NEW RESULTS FOR $2\nu\nu\beta\beta$ DECAY WITHIN A FULLY RENORMALIZED pnQRPA APPROACH WITH RESTORED GAUGE SYMMETRY

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A many body Hamiltonian involving the mean field for a projected spherical single particle basis, the pairing interactions for alike nucleons, a repulsive dipole-dipole proton-neutron interaction in the particle-hole (ph) channel and an attractive dipole-pairing interaction is treated by the gauge restored of a fully renormalized proton-neutron quasiparticle random phase approximation (GRFRpnQRPA) approach. The resulting wave functions and energies for the mother and the daughter nuclei are used to calculate the $2\nu\beta\beta$ decay rate and the process half life. The formalism is applied for the decays of ^{148}Nd, ^{150}Nd, ^{154}Sm and ^{160}Gd. The results are compared with those obtained by traditional methods. The Ikeda sum rule (ISR) is obeyed.

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1. INTRODUCTION

The $2\nu\beta\beta$ process is interesting by its own but is also very attractive because it constitutes a test for the nuclear matrix elements (m.e.) which are used for the process of $0\nu\beta\beta$ decay. The discovery of this process may provide an answer to the fundamental question, whether neutrino is a Mayorana or a Dirac particle. The subject development is described by several review papers [1–7]. The present paper refers to the $2\nu\beta\beta$ process, which can be conceived as consisting of two consecutive and virtual single β^- decays. The formalism yielding closest results to the experimental data is the proton-neutron random phase approximation (pnQRPA) which includes the particle-hole (ph) and particle-particle (pp) as independent two body interactions. The second leg of the $2\nu\beta\beta$ process is very sensitive to changing the relative strength of the later interaction, denoted hereafter by g_{pp}. It is worth mentioning that the ph interaction is repulsive while the pp one is attractive. Consequently, there is a critical value of g_{pp} for which the first root of the pnQRPA equation vanishes. Actually, this is the signal that the pnQRPA approach is no longer valid. Moreover, the g_{pp} value which corresponds to a transition amplitude which agrees with the corresponding experimental data is close to the mentioned critical value. That means that the result is not stable to adding corrections to the RPA picture. The first improvement for the pnQRPA was achieved by one of us (AAR) in Refs. [8, 9], by using a boson expansion (BE) procedure. Later on another procedure showed up, which renormalized the dipole two quasiparticle operators by replacing the scalar components of their commutators by their average values [10]. Such a renormalization is, however, inconsistently achieved since the scattering operators are not renormalized. This lack of consistency was removed in [11, 12] where a fully renormalized pnQRPA (FRpnQRPA) is proposed.

Unfortunately, all higher pnQRPA procedures mentioned above have a common drawback of violating the Ikeda sum rule (ISR) by an amount of about 20-30% [13]. It is believed that such a violation is caused by the gauge symmetry breaking. Consequently, a method of restoring this symmetry was formulated by two of us (A. A. R. and C. M. R.) in [14].

In this paper the results of [14] are improved in two respects: a) aiming at providing a unitary description of the process for the situations when the involved nuclei are spherical or deformed, here we use the projected spherical single particle basis defined in [15] and used for double beta decay in [16,17]; b) the space of proton-neutron dipole configurations is split in three subspaces, one being associated to the single β^- decay, one to the single β^+ process, and one spanned by the unphysical states. A compact expression for the dispersion equation for energies is obtained from the linearized equations of motion of the basic transition operators corresponding to the two coupled processes. The numerical application is made for the $2\nu\beta\beta$ processes ^{148}Nd→^{148}Sm, ^{150}Nd→^{150}Sm, ^{154}Sm →^{154}Gd and ^{160}Gd →^{160}Dy. Although for these isotopes experimental relevant data are not yet available, their choice is motivated by the corresponding Q values which, indeed, recommend them as possible double beta emitters.

Results are described according to the following plan. The single particle basis is briefly presented in Section II. The model Hamiltonian is given in Section III. The FRpnQRPA approach is discussed in Section IV, while the projected gauge of FRpnQRPA (GRFRpnQRPA) is the objective of Section V. The Gamow-Teller (GT) amplitude for the $2\nu\beta\beta$ process is given in Section VI. Numerical applications are shown in Section VII while the final conclusions are drawn in Section VIII.

2. PROJECTED SINGLE PARTICLE BASIS

In [15], one of us, (A.A.R.), introduced an angular momentum projected single particle basis which seems to be appropriate for the description of the single particle motion in a deformed mean field generated by the particle-core interaction. This single particle basis has been used to study the collective M1 states in deformed nuclei [15, 18, 19] as well as the rate of double beta process [16, 20, 21].

In order to fix the necessary notations and moreover for the sake of a self-contained presentation, we describe briefly the main ideas underlying the construction of the projected single particle basis.

The single particle mean field is determined by a particle-core Hamiltonian:

$$\tilde{H} = H_{sm} + H_{core} - M\omega_0^2 r^2 \sum_{\lambda=0,2} \sum_{-\lambda \leq \mu \leq \lambda} \alpha^*_{\lambda\mu} Y_{\lambda\mu}, \qquad (1)$$

where H_{sm} denotes the spherical shell model Hamiltonian, while H_{core} is a harmonic quadrupole boson (b^+_μ) Hamiltonian associated to a phenomenological core. The interaction of the two subsystems is accounted for by the third term of the above equation, written in terms of the shape coordinates $\alpha_{00}, \alpha_{2\mu}$. The quadrupole coordinates are related to the quadrupole boson operators by the canonical transformation:

$$\alpha_{2\mu} = \frac{1}{k\sqrt{2}}(b^\dagger_{2\mu} + (-)^\mu b_{2,-\mu}), \qquad (2)$$

where k is an arbitrary C number. The monopole shape coordinate is to be determined from the volume conservation condition.

Averaging $\tilde{H}$ on a given eigenstate of H_{sm}, denoted as usual by $|nljm\rangle$, one obtains a deformed quadrupole boson Hamiltonian which admits the axially symmetric coherent state

$$\Psi_g = exp[d(b^+_{20} - b_{20})]|0\rangle_b, \qquad (3)$$

as eigenstate. $|0\rangle_b$ stands for the vacuum state of the boson operators and d is a real parameter which simulates the nuclear deformation. On the other hand, averaging $\tilde{H}$ on Ψ_g, one obtains a single particle mean field operator for the single particle motion, similar to the Nilsson Hamiltonian. Concluding, by averaging the particle-core Hamiltonian with a factor state the rotational symmetry is broken and the mean field mentioned above may generate, by diagonalization, a deformed basis for treating the many body interacting systems. However, this standard procedure is tedious since the final many body states should be projected over the angular momentum.

Our procedure defines first a spherical basis for the particle-core system, by projecting out the angular momentum from the deformed state

$$\Psi^{pc}_{nlj} = |nljm\rangle\Psi_g. \qquad (4)$$

The projected states are obtained, in the usual manner, by acting on these deformed states with the projection operator

$$P^I_{MK} = \frac{2I+1}{8\pi^2}\int {D^I_{MK}}^*(\Omega)\hat{R}(\Omega)d\Omega. \qquad (5)$$

We consider the subset of projected states :

$$\Phi^{IM}_{nlj}(d) = \mathcal{N}^I_{nlj} P^I_{MI}[|nljI\rangle\Psi_g] \equiv \mathcal{N}^I_{nlj}\Psi^{IM}_{nlj}(d), \qquad (6)$$

which are orthonormalized and form a basis for the particle-core system. This basis exhibits useful properties which have been presented in some of our previous publications.
To the projected spherical states, one associates the "deformed" single particle energies defined as the average values of the particle-core Hamiltonian $H' = \tilde{H} - H_{core}$:

$$\epsilon^I_{nlj} \quad = \quad \langle \Phi^{IM}_{nlj}(d)|H'|\Phi^{IM}_{nlj}(d)\rangle. \tag{7}$$

Since the core contribution to this average value does not depend on the quantum numbers of the single particle energy levels, it produces a constant shift for all energies. For this reason such a term is omitted in (7). The deformation dependence of the new single particle energies is similar to that shown by the Nilsson model [22]. Therefore, the average values ϵ^I_{nlj} may be viewed as approximate single particle energies in deformed Nilsson orbits [22]. We may account for the deviations from the exact eigenvalues by considering, at a later stage when a specific treatment of the many body system is performed, the exact matrix elements of the two body interaction.

Although the energy levels are similar to those of the Nilsson model, the quantum numbers in the two schemes are different. Indeed, here we generate from each j a multiplet of $(2j+1)$ states distinguished by the quantum number I, which plays the role of the Nilsson quantum number Ω and runs from 1/2 to j. Moreover, the energies corresponding to the quantum numbers K and $-K$ are equal to each other. On the other hand, for a given I there are $2I+1$ degenerate sub-states while the Nilsson states are only double degenerate. As explained in [11], the redundancy problem can be solved by changing the normalization of the model functions:

$$\langle \Phi^{IM}_\alpha|\Phi^{IM}_\alpha\rangle = 1 \Longrightarrow \sum_M \langle \Phi^{IM}_\alpha|\Phi^{IM}_\alpha\rangle = 2. \tag{8}$$

Due to this weighting factor the particle density function is providing the consistency result that the number of particles which can be distributed on the (2I+1) sub-states is at most 2, which agrees with the Nilsson model. Here α stands for the set of shell model quantum numbers nlj. Due to this normalization, the states Φ^{IM}_α used to calculate the matrix elements of a given operator should be multiplied with the weighting factor $\sqrt{2/(2I+1)}$.

The projected states might be thought of as eigenstates of an effective rotational invariant fermionic one-body Hamiltonian H_{eff}, with the corresponding energies given by (7):

$$H_{eff}\Phi^{IM}_\alpha = \epsilon^I_\alpha(d)\Phi^{IM}_\alpha. \tag{9}$$

As shown in [15] in the vibrational limit, $d \rightarrow 0$, the projected spherical basis goes to the spherical shell model basis and ϵ^I_{nlj} to the eigenvalues of H_{sm}.

A fundamental result obtained in [13] for the product of two single particle states, which comprises a product of two core components, deserves to be mentioned. Therein we have proved that the matrix elements of a two body interaction corresponding to the present scheme are very close to the matrix elements corresponding to spherical states projected from a deformed product state with one factor being a product of two spherical single particle states, and a second factor consisting of a unique collective core wave function. The small discrepancies of the two types of matrix elements could be washed out by using slightly different strengths for the two body interaction in the two methods. Due to this property the basis (6) might be used for studying any two-body interaction.

3. THE MODEL HAMILTONIAN

We suppose that the states describing the nuclei involved in a $2\nu\beta\beta$ process are described by a many body Hamiltonian which may be written in the projected spherical basis as:

$$\begin{aligned} H = & \sum_{\tau,\alpha,I,M} \frac{2}{2I+1}(\epsilon_{\tau\alpha I} - \lambda_{\tau\alpha})c^{\dagger}_{\tau\alpha IM}c_{\tau\alpha IM} - \sum_{\tau,\alpha,I,I'} \frac{G_\tau}{4} P^{\dagger}_{\tau\alpha I}P_{\tau\alpha I'} \\ & +2\chi \sum_{pn;p'n';\mu} \beta^{-}_{\mu}(pn)\beta^{+}_{-\mu}(p'n')(-)^{\mu} - 2\chi_1 \sum_{pn;p'n';\mu} P^{-}_{\mu}(pn)P^{+}_{-\mu}(p'n')(-)^{\mu}, \end{aligned} \tag{10}$$

where $c^{\dagger}_{\tau\alpha IM}(c_{\tau\alpha IM})$ denotes the creation (annihilation) operator of one nucleon of the type $\tau(=p,n)$ in the state Φ^{IM}_{α}, with α being an abbreviation for the set of quantum numbers nlj. The Hamiltonian H contains the mean field term, the pairing interaction for alike nucleons and the Gamow-Teller dipole-dipole interaction in the ph and pp channels, characterized by the strengths χ and χ_1, respectively.

In order to simplify the notations, hereafter the set of quantum numbers $\alpha(=nlj)$ will be omitted. The two body interaction consists of three terms, the pairing, the dipole-dipole particle-hole (ph) and the particle-particle (pp) interactions. The corresponding strengths are denoted by $G_\tau(\tau = p,n), \chi, \chi_1$, respectively. All of them are separable interactions, with the factors defined by the following expressions:

$$\begin{aligned} P^{\dagger}_{\tau I} &= \sum_{M} \frac{2}{2I+1} c^{\dagger}_{\tau IM} c^{\dagger}_{\widetilde{\tau IM}}, \\ \beta^{-}_{\mu}(pn) &= \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}} \langle pIM|\sigma_\mu|nI'M'\rangle \frac{\sqrt{2}}{\hat{I'}} c^{\dagger}_{pIM} c_{nI'M'}, \\ P^{-}_{1\mu}(pn) &= \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}} \langle pIM|\sigma_\mu|nI'M'\rangle \frac{\sqrt{2}}{\hat{I'}} c^{\dagger}_{pIM} c^{\dagger}_{\widetilde{nI'M'}}. \end{aligned} \tag{11}$$

The remaining operators from (10) can be obtained from the above defined operators,

by Hermitian conjugation.
Passing to the quasiparticle representation through the Bogoliubov-Valatin transformation

$$a^{\dagger}_{\tau IM} = U_{\tau I} c^{\dagger}_{\tau IM} - s_{IM} V_{\tau I} c_{\tau I-M}, s_{IM} = (-)^{I-M}, U^2_{\tau I} + V^2_{\tau I} = 1; \tau = p,n, \quad (12)$$

the first two terms of H are replaced by the independent quasiparticles term, $\sum E_{\tau I} a^{\dagger}_{\tau IM} a_{\tau IM}$, while the ph and pp interactions are expressed in terms of the dipole two qp and the qp density operators:

$$\begin{aligned} A^{\dagger}_{1\mu}(pn) &= \sum C^{I_p\ I_n\ 1}_{m_p\ m_n\ \mu} a^{\dagger}_{pI_pm_p} a^{\dagger}_{nI_nm_n}, \, A_{1\mu}(pn) = \left(A^{\dagger}_{1\mu}(pn)\right)^{\dagger}, \\ B^{\dagger}_{1\mu}(pn) &= \sum C^{I_p\ I_n\ 1}_{m_p\ -m_n\ \mu} a^{\dagger}_{pj_pm_p} a_{nI_nm_n} (-)^{I_n-m_n}, \, B_{1\mu}(pn) = \left(B^{\dagger}_{1\mu}(pn)\right)^{\dagger}. \end{aligned} \quad (13)$$

4. THE FULLY RENORMALIZED pnQRPA

In [11], we showed that all these operators can be renormalized as suggested by the commutation equations:

$$\begin{aligned} \left[A_{1\mu}(k), A^{\dagger}_{1\mu'}(k')\right] &\approx \delta_{k,k'}\delta_{\mu,\mu'}\left[1 - \frac{\hat{N}_n}{\hat{I}^2_n} - \frac{\hat{N}_p}{\hat{I}^2_p}\right], \\ \left[B^{\dagger}_{1\mu}(k), A^{\dagger}_{1\mu'}(k')\right] &\approx \left[B^{\dagger}_{1\mu}(k), A_{1\mu'}(k')\right] \approx 0, \\ \left[B_{1\mu}(k), B^{\dagger}_{1\mu'}(k')\right] &\approx \delta_{k,k'}\delta_{\mu,\mu'}\left[\frac{\hat{N}_n}{\hat{I}^2_n} - \frac{\hat{N}_p}{\hat{I}^2_p}\right], \, k = (I_p, I_n). \end{aligned} \quad (14)$$

Indeed, denoting by $C^{(1)}_{I_p,I_n}$ and $C^{(2)}_{I_p,I_n}$ the averages of the right hand sides of (14) with the renormalized RPA vacuum state, the renormalized operators defined as

$$\bar{A}_{1\mu}(k) = \frac{1}{\sqrt{C^{(1)}_k}} A_{1\mu}, \, \bar{B}_{1\mu}(k) = \frac{1}{\sqrt{|C^{(2)}_k|}} B_{1\mu}, \quad (15)$$

obey boson like commutation relations:

$$\left[\bar{A}_{1\mu}(k), \bar{A}^{\dagger}_{1\mu'}(k')\right] = \delta_{k,k'}\delta_{\mu,\mu'}, \, \left[\bar{B}_{1\mu}(k), \bar{B}^{\dagger}_{1\mu'}(k')\right] = \delta_{k,k'}\delta_{\mu,\mu'} f_k, \, f_k = sign(C^{(2)}_k).$$

Further, these operators are used to define the phonon operator:

$$\begin{aligned} C^{\dagger}_{1\mu} = \sum_k \Big[& X(k)\bar{A}^{\dagger}_{1\mu}(k) + Z(k)\bar{D}^{\dagger}_{1\mu}(k) - Y(k)\bar{A}_{1-\mu}(k)(-)^{1-\mu} \\ & - W(k)\bar{D}_{1-\mu}(k)(-)^{1-\mu}\Big], \end{aligned} \quad (16)$$

where $\bar{D}^{\dagger}_{1\mu}(k)$ is equal to $\bar{B}^{\dagger}_{1\mu'}(k')$ or $\bar{B}_{1\mu}(k)$ depending on whether f_k is + or -. The phonon amplitudes are determined by the equations:

$$\left[H, C^{\dagger}_{1\mu}\right] = \omega C^{\dagger}_{1\mu}, \quad \left[C_{1\mu}, C^{\dagger}_{1\mu'}\right] = \delta_{\mu\mu'}. \tag{17}$$

Interesting properties for these equations and their solutions are discussed in our previous publications [11, 12].

5. GAUGE PROJECTION OF THE FULLY RENORMALIZED pnQRPA

The renormalized ground state can be excited by the phonon operator defined by the FRpnQRPA approach to a state which is a superposition of components describing the neighboring nuclei $(N-1, Z+1), (N+1, Z-1), (N+1, Z+1), (N-1, Z-1)$. The first two components conserve the total number of nucleons (N+Z) but violate the third component of isospin, T_3. By contrast, the last two components violates the total number of nucleons but preserve T_3. Actually, the last two components contribute to the violation of the ISR. One can construct linear combinations of the basic operators $A^{\dagger}, A, B^{\dagger}, B$ which excite the nucleus (N, Z) to the nuclei $(N-1, Z+1), (N+1, Z-1), (N+1, Z+1), (N-1, Z-1)$, respectively. These operators are:

$$\begin{aligned}
\mathcal{A}^{\dagger}_{1\mu}(pn) &= U_p V_n A^{\dagger}_{1\mu}(pn) + U_n V_p A_{1,-\mu}(pn)(-)^{1-\mu} \\
&\quad + U_p U_n B^{\dagger}_{1\mu}(pn) - V_p V_n B_{1,-\mu}(pn)(-)^{1-\mu}, \\
\mathcal{A}_{1\mu}(pn) &= U_p V_n A_{1\mu}(pn) + U_n V_p A^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu} \\
&\quad + U_p U_n B_{1\mu}(pn) - V_p V_n B^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu}, \\
\mathbf{A}^{\dagger}_{1\mu}(pn) &= U_p U_n A^{\dagger}_{1\mu}(pn) - V_p V_n A_{1,-\mu}(pn)(-)^{1-\mu} \\
&\quad - U_p V_n B^{\dagger}_{1\mu}(pn) - V_p U_n B_{1,-\mu}(pn)(-)^{1-\mu}, \\
\mathbf{A}_{1\mu}(pn) &= U_p U_n A_{1\mu}(pn) - V_p V_n A^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu} \\
&\quad - U_p V_n B_{1\mu}(pn) - V_p U_n B^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu}.
\end{aligned}$$

Indeed, in the particle representation these operators have the expressions:

$$\begin{aligned}
&\mathcal{A}^{\dagger}_{1\mu}(pn) = -\left[c^{\dagger}_p c_{\widetilde{n}}\right]_{1\mu}, \quad \mathcal{A}_{1\mu}(pn) = -\left[c^{\dagger}_p c_{\widetilde{n}}\right]^{\dagger}_{1\mu}, \\
&\mathbf{A}^{\dagger}_{1\mu}(pn) = \left[c^{\dagger}_p c^{\dagger}_n\right]_{1\mu}, \quad \mathbf{A}_{1\mu}(pn) = \left[c^{\dagger}_p c^{\dagger}_n\right]^{\dagger}_{1\mu}.
\end{aligned} \tag{18}$$

In terms of the new operators the many body Hamiltonian is:

$$H = \sum_{\tau jm} E_{\tau j} a^{\dagger}_{\tau jm} a_{\tau jm} + 2\chi \sum_{pn,p'n';\mu} \sigma_{pn;p'n'} \mathcal{A}^{\dagger}_{1\mu}(pn)\mathcal{A}_{1\mu}(p'n')$$
$$- 2\chi_1 \sum_{pn,p'n';\mu} \sigma_{pn;p'n'} \mathbf{A}^{\dagger}_{1\mu}(pn)\mathbf{A}_{1\mu}(p'n'), \quad (19)$$
$$\sigma_{pn;p'n'} = \frac{2}{3\hat{I}_n\hat{I}_{n'}} \langle I_p||\sigma||I_n\rangle\langle I_{p'}||\sigma||I_{n'}\rangle.$$

Here $E_{\tau I}$ denotes the quasiparticle energy. Since we are interested in describing the harmonic modes which preserve the total number of nucleons, we ignore the χ_1 term.

However, aiming at a quantitative description of the double beta process, the presence of an attractive proton-neutron interaction is necessary. Due to this reason we replace the pp interaction, which is ineffective anyway, with a dipole pairing interaction:

$$\Delta H = -X_{dp} \sum_{\substack{pn;p' \\ n';\mu}} \left(\beta^-_{\mu}(pn)\beta^-_{-\mu}(p'n') + \beta^+_{-\mu}(p'n')\beta^+_{\mu}(pn)\right)(-1)^{1-\mu}. \quad (20)$$

In this way the model Hamiltonian written in the quasiparticle representation becomes:

$$H = \sum_{\tau jm} E_{\tau j} a^{\dagger}_{\tau jm} a_{\tau jm} + 2\chi \sum_{pn;p'n';\mu} \sigma_{pn;p'n'} \mathcal{A}^{\dagger}_{1\mu}(pn)\mathcal{A}_{1\mu}(p'n')$$
$$-X_{dp} \sum_{\substack{pn;p' \\ n';\mu}} \sigma_{pn;p'n'} \left(\mathcal{A}^{\dagger}_{1\mu}(pn)\mathcal{A}^{\dagger}_{1,-\mu}(p'n') + \mathcal{A}_{1,-\mu}(p'n')\mathcal{A}_{1\mu}(pn)\right)(-)^{1-\mu}. \quad (21)$$

The equations of motion of the operators defining the phonon operator are determined by the commutation relations:

$$\left[\mathcal{A}_{1\mu}(pn), \mathcal{A}^{\dagger}_{1\mu'}(p'n')\right] \approx \delta_{\mu\mu'}\delta_{j_p j_{p'}}\delta_{j_n j_{n'}}\left[U_p^2 - U_n^2 + \frac{U_n^2 - V_n^2}{\hat{I}_n^2}\hat{N}_n - \frac{U_p^2 - V_p^2}{\hat{I}_p^2}\hat{N}_p\right]. \quad (22)$$

The average of the r.h.s. of this equation with the *GRFRpnQRPA* vacuum state is denoted by

$$D_1(pn) = U_p^2 - U_n^2 + \frac{1}{2I_n+1}(U_n^2 - V_n^2)\langle\hat{N}_n\rangle - \frac{1}{2I_p+1}(U_p^2 - V_p^2)\langle\hat{N}_p\rangle. \quad (23)$$

The equations of motion show that the two qp energies are renormalized too:

$$E^{ren}(pn) = E_p(U_p^2 - V_p^2) + E_n(V_n^2 - U_n^2). \quad (24)$$

The space of pn dipole states, $\mathcal{S}$, is written as a sum of three subspaces defined as:

$$\begin{aligned}
&\mathcal{S}_+ = \{(p,n)|D_1(pn) > 0,\ E^{ren}(pn) > 0,\},\\
&\mathcal{S}_- = \{(p,n)|D_1(pn) < 0,\ E^{ren}(pn) < 0,\},\\
&\mathcal{S}_{sp} = \mathcal{S} - (\mathcal{S}_+ + \mathcal{S}_-),\\
&\mathcal{N}_\pm = \dim(\mathcal{S}_\pm),\ \mathcal{N}_{sp} = \dim(\mathcal{S}_{sp}),\\
&\mathcal{N} = \mathcal{N}_+ + \mathcal{N}_- + \mathcal{N}_{sp}.
\end{aligned} \tag{25}$$

The fourth line of the above equations specifies the dimensions of these subspaces. In $\mathcal{S}_+$ one defines the renormalized operators:

$$\bar{\mathcal{A}}^\dagger_{1\mu}(pn) = \frac{1}{\sqrt{D_1(pn)}}\mathcal{A}^\dagger_{1\mu}(pn),\ \bar{\mathcal{A}}_{1\mu}(pn) = \frac{1}{\sqrt{D_1(pn)}}\mathcal{A}_{1\mu}(pn), \tag{26}$$

while in $\mathcal{S}_-$ the renormalized operators are:

$$\bar{\mathcal{F}}^\dagger_{1\mu}(pn) = \frac{1}{\sqrt{|D_1(pn)|}}\mathcal{A}_{1\mu}(pn),\ \bar{\mathcal{F}}_{1\mu}(pn) = \frac{1}{\sqrt{|D_1(pn)|}}\mathcal{A}^\dagger_{1\mu}(pn). \tag{27}$$

Indeed, the operator pairs $\mathcal{A}_{1\mu}, \mathcal{A}^\dagger_{1\mu}$ and $\mathcal{F}_{1\mu}, \mathcal{F}^\dagger_{1\mu}$ satisfy commutation relations of boson type. An RPA treatment within $\mathcal{S}_{sp}$ would yield either vanishing or negative energies. The corresponding states are therefore spurious. FRpnQRPA with the gauge symmetry projected defines the phonon operator as:

$$\begin{aligned}
\Gamma^\dagger_{1\mu} = \sum_k \Big[& X(k)\bar{\mathcal{A}}^\dagger_{1\mu}(k) + Z(k)\bar{\mathcal{F}}^\dagger_{1\mu}(k) - Y(k)\bar{\mathcal{A}}_{1-\mu}(k)(-)^{1-\mu}\\
& - W(k)\bar{\mathcal{F}}_{1-\mu}(k)(-)^{1-\mu}\Big],
\end{aligned} \tag{28}$$

with the amplitudes determined by the equations:

$$\left[H, \Gamma^\dagger_{1\mu}\right] = \omega\Gamma^\dagger_{1\mu},\ \left[\Gamma_{1\mu}, \Gamma^\dagger_{1\mu'}\right] = \delta_{\mu,\mu'}. \tag{29}$$

Thus, the phonon amplitudes are obtained by solving the GRFRpnQRPA equations:

$$\begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12}\\ A_{21} & A_{22} & B_{21} & B_{22}\\ -B_{11} & -B_{12} & -A_{11} & -A_{12}\\ -B_{21} & -B_{22} & -A_{21} & -A_{22}\end{pmatrix} \begin{pmatrix} X(pn)\\ Z(pn)\\ Y(pn)\\ W(pn)\end{pmatrix} = \omega \begin{pmatrix} X(p_1n_1)\\ Z(p_1n_1)\\ Y(p_1n_1)\\ W(p_1n_1)\end{pmatrix}, \tag{30}$$

where the following notations have been used:

$$
\begin{aligned}
&(A_{11}) = E^{ren}(pn)\delta_{pn;p_1n_1} + 2\chi\sigma^{(1)T}_{p_1n_1;pn},\\
&(A_{12}) = 0, \quad (B_{11}) = 0,\\
&(B_{12}) = 2\chi\sigma^{(1)T}_{p_1n_1;pn}, \quad (B_{21}) = 2\chi\sigma^{(1)T}_{p_1n_1;pn}\\
&(A_{21}) = 0, \quad (B_{22}) = 0,\\
&(A_{22}) = |E^{ren}(pn)|\delta_{pn;p_1n_1} + 2\chi\sigma^{(1)T}_{p_1n_1;pn},
\end{aligned}
\tag{31}
$$

Matrix dimension for A_{11} and B_{11} is $\mathcal{N}_+ \times \mathcal{N}_+$, while for A_{22} and B_{22} is $\mathcal{N}_- \times \mathcal{N}_-$. The off diagonal sub-matrices A_{12} and B_{12} have the dimension $\mathcal{N}_+ \times \mathcal{N}_-$, while A_{12} and B_{12} are of the $\mathcal{N}_- \times \mathcal{N}_+$ type.

In order to solve Eqs.(30) we need to know $D_1(pn)$ and, therefore, the averages of the qp's number operators, $\hat{N}_p$ and $\hat{N}_n$. These are written first in particle representation and then the particle number conserving term is expressed as a linear combination of $\mathcal{A}^\dagger\mathcal{A}$ and $\mathcal{F}^\dagger\mathcal{F}$ chosen such that their commutators with $\mathcal{A}^\dagger$, $\mathcal{A}$ and $\mathcal{F}^\dagger$, $\mathcal{F}$ are preserved. The final result is:

$$
\begin{aligned}
&\langle\hat{N}_p\rangle = V_p^2(2I_p+1) + 3(U_p^2 - V_p^2)\\
&\times[\sum_{\substack{n',k\\(p,n')\in\mathcal{S}_+}} D_1(p,n')(Y_k(p,n'))^2 - \sum_{\substack{n',k\\(p,n')\in\mathcal{S}_-}} D_1(p,n')(W_k(p,n'))^2],\\
&\langle\hat{N}_n\rangle = V_n^2(2I_n+1) + 3(U_n^2 - V_n^2)\\
&\times[\sum_{\substack{p',k\\(p',n)\in\mathcal{S}_+}} D_1(p',n)(Y_k(p',n))^2 - \sum_{\substack{p',k\\(p',n)\in\mathcal{S}_-}} D_1(p',n)(W_k(p',n))^2].
\end{aligned}
\tag{32}
$$

Eqs. (30), (32) and (23) are to be simultaneously considered and solved iteratively. It is worth mentioning that using the quasiparticle representation for the basic operators $\mathcal{A}^\dagger_{1\mu}$, $\mathcal{F}^\dagger_{1\mu}$, $\mathcal{A}_{1,-\mu}(-)^{1-\mu}$, $\mathcal{F}_{1,-\mu}(-)^{1-\mu}$, one obtains for $\Gamma^\dagger_{1\mu}$ an expression which involves the scattering pn operators. Thus, the present approach is, indeed, the *GRFRpnQRPA*.

6. THE $2\nu\beta\beta$ PROCESS

The formalism presented above was used to describe the $2\nu\beta\beta$ process. If the energy carried by leptons in the intermediate state is approximated by the sum of the rest energy of the emitted electron and half the Q-value of the double beta decay process

$$\Delta E = \frac{1}{2}Q_{\beta\beta} + m_e c^2, \tag{33}$$

the reciprocal value of the $2\nu\beta\beta$ half life can be factorized as:

$$(T_{1/2}^{2\nu\beta\beta})^{-1} = F|M_{GT}(0_i^+ \to 0_f^+)|^2, \tag{34}$$

where F is an integral on the phase space, independent of the nuclear structure, while M$_{GT}$ stands for the Gamow-Teller transition amplitude and has the expression :

$$M_{GT} = \sqrt{3}\sum_{k,k'} \frac{{}_i\langle 0||\beta_i^+||1_k\rangle_{ii}\langle 1_k|1_{k'}\rangle_{ff}\langle 1_{k'}||\beta_f^+||0\rangle_f}{E_k + \Delta E + E_{1^+}}. \tag{35}$$

In the above equation, the denominator consists of three terms: a) ΔE, which was already defined, b) the average value of the k-th *PRFRpnQRPA* energy normalized to the particular value corresponding to $k = 1$, and c) the experimental energy for the lowest 1^+ state. The indices carried by the β^+ operators indicate that they act in the space spanned by the *GRFRpnQRPA* states associated to the initial (i) or final (f) nucleus. The overlap m.e. of the single phonon states excited from the initial and final nuclei respectively, are calculated within *GRFRpnQRPA*. In (35), the Rose convention for the reduced m.e. is used [23].

Note that if we restrict the pn space to $\mathcal{S}_+$ and, moreover, the dipole-pairing interaction is ignored, M_{GT} vanishes due to the second leg of the transition. Indeed, the m.e. associated to the daughter nucleus is of the type ${}_f\langle 0|(c_n^\dagger c_p)_{1\mu}(c_n^\dagger c_p)_{1\mu}|0\rangle_f$, which is equal to zero due to the Pauli principle restriction. In this case the equations of motion are of Tamm Dankoff type and therefore the ground state correlations are missing. In order to induce the necessary correlations we have either to extend the formalism in the space $\mathcal{S}_-$, or to allow the ph excitations to interact via a pairing like force. Also, we remark that the operator $\bar{\mathcal{A}}_{1\mu}^\dagger$ plays the role of a β^- transition operator, while when $\bar{\mathcal{F}}_{1\mu}^\dagger$ or $\mathcal{A}_{1\mu}$ is applied on the ground state of the daughter nucleus, it induces a β^+ transition. Therefore, the 2β decay cannot be described by considering the β^- transition alone.

7. NUMERICAL APPLICATION

For illustration, we present the results for the transitions of four emitters: ^{148}Nd, ^{150}Nd, ^{154}Sm and ^{160}Gd. For these cases the energy corrections involved in (35) are:

$$\begin{aligned}
\Delta E(^{148}Nd) &= 1.476\ \text{MeV}, \quad E_{1^+}(^{148}Pm) = 0.137\ \text{MeV},\\
\Delta E(^{150}Nd) &= 2.196\ \text{MeV}, \quad E_{1^+}(^{150}Pm) = 0.137\ \text{MeV},\\
\Delta E(^{154}Sm) &= 1.530\ \text{MeV}, \quad E_{1^+}(^{154}Eu) = 0.046\ \text{MeV},\\
\Delta E(^{160}Gd) &= 0.046\ \text{MeV}, \quad E_{1^+}(^{160}Tb) = 0.139\ \text{MeV}.
\end{aligned} \tag{36}$$

The parameters defining the single particle energies are those of the spherical shell model, the deformation parameter d and the parameter k relating the quadrupole coordinate with the quadrupole bosons as shown in (2).

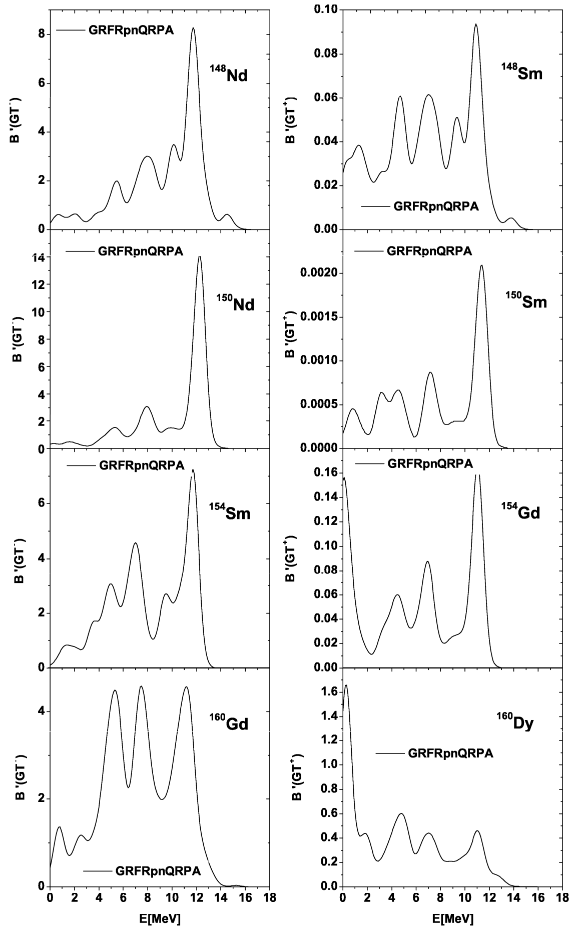

Fig. 1 – One third of the single β^- (left column) and one third of the β^+ (right column) strengths for the mother and daughter nuclei respectively, denoted by $B'(GT^-)$ and $B'(GT^+)$, folded by a Gaussian function with a width of 1 MeV, are plotted as functions of the corresponding energies yielded by the present formalism. If the ISR is obeyed, the difference between the two B' strengths associated to the mother nucleus should be equal to N-Z.

Table 1.

The number of single particle proton states lying above the (Z,N) core is given. The single particle space for neutrons is identical to that for protons. D_1 and D_2 are the dimensions of the spaces $\mathcal{S}_+, \mathcal{S}_-, \mathcal{S}$ defined in the text, for the mother and daughter nuclei, respectively. The dimension of the pnQRPA matrix is equal to the sum of the $\mathcal{S}_+$ and $\mathcal{S}_-$ dimensions. Also, the number of iterations necessary for the iterative procedure convergence, are listed.

Nucleus	^{148}Nd	^{150}Nd	^{154}Sm	^{160}Gd
The (Z,N) core	(40,40)	(40,40)	(40,40)	(40,40)
Number of states	51	57	57	59
D_1	(158,3,203)	(203,2,246)	(203,0,249)	(216,1,253)
D_2	(168,1,203)	(197,1,246)	(204,3,249)	(215,0,253)
Number of iterations	5	4	9	14

These are fixed as described in [17]. The core system for the four decays is defined by (Z,N) listed in Table 1. Therein one may find also the number of single particle states used in our calculations. The dimensions for the spaces ($\mathcal{S}_+, \mathcal{S}_-, \mathcal{S}$) for the mother ($D_1$) and daughter ($D_2$) are also listed. The strength of the dipole pn two-body interaction was taken to be

$$\chi = \frac{5.2}{A^{0.7}}\ \text{MeV}. \tag{37}$$

This expression was obtained by fitting the positions of the GT resonances in ^{40}Ca, ^{90}Zr and ^{208}Pb [24]. The values obtained for the nuclei involved in the two processes are given in Table 2.

Using these input data we calculated the distribution of the $\beta^{\pm}$ strengths with the result shown in Fig.1. The energy intervals where both distributions are large, contribute significantly to the double beta transition amplitude. The β^- strength is fragmented among the $pnQRPA$ states reflecting the fact that the single particle states are deformed. For the first two transitions the β^- strength has a dominant peak, which is just the GT resonance. For ^{154}Sm and ^{160}Gd, one and two additional peaks show up at lower energy and with a height comparable to that of the GT resonance. The β^+ strength is also fragmented but exhibits a single dominant peak located at an energy close to the GT resonance centroid. For the transitions of ^{154}Gd and ^{160}Dy an important amount of strength is accumulated in the low part of the spectrum. Actually this appear to be an effect caused by the scattering terms from the phonon operator.

Calculating first the GT transition amplitude and then the Fermi integral with $G_A = 1.254$, as in [4], we obtained the results given in Table 3.

Another interesting result concerns the summed strength for the β^- and β^+

Table 2.

The deformation parameter d, the pairing interaction strengths for protons (G_p) and neutrons (G_n), the GT dipole (χ) and dipole-pairing (X_{dp}) strengths interactions used in our calculations. We also give the parameter k relating the quadrupole coordinates and bosons (this is involved in the expression of the single particle energies).

	d	k	G_p[MeV]	G_n[MeV]	ISR	χ [MeV]	X_{dp} [MeV]
^{148}Nd	1.555	14.	0.11	0.2516	28.02	0.157	0.142
^{148}Sm	0.1555	14.	0.11	0.225	24.04	0.157	0.142
^{150}Nd	1.952	16.	0.10	0.254	30.05	0.156	0.016
^{150}Sm	1.952	16.	0.11	0.235	26.08	0.156	0.016
^{154}Sm	2.29	16.	0.10	0.316	30.08	0.153	0.138
^{154}Gd	2.29	14.	0.11	0.27	26.01	0.153	0.138
^{160}Gd	2.714	10.	0.11	0.3	32.07	0.149	0.298
^{160}Dy	2.714	8.	0.11	0.2578	28.02	0.149	0.298

Table 3.

The Gamow-Teller amplitude for the $2\nu\beta\beta$ decay, in units of MeV^{-1}, and the corresponding half life ($T_{1/2}$), in units of yr, are listed for two ground to ground transitions. The experimental half-life for the transition of ^{150}Nd ($^{a)}$ [26]) is also given. Comparison is also made with the theoretical results from the last two columns, reported in [16] and [25], respectively.

	M_{GT}	$T_{1/2}$[yr]			
	[MeV^{-1}]	present	Exp.	Raduta *et al* [16]	Wu *et al* [25]
^{148}Nd→^{148}Sm	0.422	2.00×10^{19}	-	2.33×10^{19}	1.19×10^{21}
^{150}Nd→^{150}Sm	0.042	2.50×10^{19}	$\geq 1.8\times10^{19\ a)}$	2.63×10^{17}	1.66×10^{19}
^{154}Sm→^{154}Gd	0.303	2.02×10^{21}	-	8.76×10^{20}	1.49×10^{22}
^{160}Gd→^{160}Dy	0.111	1.02×10^{21}	-	2.013×10^{20}	2.81×10^{21}

Table 4.

The calculated summed strengths for the β^- (first row) and β^+ (second row) are presented.

Nucleus	^{148}Nd	^{150}Nd	^{154}Sm	^{160}Gd
$0.6\sum B(GT)^-$	51.74	54.11	54.68	57.93
$0.6\sum B(GT)^+$	1.29	0.02	0.54	0.21

Table 5.

The log ft values characterizing the β^+/EC and β^- processes associated to intermediate odd-odd nuclei.

Mother nucleus		odd-odd nucleus		Daughter nucleus
^{148}Nd	$\overset{\beta^+/EC}{\longleftarrow}$ 6.8	^{148}Pm	$\overset{\beta^-}{\longrightarrow}$ 7.33	^{148}Sm
^{150}Nd	$\overset{\beta^+/EC}{\longleftarrow}$ 5.55	^{150}Pm	$\overset{\beta^-}{\longrightarrow}$ 8.46	^{150}Sm
^{154}Sm	$\overset{\beta^+/EC}{\longleftarrow}$ 5.52	^{154}Eu	$\overset{\beta^-}{\longrightarrow}$ 5.13	^{154}Gd
^{160}Gd	$\overset{\beta^+/EC}{\longleftarrow}$ 5.25	^{160}Tb	$\overset{\beta^-}{\longrightarrow}$ 4.20	^{160}Dy

transition, denoted conventionally, by $\sum B_{GT-}$ and $\sum B_{GT-}$, respectively. It is well known that experimentally only a fraction (about 60%) of the GT strength can be measured and, therefore, in order to get a fair comparison of the theoretical results and the corresponding data we have to quench the calculated strength by a factor 0.6. The results are presented in Table 4.

The intermediate odd-odd nuclei involved in the double beta process can, in principle, perform the transition β^+/EC, which results in feeding the mother nucleus of each transition. On the other hand, they can perform a β^- transition to the daughter nucleus. For some transitions of this type the $\log\, ft$ values are measured. The corresponding theoretical results are obtained by means of the expression:

$$ft_{\mp} = \frac{6160}{[_l\langle 1_1||\beta^{\pm}||0\rangle_l g_A]^2},\; l = i, f. \tag{38}$$

In order to take account of the effect of distant states responsible for the ”missing strength” in the giant GT resonance [4] we chose $g_A = 1.0$. In a previous publication [17], where a standard *pnQRPA* approach was used, the strengths of the ph and pp interactions have been fixed in order to reproduce the $\log\, ft$ values characterizing the two transitions of the intermediate odd-odd nucleus. Similarly, here the strengths of the two body proton-neutron interactions, χ and X_{dp}, could be fixed by fitting the $\log\, ft$ values associated to the two single beta transitions. Unfortunately, there are not enough available data to enable a fitting procedure. In Table VI the results of our calculations for the mentioned $\log\, ft$ values are listed. In our calculations X_{dp} was fixed so that the $\log\, ft$ value for the β^+/EC process be close to the corresponding experimental data in that region.

8. CONCLUSIONS

Summarizing the results of this paper, one may say that restoring the gauge symmetry from the fully renormalized *pnQRPA* provides a consistent and realistic description of the transition rate and, moreover, the ISR is obeyed. As shown in this paper, it seems that there is no need to include the *pp* interaction in the many body treatment of the process. Indeed, in the framework of a *pnQRPA* approach this interaction violates the total number of particle and consequently the gauge projection process makes it ineffective. The proton neutron correlations in the ground state are however determined by an attractive dipole pairing interaction. The results of our calculations are compared with those obtained by different methods as well as with the available experimental data.

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