ADOMIAN DECOMPOSITION METHOD AND NON-ANALYTICAL SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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A new Adomian decomposition method is proposed to approximately solve fractional differential equations. The iteration procedure is based on a fractional Taylor series. An example is illustrated to show the presented method's efficiency and convenience.

Key words: Fractional differential equations; Adomian decomposition method; Fractional Taylor series.

1. INTRODUCTION

In the last three decades, scientists and applied mathematicians have found *fractional differential equations* (FDEs) useful in various fields: rheology, quantitative biology, physiology, electrochemistry, scattering theory, diffusion, transport theory, probability, potential theory and elasticity.

Recently, Kolwankar and Gangal proposed a local fractional derivative which is a potential tool to investigate the local behavior of fractional differential equations [1–3]. Since then, many works for such non-differential functions have been started and finding accurate and efficient methods for solving such fractional equations has been an active research undertaking. Although there are some versions of the Adomian decomposition method for fractional differential equations [4–7], they are not suitable for local fractional equations since the fractional derivatives *i.e.*, the Caputo derivative, the Riemann-Liouville derivative are non-local operators and also Adomian decomposition method with the Caputo derivative only can deals with smooth initial boundary problems.

In this paper, using Jumarie-Kolwankar's fractional Taylor series, Adomian decomposition method is extended to solve non-smooth initial boundary problems [8].

2. SOME PROPERTIES OF FRACTIONAL CALCULUS

2.1. FRACTIONAL CALCULUS AND SOME PROPERTIES

Based on Cantor-like sets, Kolwankar and Gangal proposed the concept of a local fractional derivative

$$D_{x_0}^{\alpha} f(x) = \lim_{x \to x_0} \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{x_0}^{x} (x-\xi)^{n-\alpha-1} (f(\mathrm{x}) - f(x_0))\, \mathrm{d}\xi, \tag{1}$$

where the derivative on the right-hand side is the Riemann-Liouville fractional derivative. With this concept, the local behavior of a fractional Fokker-Planck equation was investigated [3]. Recently, in order to cancel the initial point effect, Jumarie proposed a modified Riemann-Liouville derivative [8]. Assume $f\colon R \to R, x \to f(x)$ denotes a continuous (but not necessarily differentiable) function. Jumarie's derivative as

$${}^{mRL}D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{x_0}^{x} (x-\xi)^{-\alpha} (f(\xi) - f(x_0))\, \mathrm{d}\xi, \tag{2}$$

where $x \in [0,1]$, x_0 is an initial point and $0 \le \alpha < 1$. Both Jumarie's derivative and Kolwankar's derivative deal with non-smooth or fractal initial boundary problems and coarse-graining disturbance. They have the same results for fractional differentiable functions, *i.e.*, fractional Hamilton system [12], fractional variational derivative [13], although they are defined from different physical backgrounds.

(a) Fractinal Leibniz product law

From the proposition Eq. (3), we can readily find that

$$D_x^{\alpha}(uv) = u^{(\alpha)} v + u v^{(\alpha)}.$$

Readers must note that u, v should be analytical of fractional order. The law only holds for fractional differentiable functions.

(b) Fractional Leibniz Formulation

Replace $g(x)$ with $D_x^{\alpha} f(x)$ in Eq. (4), we than obtain that

$$ {}_0I_x^{\alpha} D_x^{\alpha} f(x) = f(x) - f(0), \; 0 < \alpha \leq 1, $$

and

$$ D_x^{\alpha} {}_0I_x^{\alpha} f(x) = f(x), \; 0 < \alpha \leq 1. $$

As a result, generalized integration by parts can be used during the local fractional calculus operations

$$ {}_aI_b^{\alpha} u^{(\alpha)} v = (uv)|_a^b - {}_aI_b^{\alpha} uv^{(\alpha)}. \tag{5} $$

(c) Jumarie-Kolwankar's Taylor series

If $f(x)$ is a $k\alpha$-differentiable function and k is an arbitrary positive integer, Jumarie-Kolwankar's Taylor series [8] can be presented as

$$ f(x) = \sum_{i=0}^{\infty} \frac{x^{k\alpha}}{(k\alpha)!} f^{(k\alpha)}(0). \tag{6} $$

Here $f^{(k\alpha)}(x)$ is $k\alpha$-differentiable near the point $x = 0$. We remark that, the fractional Taylor series is important here as it is used to take the initial values of the FADM's iteration process. Some recent results for Jumarie's derivative can be found in Refs. [12–15].

2.2. ADOMIAN DECOMPOSITION METHOD WITH NON-DIFFERENTIABLE FUNCTIONS

Many methods of mathematical physics have been developed to solve differential equations, among which the Adomian decomposition method is an efficient approximation technique to solve nonlinear models with initial-boundary value problems [16]. In this section, Adomian decomposition method is extended to fractional case in sense of Kolwankar's derivative.

Consider a fractional nonlinear differential equation in the form

$$ L^{\alpha}(y) - N(y) = f, y = y(x), \tag{7} $$

where $L^{\alpha} = \frac{\mathrm{d}^{n\alpha}}{\mathrm{d}x^{n\alpha}} = \underbrace{D_x^{\alpha} ... D_x^{\alpha}}_{n}$ is the fractional derivative of $n\alpha$-order, then the corresponding $L^{-\alpha}$ operator can be written in the form

$$L^{-\alpha}(\cdot)=\frac{1}{\Gamma^{n}(1+\alpha)}\int_0^x\int_0^{t_n}\int_0^{t_{n-1}}\dots\int_0^{t_2}(\cdot)(\mathrm{d}t_1)^{\alpha}\dots(\mathrm{d}t_{n-1})^{\alpha}(\mathrm{d}t_n)^{\alpha}. \tag{8}$$

$\frac{1}{\Gamma(1+\alpha)}\int_0^x(\mathrm{d}t_n)^{\alpha}$ is the Riemann-Liouville integration.

The nonlinear term, $N(y)$, is expressed by an infinite series of the Adomian polynomials:

$$N(y)=\sum_{n=0}^{\infty}A_n, \tag{9}$$

$$A_n(y_0,y_1,\dots y_n)=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}[N[\sum_{k=0}^{\infty}\lambda^k y_k]]\bigg|_{\lambda=0},\ n\geq 0. \tag{10}$$

Using the Maclaurin series of fractional order [9], we have

$$y(x)=\sum_{k=0}^{n-1}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}y^{(k\alpha)}(0)+L^{-\alpha}N(y)+L^{-\alpha}f,\ 0<\alpha\leq 1, \tag{11}$$

where $N(y)=\sum_{n=0}^{\infty}A_n$, and $A_n(y_0,y_1,\dots y_n)=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}[N[\sum_{k=0}^{\infty}\lambda^k y_k]]\bigg|_{\lambda=0}$, $n\geq 0.$

The recent development of the method can be found in Ref. [17].

3. APPLICATION OF THE METHOD

The fractional nonlinear differential equation [18] was solved approximately when the fractional order is $\alpha=\frac{1}{2}$,

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t}+\frac{\mathrm{d}^{\frac{1}{2}}y(t)}{\mathrm{d}t^{\frac{1}{2}}}-2y^2=0,\ 0<t. \tag{12}$$

A more general case can be given as

$$\frac{\mathrm{d}^{(2\alpha)}y(t)}{\mathrm{d}t^{2\alpha}}+\frac{\mathrm{d}^{\alpha}y(t)}{\mathrm{d}t^{\alpha}}-2y^2=0,\ 0<t,0<\alpha<1. \tag{13}$$

with the initial values problem $y(0)=0$ and $\frac{\mathrm{d}^{\alpha} y(t)}{\mathrm{d}t^{\alpha}}|_{t=0}=1$.

Eq. (13) can return back as an ordinary equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}+\frac{\mathrm{d}y(t)}{\mathrm{d}t}-2y^2=0,\ 0<t, \tag{14}$$

with a smooth initial value problem $y(0)=0$ and $\frac{\mathrm{d}y(t)}{\mathrm{d}t}|_{t=0}=1$.

Using Jumarie-Kolwankar's Taylor series [8], we can determine the initial value or a trial function

$$y_0=y(0)+\frac{y^{(\alpha)}(0)}{\Gamma(1+\alpha)}t^{\alpha}=\frac{t^{\alpha}}{\Gamma(1+\alpha)}.$$

The generalized iteration procedures can be given as

$$y_{n+1}=-\frac{1}{\Gamma^2(1+\alpha)}\int_0^t\int_0^t y_n^{(\alpha)}(\mathrm{d}\tau)^{\alpha}(\mathrm{d}\tau)^{\alpha}+$$
$$+\frac{1}{\Gamma^2(1+\alpha)}\int_0^t\int_0^t A_n(\mathrm{d}\tau)^{\alpha}(\mathrm{d}\tau)^{\alpha},\ 0\le n,$$

where A_n is a Adomian polynomials

$$A_0=\phi(y_0)=2y_0{}^2,$$
$$A_1=y_1\phi'(y_0)=4y_0y_1,$$
$$A_2=y_2\phi'(y_0)+\frac{y_1{}^2}{2!}\phi''(y_0)=4y_0y_1y_2+2y_1{}^2,$$
....

In this example, the nonlinear team is $\phi(y)=2y^2$.

As a result, we can derive

$$y_1 = -\frac{1}{\Gamma^2(1+\alpha)}\int_0^t\int_0^t y_0^{(\alpha)}(\mathrm{d}\tau)^\alpha(\mathrm{d}\tau)^\alpha + \frac{1}{\Gamma^2(1+\alpha)}\int_0^t\int_0^t 2y_0^2(\mathrm{d}t)^\alpha(\mathrm{d}t)^\alpha$$
$$= -\frac{1}{\Gamma^2(1+\alpha)}\int_0^t\int_0^t 1(\mathrm{d}\tau)^\alpha(\mathrm{d}\tau)^\alpha + \frac{1}{\Gamma^2(1+\alpha)}\int_0^t\int_0^t \frac{4\tau^{2\alpha}}{\Gamma(1+2\alpha)}(\mathrm{d}\tau)^\alpha(\mathrm{d}\tau)^\alpha$$
$$= -\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{4t^{4\alpha}}{\Gamma(1+4\alpha)},$$

$$y_2 = -\frac{1}{\Gamma^2(1+\alpha)}\int_0^t\int_0^t y_1^{(\alpha)}(\mathrm{d}\tau)^\alpha(\mathrm{d}\tau)^\alpha + \frac{1}{\Gamma^2(1+\alpha)}\int_0^t\int_0^t 4y_1y_0(\mathrm{d}\tau)^\alpha(\mathrm{d}\tau)^\alpha$$
$$= \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{16t^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{80t^{7\alpha}}{\Gamma(1+7\alpha)}.$$

If the third-order approximation is sufficient, we obtain the approximate solution

$$y(t) \approx \sum_{n=0}^{2} y_n(t) = y_0(t) + y_1(t) + y_2(t)$$
$$= \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{4t^{4\alpha}}{\Gamma(1+4\alpha)} - \frac{16t^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{80t^{7\alpha}}{\Gamma(1+7\alpha)}.$$

We assume $\alpha = 0.5$ in the 3rd order approximation which can be shown through Figure. 1.

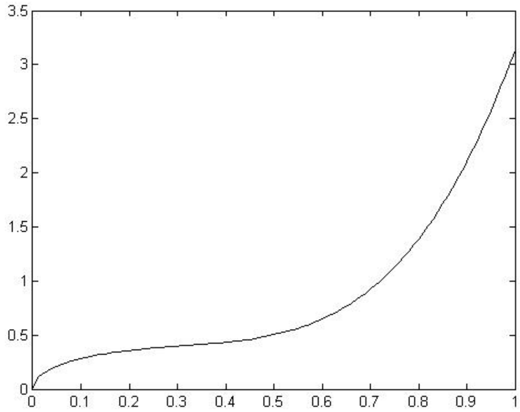

Fig.1 – Third-order approximation solution of half derivative.

Fig.1 shows 3rd-order approximate solution of Eq. (13) *vs* the approximate solution of the ordinary equation (14) solved by the Runge-Kutta method [19].

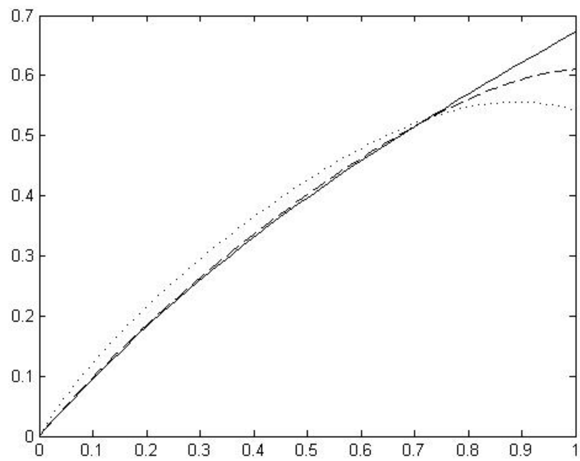

Fig. 2 – Curves of the 3rd order approximate solutions with different values of α **.**

The discontinuous line (--) is the approximate solution when $\alpha = 0.99$ and the dotted line (…) when $\alpha = 0.9$ .The continuous line is the approximate solution of Eq. (14). We can find that the approximate solution of fractional order tends to the approximate one of the ordinary differential equation (14) when $\alpha \to 1$.

4. CONCLUSION

The Adomian decomposition method for fractional differential equations has been extensively worked out for many years. However, all the previous versions are employed for nonlocal fractional derivatives. In this study, the approximate solution of a fractional differential equation is investigated by a new Adomian decomposition method. This approach can be used to solve other fractional differential equations defined on the Cantor-like set and describe the pointwise behaviors of fractional dynamical systems on a large scale.

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