THE SOLUTIONS TO THE SALPETER EQUATION FOR THE P STATES

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Applying a new approach to solve the instantaneous Bethe-Salpeter equation (or the Salpeter equation), the P states 0^{++}, 1^{+-}, 1^{++} and 2^{++} of positronium are solved as examples. The new approach exhibits features different from those obtained when the Salpeter wave functions are represented in Cartesian coordinates. In this paper, the Salpeter wave functions are represented in spherical coordinates in which different wave contributions can be separated clearly and the numerical results are obtained by solving the coupled scalar equations.

Key words: Salpeter equation, wave function, spherical coordinates, P state, relativistic corrections.

1. INTRODUCTION

The Bethe-Salpeter (BS) equation [1, 2] is a good tool to deal with various bound systems. Many methods are applied to solve analytically or numerically the Bethe-Salpeter equation and the instantaneous Bethe-Salpeter equation (also called the Salpeter equation) [3–6].

In papers [7, 8], we have presented an approach to solve the Salpeter equation. The S states $0^{-+}({}^0S_0)$ and $1^{--}({}^3S_1 - {}^3D_1)$ were investigated in detail. We have shown that for the state 0^{-+} the S-wave components play dominant roles. For the state 1^{--} both the S-wave components and the D-wave components play dominant roles. When the S-wave components are important, the D-wave components become very small, and when the S-wave components are small, the D-wave components become dominant. The results obtained are coincident with the phenomenological understanding, see Table 1.

The P-states are very important in positronium physics and in heavy quarkonium physics. The studies of Lyman spectra involve the P-states. The P-states are also needed in positronium interactions with atoms. Lots of S states and P states of heavy quarkonia are investigated which are important to improve the understanding of Quantum Chromodynamics (QCD).

Taken as examples, the P states of positronium $0^{++}({}^3P_0)$, $1^{+-}({}^1P_1)$, $1^{++}({}^3P_1)$,

and $2^{++}(^3P_2 -^3 F_2)$ are investigated adopting the method proposed in papers [7, 8]. The method and some results obtained in this paper can be applied to quarkonia. Firstly, the Salpeter wave functions for the P states are given in spherical coordinates in which different wave contributions will be presented explicitly and can be separated from each other clearly. Then the coupled radial equations are obtained. Finally, expanding the scalar functions with the analytic solutions to the Schrödinger equation [9] with Coulomb potential, the numerical results are obtained by solving eigenvalue problem.

The paper is organized as follows: In section 2, the Salpeter equation is given. In section 3, the solutions for the P-states of positronium $0^{++}(^3P_0)$, $1^{+-}(^1P_1)$, $1^{++}(^3P_1)$ and $2^{++}(^3P_2 -^3 F_2)$ are presented. The conclusion is given in section 4.

Table 1.

The physically existing states for the bound state composed of one fermion and one antifermion, $J^{PC}(^{2*S+1}L_J)$. / means that the states $|q\bar{q}\rangle$ with exotic quantum numbers cannot be described by the Salpeter equation.

Spin of bound state (J)	Parity (J^P)	Wave state (L_J)	Charge parity (J^{PC})	Total spin of constituents $(^{2*S+1}L_J)$
J=0	0^-	S_0	0^{--}	/
			0^{-+}	$^1S_0$
	0^+	P_0	0^{+-}	/
			0^{++}	$^3P_0$
J=1	1^-	$S_1 - D_1$	1^{--}	$^3S_1 -^3 D_1$
			1^{-+}	/
	1^+	P_1	1^{+-}	$^1P_1$
			1^{++}	$^3P_1$
J=2	2^-	D_2	2^{-+}	$^1D_2$
			2^{--}	$^3D_2$
	2^+	$P_2 - F_2$	2^{+-}	/
			2^{++}	$^3P_2 -^3 F_2$
...	...	...	...	...

2. THE SALPETER EQUATION

The general form of the Bethe-Salpeter equation for a fermion-antifermion bound state in momentum space [1, 2] reads

$$(\not{p}_1 - m_1)\chi_P(q)(\not{p}_2 + m_2) = \int \frac{d^4k}{(2\pi)^4} K(P,q,k)\chi_P(k)\,, \tag{1}$$

where $\chi_P(q)$ is the Bethe-Salpeter wave function, p_1 and p_2 are four momenta for constituent a and constituent b respectively, which can be incorporated into another set of variables: the momentum P for the bound state and the relative momentum q,

$$P = p_1 + p_2, \quad q = \eta_2 p_1 - \eta_1 p_2, \quad p_1 = \eta_1 P + q, \quad p_2 = \eta_2 P - q, \tag{2}$$

where $\eta_{1,2} = m_{1,2}/(m_1 + m_2)$, $\eta_1 + \eta_2 = 1$. For positronium system, $m_1 = m_2$. The Bethe-Salpeter wave function $\chi_P(q)$ is normalized

$$i \int \frac{d^4q}{(2\pi)^4} \frac{d^4q'}{(2\pi)^4} Tr \left[\bar{\chi}_P(q) \frac{\partial}{\partial P^\mu} \left(G_0^{-1}(P,q,q') - K(P,q,q') \right) \chi_P(q') \right] = 2P_\mu, \tag{3}$$

where $G_0^{-1} = (\not{p}_1 - m_1)(\not{p}_2 + m_2)(2\pi)^4 \delta^4(q - q')$, $\bar{\chi}_P(q) = \gamma^0 \chi_P^\dagger(q) \gamma^0$.

The center of mass system frame (C.M.S.) is chosen, $P^\mu = (M, \vec{P})$, $\vec{P} = 0$. M is the mass of the bound state. If kernel $K(P,q,k)$ in C.M.S. frame behaves like $K(P,q,k) = iV(\vec{q},\vec{k})$, the Bethe-Salpeter equation is then called the instantaneous Bethe-Salpeter equation. At the lowest order, the kernel for positronium system takes the form

$$V(\vec{q},\vec{k}) = \gamma^0 \frac{4\pi\alpha}{(\vec{q} - \vec{k})^2} \gamma^0. \tag{4}$$

Indeed, there exist many physical systems, for example positronium, $e^\pm \mu^\pm$ system and heavy quarkonia, which can be well described by the Salpeter equation.

The Salpeter wave function is defined as

$$\varphi_P(\vec{q}) = \int \frac{dq^0}{2\pi} \chi_P(q). \tag{5}$$

Using equations (1) and (5), the Salpeter equation is obtained by contour integration

$$\varphi_P(\vec{q}) = \frac{\Lambda_1^+(\vec{q})\gamma^0 \eta(\vec{q}) \gamma^0 \Lambda_2^-(-\vec{q})}{(M - \omega_1 - \omega_2)} - \frac{\Lambda_1^-(\vec{q})\gamma^0 \eta(\vec{q}) \gamma^0 \Lambda_2^+(-\vec{q})}{(M + \omega_1 + \omega_2)}, \tag{6}$$

$$\eta(\vec{q}) = \int \frac{d^3\vec{k}}{(2\pi)^3} V(\vec{q},\vec{k}) \varphi_P(\vec{k}),$$

where $\omega_i = \sqrt{m_i^2 + \vec{q}_i^2}$, $\Lambda_i^\pm(\vec{p_i})$ are the energy projection operators

$$\Lambda_i^\pm(\vec{p_i}) = \frac{\omega_i(\vec{p_i}) \pm H_i(\vec{p_i})}{2\omega_i(\vec{p_i})}, \quad H_i(\vec{p_i}) = \gamma_0(\vec{\gamma} \cdot \vec{p_i} + m_i).$$

Applying the energy projection operators to both sides of equation (6), the coupled equations can be obtained

$$(M - \omega_1 - \omega_2) \varphi_P^{+-}(\vec{q}) = \Lambda_1^+(\vec{q}) \gamma^0 \int \frac{d^3\vec{k}}{(2\pi)^3} V(\vec{q},\vec{k}) \varphi_P(\vec{k}) \gamma^0 \Lambda_2^-(-\vec{q}), \tag{7}$$

$$(M + \omega_1 + \omega_2) \varphi_P^{-+}(\vec{q}) = -\Lambda_1^-(\vec{q}) \gamma^0 \int \frac{d^3\vec{k}}{(2\pi)^3} V(\vec{q},\vec{k}) \varphi_P(\vec{k}) \gamma^0 \Lambda_2^+(-\vec{q}), \tag{8}$$

$$\varphi_P^{++}(\vec{q}) = \varphi_P^{--}(\vec{q}) = 0, \tag{9}$$

where $\varphi_P^{\pm\pm}(\vec{q}) = \Lambda^{\pm}(\vec{q})\varphi_P(\vec{q})\Lambda^{\pm}(-\vec{q})$. Equation (9) imposes constraints on the Salpeter wave function $\varphi_P(\vec{q})$. These constraints rule out the exotic states with $P = (-1)^J$ and $C = (-1)^{J-1}$, i.e., such states cannot be described by the Salpeter equation [10]. This conclusion is consistent with the results obtained by L-S coupling analysis in Table 1. The Salpeter wave function $\varphi_P(\vec{q})$ is normalized as

$$\int \frac{d^3q}{(2\pi)^3} Tr\left[\varphi^{\dagger}(\vec{q})\Lambda_1^+(\vec{q})\varphi(\vec{q})\Lambda_2^-(-\vec{q}) - \varphi^{\dagger}(\vec{q})\Lambda_1^-(\vec{q})\varphi(\vec{q})\Lambda_2^+(-\vec{q})\right] = 2M. \quad (10)$$

3. THE SOLUTIONS FOR THE P STATES

In this section, the P states, $0^{++}(^3P_0)$, $1^{+-}(^1P_1)$, $1^{++}(^3P_1)$, and $2^{++}(^3P_2 - ^3F_2)$ are investigated.

3.1. THE SOLUTION FOR THE 0^{++} STATE

The 0^{++} state is both a spin triplet and a P state, $^3P_0$. The general form of the Salpeter wave function for the 0^{++} state reads

$$\phi = \gamma^0 f_1 + f_2 + \not{q} f_3 + \gamma^0 \not{\hat{q}} f_4, \quad (11)$$

where $\hat{\not{q}}$ is defined in spherical coordinates,

$$\hat{\not{q}} = \hat{q}^+ \cdot \gamma^- + \hat{q}^- \cdot \gamma^+ - \hat{q}^{\Delta} \cdot \gamma^{\Delta}, \quad (12)$$

$$\gamma^+ = \frac{-(\gamma_1 + i\gamma_2)}{\sqrt{2}}, \quad \gamma^- = \frac{(\gamma_1 - i\gamma_2)}{\sqrt{2}}, \quad \gamma^{\Delta} = \gamma_3, \quad (13)$$

$$\hat{q}^+ = \sqrt{\frac{4\pi}{3}} Y_{11}, \quad \hat{q}^- = \sqrt{\frac{4\pi}{3}} Y_{1-1}, \quad \hat{q}^{\Delta} = \sqrt{\frac{4\pi}{3}} Y_{10}. \quad (14)$$

In equation (11) and the following equations, we introduce the shorthand notations: $\phi \equiv \phi(\vec{q})$, $f_i \equiv f_i(|\vec{q}|)$, $m \equiv m_1 = m_2$, $\omega \equiv \omega_{1,2}(\vec{q})$, $Y_{lm} \equiv Y_{lm}(\theta, \varphi)$.

As shown in Table 1, the P-wave components are expected to dominate in the 0^{++} state, i.e., in equation (11), the S-wave terms $f_{1,2}$ are small terms and the P-wave terms $f_{3,4}$ are big.

Imposing the constraints (9) on wave function (11) gives

$$f_1 = 0, \quad f_2 = -\frac{|\vec{q}|}{m} f_3. \quad (15)$$

Substituting constraints (15) into equation (11), the wave function for the 0^{++} state reduces to

$$\phi = -\frac{|\vec{q}|}{m} f_3 + (f_3 + \gamma^0 f_4)\not{\hat{q}}. \quad (16)$$

As expected, the P-wave terms f_3 and f_4 are big terms; f_1 is zero, f_2 is of order of $v = |\vec{q}|/m$, a correction to f_3 by the constraints (15).

From equations (7), (8) and (16), the coupled radial equations are obtained after carrying out angular integration

$$\begin{aligned} Mf_3 &= 2mf_4 - \frac{\alpha}{\pi}\int \frac{|\vec{k}|}{|\vec{q}|}d|\vec{k}|\frac{m}{\omega}Q_1 f_4, \\ Mf_4 &= 2\frac{\omega^2}{m}f_3 - \frac{\alpha}{\pi}\int \frac{|\vec{k}|}{|\vec{q}|}d|\vec{k}|\left(\frac{|\vec{q}||\vec{k}|}{m\omega}Q_0 + \frac{m}{\omega}Q_1\right)f_3, \end{aligned} \tag{17}$$

where $Q_n \equiv Q_n(\frac{\vec{q}^2+\vec{k}^2}{2|\vec{q}||\vec{k}|})$ is the Legendre function of the second kind. In equation (17) and the following part of this paper, $f_i \equiv f_i(|\vec{q}|)$ if they are not to be integrated, $f_i \equiv f_i(|\vec{k}|)$ if the functions f_i are to be integrated.

The normalization condition for the 0^{++} state (16) reads

$$\int \frac{|\vec{q}|^2 d|\vec{q}|}{(2\pi)^2}\frac{4\omega}{m}f_3 f_4 = M. \tag{18}$$

Expanding the scalar functions f_i in terms of a set of basis functions, the coupled equations are transformed into eigenvalue problem. In this paper, the analytic solutions to the Schrödinger equation [9] with Coulomb potential are employed as basis functions because the Salpeter equation with kernel (4) can be reduced to the Schrödinger equation with Coulomb potential under non-relativistic limit. The numerical results will converge quickly with increasing number of basis functions. The significant digits reach five when only the first four basis functions are employed to expand the scalar functions f_i. The numerical results of the radial equations (17) give eigenvalues (in unit of eV)

$$\begin{aligned} E^{(1)} = -1.700721, \quad E^{(2)} = -0.755875, \\ E^{(3)} = -0.425179, \quad E^{(4)} = -0.272115, \\ \cdots \end{aligned} \tag{19}$$

and the corresponding eigenfunctions for $E^{(i)}$

$$f_j = \sum_{nl} C^i_{j,nl} R_{nl}(|\vec{q}|), \tag{20}$$

where $C^i_{j,nl}$ are the expanding coefficients listed in Table 2. $R_{nl}(|\vec{q}|)$ are the solutions to the Schrödinger equation [9].

It is shown in Table 1 that there is one dominant expanding coefficient for each eigenvalue and others are very small. This character is attributed to the proper choice of basis functions according the kenel. If the kernel is not of the form (4), this character perhaps will not occur again even with these basis functions still employed.

In Table 2, the expanding coefficients $C^i_{j,nl}$ for the 0^{++} state are listed. n is principle quantum number, l is orbital angular momentum number, WF represents scalar function. R_{nl} are radial wave functions which are taken as basis functions, and the numbers listed in this column are the expanding coefficients $C^i_{j,nl}$. Borrowed from notation in Fortran program, $E \pm i$ means $10^{\pm i}$.

Table 2.

The expanding coefficients $C^i_{j,nl}$ for the 0^{++} state(see text).

State	{nl}	WF	R_{21}	R_{31}	R_{41}	R_{51}
0^{++}	$\{21\}$	f_3	-0.707106	-3.1E-6	-1.4E-6	-8.8E-6
		f_4	-0.707107	-3.7E-6	-1.8E-6	-1.1E-6
	$\{31\}$	f_3	-4.3E-6	0.707107	3.7E-6	1.8E-6
		f_4	-3.7E-6	0.707107	4.1E-6	2.E-6
	$\{41\}$	f_3	-2.1E-6	-4.4E-6	0.707107	3.9E-6
		f_4	-1.8E-6	-4.1E-6	0.707107	4.1E-6
	$\{51\}$	f_3	1.3E-6	2.2E-6	4.3E-6	-0.707107
		f_4	1.1E-6	2.E-6	4.1E-6	-0.707107

Up to order $v = |\vec{q}|/m$, the wave function (16) can be simplified as

$$\phi = \left[-\frac{|\vec{q}|}{m} + (1+\gamma^0)\hat{\not{q}} \right] f, \tag{21}$$

where $f_3 \simeq f_4 \simeq f = R_{n(l=1)}(|\vec{q}|)$.

3.2. THE SOLUTION FOR THE 1^{+-} STATE

The state 1^{+-} is solved following the procedure for solving the state 0^{++} in previous subsection. The state 1^{+-} is both a spin singlet and a P state, $^1P_1$. The general form of the Salpeter wave function state reads

$$\phi = \hat{q}\cdot\epsilon \left(f_1 + \hat{\not{q}} f_2 + \gamma^0 f_3 + \hat{\not{q}}\gamma^0 f_4 \right) \gamma^5, \tag{22}$$

where $\hat{q}\cdot\epsilon = \hat{q}^+\cdot\epsilon^- + \hat{q}^-\cdot\epsilon^+ - \hat{q}^\Delta\cdot\epsilon^\Delta$ is represented in spherical coordinates. As shown in Table 1, the P-wave components are expected to dominate in the 1^{+-} state, i.e. in equation (22), the S-wave and D-wave mixted terms f_2 and f_4 are small terms, the P-wave terms f_1 and f_3 are big.

Calculating the constraints $\phi^{+-} = 0$ and $\phi^{-+} = 0$ for the state 1^{+-} gives

$$f_2 = 0, \quad f_4 = \frac{|\vec{q}|}{m} f_3 . \tag{23}$$

Then the Salpeter wave function (22) is simplified as

$$\phi = \hat{q}\cdot\epsilon \left[f_1 + (1+\hat{\not{q}}\frac{|\vec{q}|}{m})\gamma^0 f_3 \right] \gamma^5 . \tag{24}$$

As expected, the term f_4 is small compared to f_3 by constraints (23).

Using equations (7), (8) and (24), the coupled radial equations can be obtained by straight calculation

$$\begin{aligned} Mf_1 &= 2\frac{\omega^2}{m}f_3 - \frac{\alpha}{\pi}\int \frac{|\vec{k}|}{|\vec{q}|}d|\vec{k}|\left(\frac{|\vec{q}||\vec{k}|}{3m\omega}Q_0 + \frac{m}{\omega}Q_1 + \frac{2|\vec{q}||\vec{k}|}{3m\omega}Q_2\right)f_3, \\ Mf_3 &= 2mf_1 - \frac{\alpha}{\pi}\int \frac{|\vec{k}|}{|\vec{q}|}d|\vec{k}|\frac{m}{\omega}Q_1 f_1. \end{aligned} \tag{25}$$

The normalization condition for the wave function (24) reads

$$\int \frac{\vec{q}^2 d|\vec{q}|}{(2\pi)^2}\frac{4\omega}{m}f_1 f_3 = M. \tag{26}$$

Expanding the scalar functions f_1 and f_3 with four basis functions, solving radial equations (25) numerically gives eigenvalues

$$\begin{aligned} E^{(1)} = -1.700713, \quad E^{(2)} = -0.755872, \\ E^{(3)} = -0.425178, \quad E^{(4)} = -0.272114, \\ \cdots \end{aligned} \tag{27}$$

and the expanding coefficients $C^i_{j,nl}$ for $f_{1,3}$ which are listed in Table 3.

Table 3.

The expanding coefficients $C^i_{j,nl}$ for the 1^{+-} state; see text.

State	$\{nl\}$	WF	R_{21}	R_{31}	R_{41}	R_{51}
1^{+-}	$\{21\}$	f_1	0.707107	8.2E-7	3.8E-7	2.3E-7
		f_3	0.707106	2.4E-7	5.7E-8	1.2E-8
	$\{31\}$	f_1	-8.4E-7	0.707107	1.E-6	4.7E-7
		f_3	-1.4E-6	0.707107	6.8E-7	2.7E-7
	$\{41\}$	f_1	-3.9E-7	-1.E-6	0.707107	1.E-6
		f_3	-7.1E-7	-1.3E-6	0.707107	8.E-7
	$\{51\}$	f_1	2.3E-7	4.7E-7	1.E-6	-0.707107
		f_3	4.5E-7	6.7E-7	1.2E-6	-0.707107

Up to order v, the wave function (24) is reduced to

$$\phi = \hat{q}\cdot\epsilon\left[1 + (1 + \hat{\not{q}}\frac{|\vec{q}|}{m})\gamma^0\right]\gamma^5 f, \tag{28}$$

where $f_1 \simeq f_3 \simeq f = R_{n(l=1)}(|\vec{q}|)$.

3.3. THE SOLUTION FOR THE 1^{++} STATE

The state 1^{++} is also a spin triplet and a P state, ${}^3P_1$. The general form of the Salpeter wave function reads

$$\phi = i\epsilon_{0ijk}\epsilon^i\hat{q}^j\left[\gamma^k\left(f_1-\gamma^0 f_2+\frac{\not{k}}{M}f_3\right)+i\epsilon^{0kmn}\hat{q}_m\gamma_n\gamma^5 f_4\right]. \tag{29}$$

Here

$$\begin{aligned} i\epsilon_{0ijk}\epsilon^i\hat{q}^j\gamma^k &= \left(\gamma^-\epsilon^+-\gamma^+\epsilon^-\right)\hat{q}^\Delta+\left(\gamma^\Delta\epsilon^--\gamma^-\epsilon^\Delta\right)\hat{q}^+ \\ &\quad+\left(\gamma^+\epsilon^\Delta-\gamma^\Delta\epsilon^+\right)\hat{q}^- \end{aligned} \tag{30}$$

is a pure P-wave contribution,

$$\begin{aligned} \epsilon_{0ijk}\epsilon^i\hat{q}^j\gamma^k\epsilon^{0kmn}\hat{q}_m\gamma_n &= -\left(\epsilon^+\gamma^++\epsilon^-\gamma^-\right)\left(\hat{q}^+\right)^2-\left(\epsilon^+\gamma^-+\epsilon^-\gamma^+\right)\left(\hat{q}^\Delta\right)^2 \\ &\quad+\left(\epsilon^\Delta\gamma^++\epsilon^+\gamma^\Delta\right)\hat{q}^-\hat{q}^\Delta+\left(\epsilon^\Delta\gamma^-+\epsilon^-\gamma^\Delta\right)\hat{q}^+\hat{q}^\Delta \\ &\quad+\left(\epsilon^+\gamma^-+\epsilon^-\gamma^+-2\epsilon^\Delta\gamma^\Delta\right)\hat{q}^+\hat{q}^-, \end{aligned} \tag{31}$$

is small and is a combination of S-wave contribution and D-wave contribution. In equation (29), f_1 and f_2 are pure P-wave terms and are big, f_3 and f_4 are small.

Imposing the constraints (9) on (29) leads to

$$f_3=0,\quad f_4=-\frac{|\vec{q}|}{m}f_2. \tag{32}$$

Using the constraints (32), the Salpeter wave function (29) is simplified as

$$\phi = i\epsilon_{0ijk}\epsilon^i\hat{q}^j\left[\gamma^k\left(f_1-\gamma^0 f_2\right)-i\epsilon^{0kmn}\hat{q}_m\gamma_n\gamma^5\frac{|\vec{q}|}{m}f_2\right], \tag{33}$$

Applying equations (7), (8) and (33), the coupled radial equations are obtained

$$\begin{aligned} Mf_1 &= 2\frac{\omega^2}{m}f_2-\frac{\alpha}{\pi}\int\frac{|\vec{k}|}{|\vec{q}|}d|\vec{k}|\left(\frac{2|\vec{q}||\vec{k}|}{3m\omega}Q_0+\frac{m}{\omega}Q_1+\frac{|\vec{q}||\vec{k}|}{3m\omega}Q_2\right)f_2, \\ Mf_2 &= 2mf_1-\frac{\alpha}{\pi}\int\frac{|\vec{k}|}{|\vec{q}|}d|\vec{k}|\frac{m}{\omega}Q_1f_1. \end{aligned} \tag{34}$$

For the state 1^{++}, the normalization condition is simplified as

$$\int\frac{\vec{q}^2d|\vec{q}|}{(2\pi)^2}\frac{8\omega}{m}f_1f_2=M. \tag{35}$$

Solving equations (34) numerically gives eigenvalues

$$\begin{aligned} &E^{(1)}=-1.700717,\quad E^{(2)}=-0.755874, \\ &E^{(3)}=-0.425179,\quad E^{(4)}=-0.272114, \\ &\cdots \end{aligned} \tag{36}$$

and the expanding coefficients listed in Table 4.

Table 4.

The expanding coefficients $C^i_{j,nl}$ for the 1^{++} state.

State	{nl}	WF	R_{21}	R_{31}	R_{41}	R_{51}
1^{++}	{21}	f_1	-0.707107	-2.3E-6	-1.1E-6	-6.7E-7
		f_2	-0.707106	-1.7E-6	-7.4E-7	-4.5E-7
	{31}	f_1	-2.3E-6	0.707107	2.5E-6	1.2E-6
		f_2	-2.9E-6	0.707107	2.2E-6	1.E-6
	{41}	f_1	-1.1E-6	-2.5E-6	0.707107	2.5E-6
		f_2	-1.4E-6	-2.8E-6	0.707107	2.3E-6
	{51}	f_1	-6.7E-7	-1.2E-6	-2.5E-6	0.707107
		f_2	-8.9E-7	-1.4E-6	-2.7E-6	0.707107

Up to order v, the wave function (33) becomes

$$\phi = i\epsilon_{0ijk}\epsilon^i\hat{q}^j\left[\gamma^k\left(1-\gamma^0\right) - i\epsilon^{0kmn}\hat{q}_m\gamma_n\gamma^5\frac{|\vec{q}|}{m}\right]f, \tag{37}$$

where $f_1 \simeq f_2 \simeq f = R_{n(l=1)}(|\vec{q}|)$.

3.4. THE SOLUTION FOR THE 2^{++} STATE

The State 2^{++} is a spin triplet state, ${}^3P_2 - {}^3F_2$. Unlike the states 0^{++}, 1^{+-} and 1^{++} which are all P-wave dominant states, like the state 1^{--} which is S-wave and D-wave mixed state, the state 2^{++} is a P-wave and F-wave mixed state. Both P-wave components and F-wave components play dominant roles in the state 2^{++}.

The general form of the Salpeter wave function for the 2^{++} state reads

$$\begin{aligned}\phi &= \epsilon_{ij}\hat{q^i}\left[\gamma^j(f_1+\gamma^0 f_2)+\hat{q^j}\hat{\not{q}}(f_7+\gamma^0 f_8)\right] \\ &\quad +\epsilon_{ij}\hat{q^i}\left[\hat{q^j}(f_3+\gamma^0 f_4)+\gamma^j\hat{\not{q}}f_5+i\epsilon^{0jmn}\hat{q_m}\gamma_n\gamma^5 f_6\right].\end{aligned} \tag{38}$$

The terms f_1 and f_2 are P-wave contributions and are big, the terms f_7 and f_8 contain both P-wave contributions and F-wave contributions and are big too, and other terms are small. These features cannot be shown explicitly in formula (38) which is written in Cartesian coordinates. To separate explicitly the P-wave contributions and F-wave contributions we rewrite the Salpeter wave function (38) in spherical coordinates,

$$\phi = A(g_1+\gamma^0 g_2)+B(g_3+\gamma^0 g_4)+Cg_5+D\gamma^0 g_6+E(g_7+\gamma^0 g_8). \tag{39}$$

Here

$$\begin{aligned}A &= (\epsilon_{++}\gamma^+ + \epsilon_{+-}\gamma^- + \epsilon_{+\Delta}\gamma^\Delta)\hat{q}^+ + (\epsilon_{-+}\gamma^+ + \epsilon_{--}\gamma^- + \epsilon_{-\Delta}\gamma^\Delta)\hat{q}^- \\ &\quad +(\epsilon_{\Delta+}\gamma^+ + \epsilon_{\Delta-}\gamma^- + \epsilon_{\Delta\Delta}\gamma^\Delta)\hat{q}^\Delta,\end{aligned} \tag{40}$$

$$B = \epsilon_{++}\sqrt{\frac{8\pi}{15}}Y_{22}+\epsilon_{--}\sqrt{\frac{8\pi}{15}}Y_{2-2}+(\epsilon_{\Delta\Delta}+\epsilon_{-+})\sqrt{\frac{16\pi}{45}}Y_{20} +2\epsilon_{\Delta+}\sqrt{\frac{4\pi}{15}}Y_{21}+2\epsilon_{\Delta-}\sqrt{\frac{4\pi}{15}}Y_{2-1}, \quad (41)$$

$$C = \sqrt{\frac{4\pi}{15}}\Big[(\epsilon_{++}\gamma^{+}+\epsilon_{+-}\gamma^{-}+\epsilon_{+\Delta}\gamma^{\Delta})(\sqrt{\frac{1}{3}}\gamma^{+}Y_{20}-\gamma^{\Delta}Y_{21}+\sqrt{2}\gamma^{-}Y_{22}) +(\epsilon_{-+}\gamma^{+}+\epsilon_{--}\gamma^{-}+\epsilon_{-\Delta}\gamma^{\Delta})(\sqrt{\frac{1}{3}}\gamma^{-}Y_{20}-\gamma^{\Delta}Y_{21}+\sqrt{2}\gamma^{+}Y_{2-2}) -(\epsilon_{\Delta+}\gamma^{+}+\epsilon_{\Delta-}\gamma^{-}+\epsilon_{\Delta\Delta}\gamma^{\Delta})(\sqrt{\frac{4\pi}{3}}\gamma^{-}Y_{20}-\gamma^{+}Y_{2-1}-\gamma^{-}Y_{21})\Big], \quad (42)$$

$$D = (\epsilon_{++}\gamma^{\Delta}+\epsilon_{+\Delta}\gamma^{-})\sqrt{\frac{8\pi}{15}}Y_{22} +(\epsilon_{\Delta+}\gamma^{\Delta}-\epsilon_{++}\gamma^{+}+\epsilon_{+-}\gamma^{-}+\epsilon_{\Delta\Delta}\gamma^{-})\sqrt{\frac{4\pi}{15}}Y_{21} -(\epsilon_{--}\gamma^{\Delta}+\epsilon_{-\Delta}\gamma^{+})\sqrt{\frac{8\pi}{15}}Y_{2-2}+(\epsilon_{-\Delta}\gamma^{-}-\epsilon_{+\Delta}\gamma^{+})\sqrt{\frac{4\pi}{5}}Y_{20} +(-\epsilon_{\Delta-}\gamma^{\Delta}-\epsilon_{\Delta\Delta}\gamma^{+}-\epsilon_{-+}\gamma^{+}+\epsilon_{--}\gamma^{-})\sqrt{\frac{4\pi}{15}}Y_{2-1}, \quad (43)$$

$$E = 2\sqrt{\frac{2\pi}{35}}\gamma^{+}\epsilon_{--}Y_{3-3}+2\sqrt{\frac{2\pi}{105}}[2\epsilon_{-\Delta}\gamma^{+}-\gamma^{\Delta}\epsilon_{--}]Y_{3-2} +\frac{2}{5}\sqrt{\frac{2\pi}{21}}[\gamma^{-}\epsilon_{--}-4\epsilon_{-\Delta}\gamma^{\Delta}+2\gamma^{+}(\epsilon_{-+}+\epsilon_{\Delta\Delta})]Y_{3-1} +\frac{2}{5}\sqrt{\frac{4\pi}{7}}[\epsilon_{-\Delta}\gamma^{-}+\epsilon_{+\Delta}\gamma^{+}-\gamma^{\Delta}(\epsilon_{-+}+\epsilon_{\Delta\Delta})]Y_{30} +\frac{2}{5}\sqrt{\frac{2\pi}{21}}[\gamma^{+}\epsilon_{++}-4\epsilon_{+\Delta}\gamma^{\Delta}+2\gamma^{-}(\epsilon_{-+}+\epsilon_{\Delta\Delta})]Y_{31} +2\sqrt{\frac{2\pi}{105}}[2\epsilon_{+\Delta}\gamma^{-}-\gamma^{\Delta}\epsilon_{++}]Y_{3-2}+2\sqrt{\frac{2\pi}{35}}\gamma^{-}\epsilon_{++}Y_{33}. \quad (44)$$

From LS coupling analysis, we know that $g_{1,2}$ corresponding to pure P-wave components and $g_{7,8}$ corresponding to pure F-wave components are large, and $g_{3,4,5,6}$ which are pure D-wave components are small. When equation (39) is derived from equation (38) one transformation is applied,

$$g_1 = f_1-\frac{2}{5}f_7, \quad g_2 = f_2-\frac{2}{5}f_8, \quad g_i = f_i, (i=3,4,5,6,7,8). \quad (45)$$

The constraints for the state 2^{++} are

$$g_4 = g_5 = 0, \quad g_3 = \frac{|\vec{q}|}{m} g_1 - \frac{3|\vec{q}|}{5m} g_7, \quad g_6 = \frac{|\vec{q}|}{m} g_2 + \frac{2|\vec{q}|}{5m} g_8. \tag{46}$$

Then the final form of the Salpeter wave function (39) reads

$$\begin{aligned} \phi &= A(g_1 + \gamma^0 g_2) + B(\frac{|\vec{q}|}{m} g_1 - \frac{3|\vec{q}|}{5m} g_7) + D(\frac{|\vec{q}|}{m} g_2 + \frac{2|\vec{q}|}{5m} g_8)\gamma^0 \\ &\quad + E(g_7 + \gamma^0 g_8). \end{aligned} \tag{47}$$

From equations (7), (8) and (47), the coupled radial equations are obtained

$$\begin{aligned} Mg_1 &= -(\frac{6\omega^2}{5m} + \frac{4}{5}m)g_2 - \frac{12}{25}\frac{|\vec{q}|^2}{m} f_8 \\ &\quad - \frac{\alpha}{\pi}\int \frac{|\vec{k}|}{|\vec{q}|} d|\vec{k}| \left[-(\frac{m}{\omega} Q_1 + \frac{3}{5}\frac{|\vec{q}||\vec{k}|}{m\omega})g_2 - \frac{6|\vec{q}||\vec{k}|}{25m\omega} Q_2 g_8 \right], \end{aligned} \tag{48}$$

$$\begin{aligned} Mg_2 &= -(\frac{6}{5}m + \frac{4}{5}\frac{\omega^2}{m})g_1 + \frac{12}{25}\frac{|\vec{q}|^2}{m} g_7 \\ &\quad - \frac{\alpha}{\pi}\int \frac{|\vec{k}|}{|\vec{q}|} d|\vec{k}| \left[-(\frac{m}{\omega} Q_1 + \frac{2}{5}\frac{|\vec{q}||\vec{k}|}{m\omega} Q_2)g_1 + \frac{6}{25}\frac{|\vec{q}||\vec{k}|}{m\omega} Q_2 g_7 \right], \end{aligned} \tag{49}$$

$$\begin{aligned} Mg_7 &= -2\frac{|\vec{q}|^2}{m} g_2 - (\frac{4}{5}\frac{\omega^2}{m} + \frac{6}{5}m)g_8 \\ &\quad - \frac{\alpha}{\pi}\int \frac{|\vec{k}|}{|\vec{q}|} d|\vec{k}| \left[-\frac{|\vec{q}||\vec{k}|}{m\omega} Q_2 g_2 - (\frac{m}{\omega} Q_3 + \frac{2}{5}\frac{|\vec{q}||\vec{k}|}{m\omega} Q_2)g_8 \right], \end{aligned} \tag{50}$$

$$\begin{aligned} Mg_8 &= 2\frac{|\vec{q}|^2}{m} g_1 - (\frac{4}{5}m + \frac{6}{5}\frac{\omega^2}{m})g_7 \\ &\quad - \frac{\alpha}{\pi}\int \frac{|\vec{k}|}{|\vec{q}|} d|\vec{k}| \left[-\frac{|\vec{q}||\vec{k}|}{m\omega} Q_2 g_1 - (\frac{m}{\omega} Q_3 + \frac{3}{5}\frac{|\vec{q}||\vec{k}|}{m\omega} Q_2)g_7 \right]. \end{aligned} \tag{51}$$

Just as expected, the P-wave contributions and F-wave contributions to the 2^{++} state are separated explicitly in the Salpeter wave function (47) and in the coupled equations (48), (49), (50), (51). The normalization condition for the Salpeter wave function (47) reads

$$\begin{aligned} \int \frac{|\vec{q}|^2 d|\vec{q}|}{(2\pi)^2} \frac{1}{75m\omega} \big[& 40\omega^2(5g_2 - 3g_8)g_7 - 2(5g_2 + 2g_7) \\ & \big(50m^2 g_2 + (35g_2 - 6g_8)|\vec{q}|^2\big)\big] = M. \end{aligned} \tag{52}$$

Expanding the scalar functions with six basis functions, solving the coupled

equations (48), (49), (50), (51) numerically gives eigenvalues

$$\begin{aligned} &E^{(1)}(R_{21}) = -1.700708, \quad E^{(2)}(R_{31}) = -0.755871, \\ &E^{(3)}(R_{43}) = -0.425178, \quad E^{(4)}(R_{41}) = -0.425177, \\ &E^{(5)}(R_{53}) = -0.272114, \quad E^{(6)}(R_{51}) = -0.272114, \\ &\cdots \end{aligned} \tag{53}$$

and eigenfunctions whose expanding coefficients are not listed because of their length.

Now we discuss the nonrelativistic limit of the Salpeter wave function (47). When the P-wave contribution is dominant,

$$g_1 \simeq -g_2 = R_{n(l=1)}(|\vec{q}|), \quad g_7 \simeq g_8 \simeq 0, \tag{54}$$

the Salpeter wave function is reduced to

$$\phi \;=\; (A - A\gamma^0 + B\frac{|\vec{q}|}{m} - D\frac{|\vec{q}|}{m}\gamma^0)g_1. \tag{55}$$

And when the F-wave contribution dominates,

$$g_1 \simeq g_2 \simeq 0, \quad g_7 \simeq -g_8 = R_{n(l=3)}(|\vec{q}|), \tag{56}$$

the wave function (47) will be reduced to

$$\phi \;=\; (B\frac{3|\vec{q}|}{5m} + D\frac{2|\vec{q}|}{5m}\gamma^0 - E + E\gamma^0)g_8. \tag{57}$$

As expected, for the state 2^{++} the P-wave contribution and the F-wave contribution can be separated completely in the nonrelativistic limit, just as for the state 1^{--}. The relativistic terms in order of v in the Salpeter wave function are also obtained. From equations (54)-(57), it is found that the F-wave components will be small when the P-wave components dominate in the state 2^{++}, and the P-wave components will be small when the F-wave components dominate. More details can be found in paper [10] in which the ${}^1S_1 - {}^1D_1$ state 1^{--} is investigated.

3.5. RESULTS

In this paper, for simplicity, only pure Coulomb interaction (4) is considered with the results of order $m_e\alpha^4$. The formalism applied in this paper can be applied to more complicated problems.

The eigenvalues of the pure Coulomb interacting positronium obtained in this paper and in [11] are compared, see Table 5. They are same to the order 10^{-5} when the first four basis functions are employed to expand the wave functions. Using the method in this paper and in [7, 8], the wave function contributions from different wave components are separated completely and explicitly.

It is exhibited in the previous four subsections that the advantage of choosing spherical coordinates is obvious, especially for the mixed state 2^{++}. The separation

Table 5.

The eigenvalues's comparison.

State	This paper	Ref. [11]
2^3P_0	-1.700721	-1.700729
2^1P_1	-1.700713	-1.700721
2^3P_1	-1.700717	-1.700725
2^3P_2	-1.700708	-1.700717

of different wave contributions in spherical coordinates provides the basis for the choosing of the basis functions.

4. CONCLUSION

In this paper, we present the solutions to the Salpeter equation for the P states of positronium, 0^{++}, 1^{+-}, 1^{++}, and 2^{++} in order to illustrate the new approach. The Salpeter wave functions are expressed in spherical coordinates and not in Cartesian coordinates. In spherical coordinates, the contributions from different wave components become explicit, especially for the states with different wave components mixted. For example for the 2^{++} state, the P-wave contributions and the F-wave contributions can be separated completely, and the corrections in the order $v = |\vec{q}|/m$ are obtained. It is expected that this approach can be applied to heavy quarkonia and the feature of the mixing of contributions from different wave components is universal.

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