APPROXIMATE SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH UNCERTAINTY

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Fractional differential equations have been caught much attention during the past decades. In this study, iteration formulae of a fractional differential equation with uncertainty are proposed and the approximate solutions for a simple case are derived via a fractional variational iteration method.

Key words: Modified Riemann-Liouville derivative; Fractional Variational Iteration Method; Fractional differential equations

1. INTRODUCTION

In the last decades, scientists and applied mathematicians have found fractional differential equations useful in various fields: rheology, quantitative biology, electrochemistry, scattering theory, diffusion, transport theory, probability potential theory and elasticity. For details, see the monographs of Kilbas et al. [1], Kiryakova [2], Lakshmikantham and Vatsala [3], Miller and Ross [4], and Podlubny [5].

Recently, Agarwal proposed the concept of solution for fractional differential equations with uncertainty [6]. One kind of differential equations with uncertainty of the type is governed by

$$\frac{\mathrm{d}^{\alpha} x(t)}{\mathrm{d}t^{\alpha}} = f(t, x(t)),\ 0 < t \le T, \tag{1}$$

where $f(t, x(t)) \times E \to E$ is continuous, and $x(0) = x_0 \in E$.

If $f(t, x(t)) \times R \to R$ and $x_0 \in R$, then Eq. (1) reduces to a fractional differential equation.

If $\alpha = 1$, then Eq. (1) is a fuzzy differential equation.
Thus, Eq. (1) is a new dynamic system called fuzzy differential equations of fractional order. Agarwal *et al.* introduced the concept of solution for such system.

In this study, we adopt the modified Riemann-Liouveille derivative [7] through this paper. The fractional derivative has been successfully used in fractional Lagrange mechanics [8], fractional variational approach [9, 10] and fractional Lie group method [11]. With a fractional variational iteration method [12, 13], approximate solutions of Eq. (1) are given.

2. PROPERTIES OF MODIFIED RIEMANN-LIOUVILLE DERIVATIVE

Comparing with the classical Caputo derivative, the definition of modified Reimann-Liouville derivative is not required to satisfy higher integer-order derivative than α. Secondly, α^{th} derivative of a constant is zero. Now we introduce some properties of the fractional derivative. Assume f: $R \to R$, $x \to f(x)$ denote a continuous (but not necessarily differentiable) function in the interval [0, 1]. Through the fractional famous Riemann Liouville integral

$$ {}_0I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)\mathrm{d}\xi, \quad \alpha > 0, \tag{2}$$

the modified Riemann-Liouville derivative is defined as [7]

$$ {}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0))\mathrm{d}\xi, \tag{3}$$

where $x \in [0,1]$ and $0 < \alpha < 1$.

In the next sections, we will use the integration with respect to $(\mathrm{d}x)^{\alpha}$ (**Lemma 2.1** of [14])

$$ {}_0I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)\mathrm{d}\xi = \frac{1}{\Gamma(\alpha+1)} \int_0^x f(\xi)(\mathrm{d}\xi)^{\alpha}, 0 < \alpha \le 1. \tag{4}$$

3. FRACTIONAL VARIATIONAL ITERATION METHOD

The variational iteration method for differential equations has been extensively worked out for many years by numerous authors. In this method, the equations are initially approximated with possible unknowns. A correction functional is established by the general Lagrange multiplier which can be identified optimally *via* the variational theory. Besides, the VIM has no restrictions or unrealistic assumptions such as linearization or small parameters that are used in the nonlinear operators. We consider the simple case of Eq. (1) to illustrate the fractional variational iteration method.

One of the linear fractional equations is

$$\frac{\mathrm{d}^{\alpha} x(t)}{\mathrm{d}t^{\alpha}} + ax(t) = d(t),\ 0<t \le T,\ 0 < \alpha \le 1,\ a \in R. \tag{5}$$

Then we can construct the following iteration form

$$x_{n+1}(t) = x_n(t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda(t,\tau) \{ \frac{\partial^{\alpha} x_n(\tau)}{\partial \tau^{\alpha}} + a x_n(\tau) - \mathrm{d}(\tau) \} (\mathrm{d}\tau)^{\alpha}, \tag{6}$$

where $\lambda(t,\tau)$ is unknown and to be determined.

With the fractional variational theory, we can find

$$\begin{aligned} \delta x_{n+1}(t) &= \delta x_n(t) + \frac{\delta}{\Gamma(1+\alpha)} \int_0^t \lambda(t,\tau) \{ \frac{\partial^{\alpha} x_n(\tau)}{\partial \tau^{\alpha}} + a x_n(\tau) - \mathrm{d}(\tau) \} (\mathrm{d}\tau)^{\alpha} \\ &= (1 + \lambda|_{t=\tau}) \delta x_n(t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t (\lambda_{\tau}^{(\alpha)} + a\lambda) \delta x_n(t) (\mathrm{d}\tau)^{\alpha}. \end{aligned} \tag{7}$$

$\lambda(t,\tau)$ must satisfy

$$1 + \lambda|_{\tau=t} = 0 \ \text{ and } \ \lambda_{\tau}^{(\alpha)} - a\lambda = 0. \tag{8}$$

As a result, $\lambda(t,\tau)$ can be identified explicitly

$$\lambda(t,\tau) = -E_{\alpha}(a(\tau - t)^{\alpha}), \tag{9}$$

where $E_{\alpha}(a(t-\tau)^{\alpha})$ is defined by the classical Mittag-Leffler function

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}. \tag{10}$$

Therefore, we can obtain the following iteration formulae for Eq. (5),

$$x_{n+1}(t) = x_n(t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t E_{\alpha}(a(\tau - t)^{\alpha}) \{ \frac{\partial^{\alpha} x_n(\tau)}{\partial \tau^{\alpha}} + a x_n(\tau) - \mathrm{d}(\tau) \} (\mathrm{d}\tau)^{\alpha}.$$

On the other hand, if $x_n(\tau)$ is handled as a restricted variation in Eq. (9), similarly, the Lagrange multiplier can be identified by

$$1 + \lambda|_{\tau=t} = 0 \ \text{ and } \ \lambda_{\tau}^{(\alpha)} = 0.$$

As a result, we can derive the generalized multiplier

$$\lambda(t,\tau) = -1 .$$

If we assume $\mathrm{d}(t) = 0$ for simplicity, we can have the iteration form

$$x_{n+1}(t) = x_n(t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \{\frac{\partial^\alpha x_n(\tau)}{\partial \tau^\alpha} - a x_n(\tau)\} (\mathrm{d}\tau)^\alpha . \tag{11}$$

Start from $x_0(t) = x(0)$, we can obtain

$$x_1(t) = x_0 - \frac{1}{\Gamma(1+\alpha)} \int_0^t \{\frac{\partial^\alpha x_0}{\partial \tau^\alpha} - a x_0\} (\mathrm{d}\tau)^\alpha$$

$$= x_0 - \frac{a x_0 t^\alpha}{\Gamma(1+\alpha)},$$

$$x_2(t) = x_1(t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \{\frac{\partial^\alpha x_1(\tau)}{\partial \tau^\alpha} - a x_1(\tau)\} (\mathrm{d}\tau)^\alpha$$

$$= x_0 - \frac{a x_0 t^\alpha}{\Gamma(1+\alpha)} + \frac{a^2 x_0 t^{2\alpha}}{\Gamma(1+2\alpha)},$$

$$x_3(t) = x_2(t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \{\frac{\partial^\alpha x_2(\tau)}{\partial \tau^\alpha} - a x_2(\tau)\} (\mathrm{d}\tau)^\alpha$$

$$= x_0 - \frac{a x_0 t^\alpha}{\Gamma(1+\alpha)} + \frac{a^2 x_0 t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{a^3 x_0 t^{3\alpha}}{\Gamma(1+3\alpha)},$$

The iteration process leads to the result

$$x(t) = \sum_{n=0}^{\infty} x_n(t) = x_0 \sum_{k=0}^{\infty} \frac{(-a)^k t^{k\alpha}}{\Gamma(1+k\alpha)} = x_0 E_\alpha(-a t^\alpha). \tag{12}$$

We can check Eq. (12) is the exact solution of the following equation

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} + a x(t) = 0,\ 0 < t \le T,\ 0 < \alpha \le 1,\ a \in R.$$

4. CONCLUSIONS

In this study, approximate solution of a fractional differential equation is investigated by fractional variational iteration method. Compared with the previous works via variational iteration method, this paper establishes a fractional functional and derives a generalized Lagrange multiplier for the given fractional differential equation. This approach can be used to solve other fractional nonlinear differential equations.

REFERENCES

1. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.
2. V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman Scientific & Technical, Harlow, 1994, copublished in the United States with John Wiley & Sons, Inc., New York.
3. V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations,Nonlinear Anal., **69**, 2677–2682 (2008).
4. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and DifferentialEquations*, John Wiley, New York, 1993.
5. I. Podlubny, *Fractional Differential Equation*, Academic Press, San Diego, 1999.
6. Ravi P. Agarwal, V. Lakshmikanthama, Juan J. Nieto, On the concept of solution for fractional differential equations with uncertainty, Nonlinear Anal., **72**, 2859–2862 (2010).
7. G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results, Comput. Math. Appl., **51**, 1367–1376 (2006).
8. G. Jumarie, Lagrangian mechanics of fractional order, Hamilton–Jacobi fractional PDE and Taylor's series of nondifferentiable functions, **32**, 969–987 (2007).
9. R. Almeida, A.B. Malinowska and D.F.M. Torres, A Fractional Calculus of Variations for Multiple Integrals with Application to Vibrating String, J. Math. Phys. **51**, 033503 (2010).
A.B. Malinowska, M.R.S Ammi and D.F.M Torres, Composition functionals in fractional calculus of variations, Commun. Frac. Calc. **1**, 41–47 (2010)
10. G.C. Wu, Variational Approach for Fractional Partial Differential Equations, arXiv:1006.4999v1
11. G.C. Wu, Fractional Lie Group Method for Anonymous Diffusion Equations, Commun. Frac. Calc. **1**, 27–31 (2010).
12. G.C. Wu, E.W.M. Lee, Fractional Variational Iteration Method and Its Application, Phys. Lett. A, **374**, 2506–2509 (2010).
13. G.C. Wu, Fractional Variational Iteration Method for Fractional Nonlinear Differential Equations, Comup. Math. Appl., Accepted.
14. G. Jumarie, Laplace's transform of fractional order via the Mittag–Leffler function and modified Riemann–Liouville derivative, Appl. Math. Lett. **22**, 1659–1664 (2009).