of the force depends on the value of α which corresponds to charge. This will provide turnovers inside the horizon and also on the surface of the horizon for an extreme black hole.

It would be interesting to explore the approximate solutions for sufficiently small charge. For this purpose, we expand $\sinh \alpha$ and $\cosh \alpha$ up to first order and neglecting the second and higher orders. Using this approximation, Eq.(8) gives

$$F = \frac{\frac{m}{r^2}(1 + \alpha)}{(1 - \frac{2m}{r})}. \quad (10)$$

The maximum value turns out to be at $r = m$ and the corresponding value will be

$$F^* = -\frac{(1 + \alpha)}{m}, \quad m > 8/3, \quad 1 + \alpha < 0. \quad (11)$$

This would yield the condition on the ratio of mass and charge to have the maximum force for this black hole. It is worthwhile to mention here that these limits on the ratio appear in the discussion of the pseudo-Newtonian force of the charged Kerr metric [4] as well of critical accretion [12].

The general results for particular choice of charge and mass (using Eqs.(8) and

(9)) can be summarized in the following table. This provides inverse behavior for

Table 1.

Maximum Force for particular values of Charge and Mass.

m	Q	r	F^*
2	0.1	1.98898	0.496247
2	0.02	1.95128	0.484949
2.5	0.1	0.49135	1.09407
2.5	0.2	2.4626	0.392304
2.5	0.15	2.47978	0.395675
3	0.025	2.999958	0.333264
5	0.025	4.99975	0.199985

maximum force corresponding to the stationary point.

The corresponding potential is found using Eqs.(4) and (6) given by

$$V = -\frac{\frac{m}{r}(1 + \frac{2m \sinh^4 \alpha}{r} + 2 \sinh^2 \alpha)}{2(1 + \frac{2m \sinh^2 \alpha}{r})}. \quad (12)$$

We note that the structure of force and potential indicate similar type of behaviour as for the charged Kerr metric [4].

The dual (magnetic) metric of Eq.(6) is given as follows

$$ds^2 = \frac{(1 - \frac{2m}{r})}{(1 - \frac{Q^2}{mr})} dt^2 + \frac{dr^2}{(1 - \frac{2m}{r})(1 - \frac{Q^2}{mr})} + r^2 d\Omega^2, \quad (13)$$

where Q is the magnetic charge of the black hole.

The structure of the ψN force (per unit mass of the test particle) for this black hole takes the following form

$$F_r = -\frac{\frac{m}{r^2}(1 - \frac{Q^2}{2m^2})}{2r(1 - \frac{2m}{r})(1 - \frac{Q^2}{mr})}, \quad F_\theta = 0. \quad (14)$$

We see that the radial component can never become zero outside the horizon and so the force cannot change character from an attractive to a repulsive force outside the black hole. The magnitude of the force is

$$F = \frac{\frac{m}{r^2}(1 - \frac{Q^2}{2m^2})}{2r(1 - \frac{2m}{r})(1 - \frac{Q^2}{mr})}. \quad (15)$$

The equation for the turnover along radial direction is

$$2mr - Q^2 - 2m^2 = 0 \quad (16)$$

satisfied for the value of r given by

$$r = \frac{Q^2 + 2m^2}{2m}. \quad (17)$$

We see that a maximum of the magnitude does occur at this value of r . Thus the maximum value of the force is

$$F^* = \frac{m}{2m^2 - Q^2} \quad (18)$$

which indicates the effect of Q on the force. It is easy to find the ratio of mass to charge from here. The corresponding potential is obtained by using Eqs.(4) and (12)

$$V = -\frac{\frac{m}{r} \left(1 - \frac{Q^2}{2m^2}\right)}{\left(1 - \frac{Q^2}{mr}\right)}. \quad (19)$$

Notice that the structure of force and potential coincide with that of the Kerr metric [4]. The general expression for the stationary point of the first metric does not provide much insight. However, we have explored it for small value of α for the purpose of discussion. We would like to point out that the charges and rotations can produce repulsive force. This has been verified for charged spacetimes using this formulation. It is interesting to mention here that for $\alpha = 0 = Q$, the results of the force and potential for both the spacetimes reduce to the Schwarzschild metric.

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