GENERALIZED COHERENT STATES BASED ON SIEGEL-JACOBI DISK

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Received February 4, 2011

We analyze some properties of the generalized squeezed states in the context of the Jacobi group. Applications referring to the Heisenberg uncertainty relations and Mandel parameter are included.

Key words: Coherent states, squeezed states, Jacobi group, Heisenberg uncertainty relations, Mandel parameter.

PACS: 03.65.-w, 02.20.-a, 02.20.Sv

1. INTRODUCTION

The semidirect product of the Heisenberg group and the symplectic group -*the Jacobi group* [1–3]- is an important object in quantum mechanics, geometric quantization, optics [4–7]. The Jacobi group was investigated by physicists under other names as *Hagen* [8], *Schrödinger* [9], or *Weyl-symplectic* group [10]. The Jacobi group governs the so called *squeezed states* [11–16] in quantum optics [6], [17], [18], [7]. Some coherent state systems based on Siegel-Jacobi domains [19] have been investigated in the framework of quantum mechanics, geometric quantization, dequantization, quantum optics, nuclear structure, and signal processing [20–26].

In [23] we have attached Perelomov's coherent states [27] to the Jacobi group $G^J_{1*} = \mathrm{H}_1(\mathbb{R}) \ltimes \mathrm{SU}(1,1)$, based on the Siegel-Jacobi disk $\mathfrak{D}^J_1 = \mathbb{C} \times \mathfrak{D}_1$, where $\mathfrak{D}_1 = \{z \in \mathbb{C} | |z| < 1\}$ is the Siegel ball, and $\mathrm{H}_1(\mathbb{R})$ denotes the real three-dimensional Heisenberg group. The present paper addresses firstly to physicists, so, the notation used in [23] will be adopted. [23] corresponds to the particular value $8\pi m = 1$ in [25], where m is the index of the non-trivial central character of the representation [2], [28].

In [25], [26] we have introduced *generalized squeezed states* attached to the Jacobi group G^J_{1*}. In Perelomov's construction [27], the cyclic vector e_0 of the representation π is an extremal weight vector, while in [25], [26], e_0 can be taken not necessarily an extremal weight vector of $(G, \pi, \mathcal{H})$. π is an unitary irreducible representation of the Lie group G on the complex separable Hilbert space $\mathcal{H}$ space with scalar product $< \cdot, \cdot >$. $e_z \in \mathcal{H}$ denotes the Perelomov coherent state vector indexed by the points z of the homogeneous manifold $M = G/H$, where H is compact subgroup of G. The generalized squeezed states of the type introduced in [25], [26] are not new [12,29,30]. Here we develop some of the ideas advanced in [25], [26] in the context of the Jacobi group [23].

The paper is organized as follows. §2 collects the main notation concerning the Jacobi algebra $\mathfrak{g}^J_{1*}$, the definition of the Perelomov coherent states attached to the Jacobi group G^J_{1*}, based on the Siegel-Jacobi disk $\mathfrak{D}^J_1$, and some results extracted from [23]. A brief recall of the notion of squeezing in quantum optics in the context of Jacobi group can be find in §3. We introduce generalized squeezed states attached to the Jacobi group, calculating the mean values of polynomial operators in the generators of the Jacobi group. The applications to displaced squeezed number states concerning the moments of momentum p and position q, in particular to the Heisenberg uncertainty relations, and the Mandel parameter, are presented in §4.

2. COHERENT STATE VECTORS ATTACHED TO THE JACOBI GROUP

The Heisenberg group is the group with the 3-dimensional real Lie algebra

$$\mathfrak{h}_1 \equiv < \mathrm{i}s1 + xa^\dagger - \bar{x}a >_{s\in\mathbb{R}, x\in\mathbb{C}}, \tag{1}$$

where $a^\dagger$ (a) are the boson creation (respectively, annihilation) operators which verify the canonical commutation relation $[a, a^\dagger] = 1$.

Let us also consider the Lie algebra of the group $\mathrm{SU}(1,1)$,

$$\mathfrak{su}(1,1) = < 2\mathrm{i}\theta K_0 + yK_+ - \bar{y}K_- >_{\theta\in\mathbb{R}, y\in\mathbb{C}}, \tag{2}$$

whose generators verify the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \ [K_-, K_+] = 2K_0. \tag{3}$$

The Jacobi algebra is defined as the the semi-direct sum $\mathfrak{g}^J_{1*} := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1)$, where

$$[a, K_+] \;=\; a^\dagger, \left[K_-, a^\dagger\right] = a, \left[K_+, a^\dagger\right] = [K_-, a] = 0, \tag{4a}$$

$$\left[K_0, a^\dagger\right] \;=\; \frac{1}{2}a^\dagger, \ [K_0, a] = -\frac{1}{2}a. \tag{4b}$$

Let us suppose that we know the derived representation $d\pi$ of the Lie algebra $\mathfrak{g}^J_{1*}$ of the Jacobi group G^J_{1*}. If $X \in \mathfrak{g}$, we denote $\boldsymbol{X} = d\pi(X)$, and $(\boldsymbol{a}^\dagger)^\dagger = \boldsymbol{a}$, $\boldsymbol{K}_0^\dagger = \boldsymbol{K}_0, \boldsymbol{K}_\pm^\dagger = \boldsymbol{K}_\mp$. We impose to the cyclic vector e_0 to verify simultaneously the conditions

$$\boldsymbol{a}e_0 = 0, \ \boldsymbol{K}_-e_0 = 0, \ \boldsymbol{K}_0e_0 = ke_0; \ k > 0, 2k = 2,3,\dots. \tag{5}$$

We have considered in the last relation in (5) the positive discrete series representations D_k^+ of $\mathrm{SU}(1,1)$ [31].

Perelomov's coherent state vectors associated to the group G^J_{1*} with Lie algebra the Jacobi algebra $\mathfrak{g}^J_{1*}$, based on Siegel-Jacobi domain $\mathfrak{D}^J_1 = H_1/\mathbb{R}\times \mathrm{SU}(1,1)/\mathrm{U}(1)$, are defined as

$$e_{z,w} := \mathrm{e}^{z\boldsymbol{a}^\dagger + w\boldsymbol{K}_+}e_0,\ z,w\in\mathbb{C},\ |w|<1. \tag{6}$$

We introduce also the normalized (*squeezed*) CS vector [12]

$$\Psi_{\alpha,w} := T(\alpha,w)e_0,\ T(\alpha,w) = D(\alpha)S(w), \alpha,w\in\mathbb{C}, |z|<1, \tag{7}$$

where D is the *displacement operator*

$$D(\alpha) := \exp(\alpha\boldsymbol{a}^\dagger - \bar{\alpha}\boldsymbol{a}) = \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\boldsymbol{a}^\dagger)\exp(-\bar{\alpha}\boldsymbol{a}), \tag{8}$$

and S denotes *the unitary squeezed operator*, $\underline{S}(z) = S(w)$,

$$\underline{S}(z) := \exp(z\boldsymbol{K}_+ - \bar{z}\boldsymbol{K}_-) = \exp(w\boldsymbol{K}_+)\exp(\eta\boldsymbol{K}_0)\exp(-\bar{w}\boldsymbol{K}_-), \tag{9a}$$

$$w = \frac{z}{|z|}\tanh(|z|),\ \eta = \log(1-w\bar{w}). \tag{9b}$$

We introduce the auxiliary operators [23]:

$$\boldsymbol{K}_+ = \frac{1}{2}(\boldsymbol{a}^\dagger)^2 + \boldsymbol{K}'_+,\ \boldsymbol{K}_- = \frac{1}{2}(\boldsymbol{a}^\dagger)^2 + \boldsymbol{K}'_-,\ \boldsymbol{K}_0 = \frac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{1}{2}) + \boldsymbol{K}'_0, \tag{10}$$

which have the properties

$$\boldsymbol{K}'_- e_0 = 0, \boldsymbol{K}'_0 e_0 = k' e_0;\ k = k' + \frac{1}{4}; \tag{11}$$

$$[\boldsymbol{K}'_\sigma, \boldsymbol{a}] = [\boldsymbol{K}'_\sigma, \boldsymbol{a}^\dagger] = 0,\ \sigma = \pm, 0, [\boldsymbol{K}'_0, \boldsymbol{K}'_\pm] = \pm\boldsymbol{K}'_\pm; [\boldsymbol{K}'_-, \boldsymbol{K}'_+] = 2\boldsymbol{K}'_0. \tag{12}$$

The meaning of the splitting (10) is given by the so called *fundamental principle* in the representation theory of the Jacobi group (cf. Theorem 2.6.1 in [2], see also §2.1 in [25]), while the physical consequences of this splitting are briefly discussed in §3.

We recall the *orthonormal system of coherent states* associated to the group $\mathrm{SU}(1,1)$:

$$e_{k,k+m} := a_{km}(\boldsymbol{K}_+)^m e_{k,k};\ a^2_{km} = \frac{\Gamma(2k)}{m!\Gamma(m+2k)}, \tag{13}$$

and *the number vectors* associated to the Heisenberg group

$$\varphi_n = (n!)^{-\frac{1}{2}}(a^+)^n\varphi_0;\quad <\varphi_{n'}, \varphi_n> = \delta_{nn'}, \boldsymbol{a}\varphi_0 = 0. \tag{14}$$

We write down the vector e_0 in (5) as

$$e_0 = e^H_0 \otimes e^{K'}_0,\ \text{where } e^H_0 \equiv \varphi_0;\ e^{K'}_0 \equiv e_{k',k'}, k = k' + \frac{1}{4}. \tag{15}$$

We extract from [23], the following results:

Proposition 1 *Perelomov's coherent state vector* (6) *can be written down as*

$$e_{z,w} = E(z,w)e_0^H \otimes \mathrm{e}^{w\boldsymbol{K}'_+}e_0^{K'}, \; E(z,w) = \mathrm{e}^{z\mathbf{a}^\dagger + \frac{\mathbf{w}}{\mathbf{2}}(\mathbf{a}^\dagger)^\mathbf{2}}, \tag{16}$$

$$E(z,w)\mathrm{e}_0^H = \sum \frac{1}{(n!)^{1/2}}(-\frac{w}{2})^{n/2}H(\frac{iz}{\sqrt{2w}})\varphi_n, \tag{17}$$

$$\mathrm{e}^{w\boldsymbol{K}'_+}e_0^{K'} = \sum \frac{w^m e_{k',k'+m}}{m! a_{k'm}}, \tag{18}$$

where H_n are the Hermite polynomials.

The base of functions associated to the CS-vectors attached to the Jacobi group G^J_{1*}, defined on the manifold $\mathfrak{D}^J_1$, consists of the functions

$$f_{\varphi_n e_{k',k'+m}}(z,w) := (e_{\bar{z},\bar{w}}, \varphi_n \otimes e_{k',k'+m}) = f_{e_{k',k'+m}}(w)\frac{P_n(z,w)}{\sqrt{n!}}, \; z \in \mathbb{C}, \tag{19}$$

$$P_n(z,w) = n!\sum_{k=0}^{[\frac{n}{2}]}(\frac{w}{2})^k\frac{z^{n-2k}}{k!(n-2k)!}, \tag{20}$$

$$f_{e_{k,k+n}}(w) = \sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}}w^n, \; |w| < 1. \tag{21}$$

The reproducing kernel $K(z,w;\bar{z}',\bar{w}') := (e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'})$

$$K(z,w;\bar{z}',\bar{w}') = (1-w\bar{w}')^{-2k}F(z,w;\bar{z}',\bar{w}'); \; F = \exp\frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2 w}{2(1-w\bar{w}')}, \tag{22}$$

admits the series expansion in the base of functions (19)

$$K(z,w;\bar{z}',\bar{w}') = \sum_{n,m} f_{\varphi_n,e_{k',k'+m}}(z,w)\bar{f}_{\varphi_n,e_{k',k'+m}}(z',w'). \tag{23}$$

The normalized squeezed state vector (7) *and the un-normalized (Perelomov's coherent state) vector* (6) *are related by the relation*

$$\Psi_{\alpha,w} = (1-w\bar{w})^k\exp(-\frac{\bar{\alpha}}{2}z)e_{z,w}, \; z = \alpha - w\bar{\alpha}. \tag{24}$$

Now we make a Remark, see also [31], [32]

Remark 1 *For $k \in \mathbb{R}$, $k > 1/2$, we use the Hilbert space $\mathcal{H}_{2,k}(\mathfrak{D}_1)$ of square integrable holomorphic functions on $\mathfrak{D}_1$ with respect to the inner product:*

$$(f,g)_k = \frac{2k-1}{\pi}\int_{|z|<1}\bar{f}(z)g(z)(1-|z|^2)^{2k-2}d\mathrm{Re}\,z d\mathrm{Im}\,z. \tag{25}$$

The functions (21) *form an orthonormal base in $\mathcal{H}_{2,k}(\mathfrak{D}_1)$ and if $f = \sum a_n z^n$, $g = \sum b_n z^n$, $|z| < 1$, then*

$$(f,g)_k = \sum \frac{\Gamma(2k)\Gamma(n+1)}{\Gamma(2k+n)}\bar{a}_n b_n. \tag{26}$$

For the universal covering group of $\widetilde{\mathrm{SU}(1,1)}$*, we use a class of Hilbert spaces indexed the real parameter* k*,* $0<|k|<1/2$*. In* [23] *we have defined a scalar product with holomorphic functions on the Siegel-Jacobi disk* $\mathfrak{D}^J_1$ *for* $k>3/4$*, and the functions* (19) *form an orthonormal basis. The condition* $k=k'+1/4$ *imposes the restriction* $k'>1/2$ *so, in fact, we should work on the covering of the group* $\mathrm{SU}(1,1)$*, and the scalar product corresponding to* (26) *must be considered instead of* (25)*.*

3. SQUEEZING IN QUANTUM OPTICS AND GENERALIZED SCHRÖDINGER CATS

The Lie algebra $\mathfrak{so}(1,2)\equiv\mathfrak{su}(1,1)$ of the group $\mathrm{SU}(1,1)$ entered in quantum optics in connection with squeezing [12–16]. The group theoretical background of squeezed states was realized latter [33]. A realization of the Lie algebra $\mathfrak{so}(1,2)$ in terms of annihilation and creation operators was given in [9,36,37].

In quantum optics on considers in the splitting (10) just the realization

$$\boldsymbol{K}_+=\frac{1}{2}(\boldsymbol{a}^\dagger)^2,\ \boldsymbol{K}_-=\frac{1}{2}\boldsymbol{a}^2,\ \boldsymbol{K}_0=\frac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}+\frac{1}{2}), \tag{27}$$

and the cyclic vector e_0 (15) is taken just $e_0=e^H_0=\varphi_0$. The Jacobi group in such a case admits the so called *Schrödinger-Weil representation* π^m_{SW} with character $\psi^m(x)=\mathrm{e}^{2\pi m x}$, $m\in\mathbb{R}$, considered in [2], where x is in the center of the Heisenberg-Weyl algebra. In [23] we have considered $8\pi m=1$.

Using the relation $[a,(a^\dagger)^n]=n(a^\dagger)^{n-1}$, we obtain $\boldsymbol{K}_0\varphi_{2p}=(p+\frac{1}{4})\varphi_{2p}$, $\boldsymbol{K}_0\varphi_{2p+1}=(p+\frac{3}{4})\varphi_{2p+1}$, *i.e.* we get the irreducible representations with $k=\frac{1}{4}$, $k=\frac{3}{4}$ of the covering group of $\mathrm{SU}(1,1)$. The representation matrix elements for $\mathrm{SU}(1,1)$, expressed in terms of hypergeometric functions, are calculated by Bargmann [31], and the case $k=1/4,3/4$ in [34], see also [35], [32].

Now we act with the unitary squeezing operator on the state φ_0

$$\underline{S}(z)\varphi_0=\mathrm{e}^{z\boldsymbol{K}_+-\bar{z}\boldsymbol{K}_-}\varphi_0=(1-|w|^2)^{\frac{1}{4}}\tilde{e}_w,\tilde{e}_w=\exp(w\boldsymbol{K}^+)\varphi_0,\boldsymbol{K}_+=\frac{1}{2}(\boldsymbol{a}^\dagger)^2,$$

and we have the well known formula [7]

$$\underline{S}(z)\varphi_0=(\cosh r)^{-1/2}\sum\frac{\sqrt{(2n)!}}{n!}(\frac{w}{2})^n\varphi_{2n},\ \text{where } z=r\mathrm{e}^{\mathrm{i}\phi};w=\frac{z}{|z|}\tanh r,$$

i.e. the squeezed vacuum state is a superposition of only even Fock states [38].

If we take into account the relation $S(w)\varphi_0=(1-w\bar{w})^{1/4}\tilde{e}_w$, where $\boldsymbol{K}_+$ is given by the first equation (10), then

$$K(z,\bar{z}'):=(\tilde{e}_{\bar{z}},\tilde{e}_{\bar{z}'})=\sum f_{e_{k,k+m}}(z)\bar{f}_{e_{k,k+m}}(z')=(1-z\bar{z}')^{-2k},\text{for } k=1/4. \tag{28}$$

Now we use the definition (7) with $e_0=\varphi_0$ instead of (15), and (24) reads

$$\tilde{\Psi}_{\alpha,w}=(1-w\bar{w})^{1/4}\exp(-\frac{\bar{\alpha}}{2}z)\tilde{e}_{z,w}, \quad (29)$$

$$\tilde{e}_{z,w}=\sum(n!)^{-\frac{1}{2}}(-\frac{w}{2})^{\frac{n}{2}}H_n(\frac{\mathrm{i}z}{\sqrt{2w}})\varphi_n,$$

$$f_{\varphi_n}(z,w)=(\tilde{e}_{\bar{z},\bar{w}},\varphi_n)=(n!)^{-\frac{1}{2}}P_n(z,w). \quad (30)$$

The scalar product is given by (22) with $k=1/4$ and the summation formula (23) is valid for the functions (30).

Let us consider the generalized squeezed states realized by

$$\Phi_{\alpha w}=T(\alpha,w)\varphi,\ T(\alpha,w)=D(\alpha)S(w), \quad (31)$$

The displaced squeezed number states are realized by (31) if φ is given by (14) [30], while for $\varphi=\varphi_0$, the vacuum state, we get *the squeezed states* (29) [11, 12, 14]. *The displaced number states* are realized by $\varphi=\varphi_n,\ w=0$ [29]. *The oscillator coherent states* are squeezed states with $\varphi=\varphi_0,\ w=0$ [39].

The *generalized Schrödinger cat states* corresponding to (31) are realized by

$$\Phi_{\alpha w\pm}=D_\pm(\alpha)S(w)\varphi,\ D_\pm(\alpha)=\left[2(1\pm\exp(-2|\alpha|^2))\right]^{-1/2}\left[D(\alpha)\pm D(-\alpha)\right].$$

It is convenient to introduce the notation

$$\hat{A}(\alpha,w)=T(\alpha,w)^\dagger AT(\alpha,w),$$

$$\tilde{A}(\alpha,w)=<\Phi_{\alpha w}|A|\Phi_{\alpha w}>,$$

$$\tilde{A}_\pm(\alpha,w)=<\Phi_{\alpha w\pm}|A|\Phi_{\alpha w\pm}>$$

for any polynomial operator A in infinitesimal generators of Jacobi algebra $\mathfrak{g}^J_{1*}$. Because of the relations $D(\alpha)^\dagger=D(-\alpha),\quad S(w)^\dagger=S(-w)$, we obtain

$$\tilde{A}(\alpha,w)=<\varphi|\hat{A}(\alpha,w)|\varphi>,\quad \tilde{A}_\pm(\alpha,w)=<\varphi|\hat{A}_\pm(\alpha,w)|\varphi>. \quad (32)$$

Using the canonical commutation relations, we obtain $D(-\alpha)\boldsymbol{a}D(\alpha)=\boldsymbol{a}+\alpha$, while the Holstein-Primakoff-Bogoliubov equations (eq. (6.10) in [23]) can be written as

$$S(-w)\boldsymbol{a}S(w)=r(\boldsymbol{a}+w\boldsymbol{a}^\dagger),\ r=(1-w\bar{w})^{-1/2}.$$

We have $\hat{\boldsymbol{a}}=r(\boldsymbol{a}+w\boldsymbol{a}^\dagger)+\alpha$. Also, it is easy to obtain the relations:

$$D(-\alpha)\boldsymbol{K}_0D(\alpha)=\boldsymbol{K}_0+\mathrm{Re}(\alpha\boldsymbol{a}^\dagger)+\frac{|\alpha|^2}{2},$$

$$D(-\alpha)\boldsymbol{K}_-D(\alpha)=\boldsymbol{K}_-+\alpha\boldsymbol{a}+\frac{\alpha^2}{2}.$$

By the commutation relations (4b), we get (see also (27) in [40]

$$S(-w)\boldsymbol{K}_0S(w) = r^2[(1+|w|^2)\boldsymbol{K}_0 + w\boldsymbol{K}_+ + \bar{w}\boldsymbol{K}_-], \tag{33a}$$

$$S(-w)\boldsymbol{K}_-S(w) = r^2(\boldsymbol{K}_- + 2w\boldsymbol{K}_0 + w^2\boldsymbol{K}_+), \tag{33b}$$

and finally, we have:

$$\hat{\boldsymbol{K}}_0(\alpha,w) = r^2[\bar{w}\boldsymbol{K}_- + \left(1+|w|^2\right)\boldsymbol{K}_0 + w\boldsymbol{K}_+] + \\ + r\,\mathrm{Re}[\alpha(\boldsymbol{a}^\dagger + \bar{w}\boldsymbol{a})] + \frac{1}{2}|\alpha|^2, \tag{34}$$

$$\hat{\boldsymbol{K}}_-(\alpha,w) = r^2[\boldsymbol{K}_- + 2w\boldsymbol{K}_0 + w^2\boldsymbol{K}_+] + \alpha r(\boldsymbol{a} + w\boldsymbol{a}^\dagger) + \frac{1}{2}\alpha^2. \tag{35}$$

If A is a polynomial P in the generators of the Jacobi algebra, $A = P(\boldsymbol{a},\boldsymbol{a}^\dagger,\boldsymbol{K}_-,\boldsymbol{K}_0,\boldsymbol{K}_+)$, then $\hat{A} = P(\hat{\boldsymbol{a}},\hat{\boldsymbol{a}}^\dagger,\hat{\boldsymbol{K}}_-,\hat{\boldsymbol{K}}_0,\hat{\boldsymbol{K}}_+)$. Using (32), (34), (35), $\tilde{A}$ and $\tilde{A}_\pm$ can be explicitly obtained as functions of α and w.

4. APPLICATIONS TO THE DISPLACED SQUEEZED NUMBER STATES

4.1. MOMENTS

With the calculation in the previous section, we get easily

$$\overline{\boldsymbol{a}^m} = <\varphi|[r(\boldsymbol{a} + w\boldsymbol{a}^\dagger) + \alpha]^m|\varphi> . \tag{36}$$

Also, with (34), (35), we have

$$\widetilde{\boldsymbol{K}^m_\varepsilon}(\alpha,w) = <\varphi|[\hat{\boldsymbol{K}}_\varepsilon(\alpha,w)]^m|\varphi>, \quad \varepsilon = +,0,- \quad , \tag{37}$$

$$\widetilde{\boldsymbol{K}_-}(\alpha,w) = r^2[\boldsymbol{k}_- + 2w\boldsymbol{k}_0 + w^2\boldsymbol{k}_+] + \alpha r(c + w\bar{c}) + \frac{1}{2}\alpha^2, \tag{38}$$

$$\widetilde{\boldsymbol{K}_0}(\alpha,w) = r^2[\bar{w}\boldsymbol{k}_- + (1+|w|^2)\boldsymbol{k}_0 + w\boldsymbol{k}_+] \\ + \frac{1}{2}r\,\mathrm{Re}[\alpha(\bar{c} + \bar{w}c)] + \frac{1}{2}|\alpha|^2,\ c = <\varphi|\boldsymbol{a}|\varphi>, \quad \boldsymbol{k}_\varepsilon = <\varphi|\boldsymbol{K}_\varepsilon|\varphi> . \tag{39}$$

Until now, the formulas in this section are general, *i.e.* do not depend on φ. We mentioned already that similar investigations where performed also earlier. The displaced squeezed number states have been studied in [41, 42], the displaced number states in [43, 44], the squeezed number states in [45].

Now we consider the displaced squeezed number states, *i.e.* we take $\varphi = \varphi_n$

given by (14) and (27) is satisfied. Then $\tilde{\boldsymbol{a}} = \alpha$, $\widetilde{\boldsymbol{a}^\dagger} = \bar{\alpha}$, and also

$$\begin{aligned} \widetilde{\boldsymbol{K}_0}(\alpha,w) &= \frac{1}{2}(n+\frac{1}{2})\frac{1+w\bar{w}}{1-w\bar{w}}+\frac{\alpha\bar{\alpha}}{2}, && (40a)\\ \widetilde{\boldsymbol{K}_+}(\alpha,w) &= \frac{1}{2}(n+\frac{1}{2})\frac{2\bar{w}}{1-w\bar{w}}+\frac{\bar{\alpha}^2}{2}, && (40b)\\ \widetilde{\boldsymbol{K}_-}(\alpha,w) &= \frac{1}{2}(n+\frac{1}{2})\frac{2w}{1-w\bar{w}}+\frac{\alpha^2}{2}. && (40c) \end{aligned}$$

Consider the quantum Hamiltonian H given by

$$H = H_0 + V(q),\ H_0 = \frac{1}{2m}\,p^2+\frac{1}{2}\,m\omega^2q^2 \tag{41}$$

with the momentum and coordinate operators given by $p = \frac{\hbar}{2d}\,i(a^\dagger - a)$; $q = d(a + a^\dagger)$, where $d = \sqrt{\hbar/2m\omega}$. V is an analytic function of q. Denote $\mu = \mathrm{Re}\,\alpha$ and $\nu = \mathrm{Im}\,\alpha$. Is convenient to introduce the real variables

$$u_+ = \frac{(1+w)(1+\bar{w})}{1-w\bar{w}}\ ,\ u_- = \frac{(1-w)(1-\bar{w})}{1-w\bar{w}}\ . \tag{42}$$

Using the relations

$$q^2 = 2d^2(\boldsymbol{K}_+ + \boldsymbol{K}_- + 2\boldsymbol{K}_0)\ ,\ p^2 = \frac{\hbar^2}{2d^2}(2\boldsymbol{K}_0 - \boldsymbol{K}_+ - \boldsymbol{K}_-), \tag{43}$$

we obtain for the moments of q and p the values

$$\begin{aligned} \tilde{q} &= 2d\mu\ ,\ \tilde{p} = \frac{\hbar}{d}\nu,\\ \widetilde{q^2} &= \frac{\hbar}{m\omega}[(n+\frac{1}{2})u_+ + 2\mu^2]\ ,\ \widetilde{p^2} = m\hbar\omega[(n+\frac{1}{2})u_- + 2\nu^2],\\ \widetilde{H_0} &= \frac{\hbar\omega}{2}(n+\frac{1}{2})(u_+ + u_-) + \mu^2 + \nu^2,\\ \frac{\widetilde{q^3}}{2\mu d^3} &= 4\mu^2 + 3(2n+1)u_+, \ldots,\ . \end{aligned}$$

The moments of p and q are polynomials of μ, ν, u_+, and u_-. In fact, *the moments of q are polynomials of μ and u_+, only.*

4.2. HEISENBERG UNCERTAINTY RELATIONS AN SQUEEZING

With the expressions obtained in the last section for mean values of p and q, we get

$$\Delta q = \sqrt{\frac{\hbar}{m\omega}(n+\frac{1}{2})u_+};\ \ \Delta p = \sqrt{\hbar m\omega(n+\frac{1}{2})u_-},$$
$$\Delta p\Delta q = (n+\frac{1}{2})\hbar\sqrt{u_+u_-}.$$

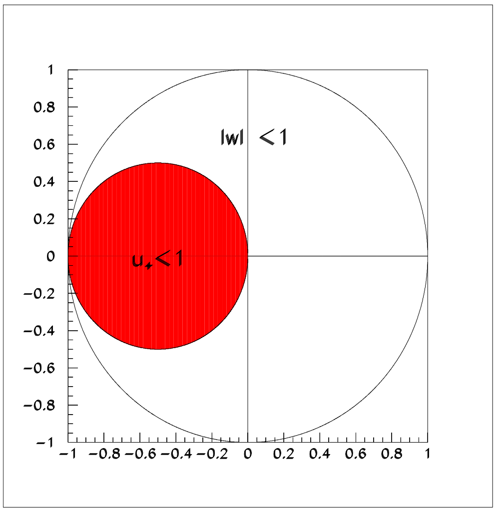

Fig. 1 – Squeezing region in the open disc $|w| < 1$.

Evidently, we have $\Delta p \Delta q \geq (n+\frac{1}{2})\hbar$.

We have **squeezing** in the region $u_+ < \frac{1}{2n+1}$, *i.e.* the interior of the circle $(x+\frac{2n+1}{2n+2})^2 + y^2 < \frac{1}{4(n+1)^2}$, $w = x + \mathrm{i}y$, $x, y \in \mathbb{R}$. For $n = 0$, we have squeezing in the region $u_+ < 1$. The last condition means $x^2 + y^2 + x < 0$, *i.e.* the interior of the circle $(x+\frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$.

4.3. MANDEL PARAMETER

The Mandel parameter

$$Q = \frac{<(\Delta n)^2> - <n>}{<n>}$$

is a convenient way to characterize the non-classical states [6]. Q is negative for sub-Poissonian states and -1 for Fock states. We obtain easily

$$\widetilde{\boldsymbol{K}}_0 = \frac{1}{2} <n> + \frac{1}{4};\ <n^2> = 4\widetilde{\boldsymbol{K}_0^2} - 2\widetilde{\boldsymbol{K}}_0 + \frac{1}{4},\ \widetilde{\boldsymbol{K}_0^2} = <\varphi|\hat{\boldsymbol{K}}_0\hat{\boldsymbol{K}}_0|\varphi>,\ \varphi = \varphi_n.$$

Writing down formula (34) as

$$\hat{\boldsymbol{K}}_0 = A\boldsymbol{K}_- + B\boldsymbol{K}_+ + C\boldsymbol{K}_0 + D\boldsymbol{a} + E\boldsymbol{a}^\dagger + F,$$

where

$$A = 2r\mu(1+\bar{w}), B = \bar{A}, C = 2r^2(1+w)(1+\bar{w}), D = 2r\mu(1+\bar{w}), E = \bar{D}, F = 2\mu^2,$$

we express the mean value of $\widetilde{\boldsymbol{K}_0^2}$ as

$$< \varphi_n|\widetilde{\boldsymbol{K}_0^2}|\varphi_n >=< \varphi_n|AB(\boldsymbol{K}_-\boldsymbol{K}_+ + \boldsymbol{K}_+\boldsymbol{K}_-) + C^2\boldsymbol{K}_0^2 + 2CF\boldsymbol{K}_0 + DE(\boldsymbol{a}\boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a}) + F^2|\varphi_n > .$$

With the relations

$$\boldsymbol{K}_+\varphi_n = \frac{\sqrt{(n+1)(n+2)}}{2}\varphi_{n+2};\ \boldsymbol{K}_-\varphi_n = \frac{\sqrt{n(n-1)}}{2}\varphi_{n-2},$$

we get successively

$$< n^2 > = 2(n^2+n+1)r^4|w|^2 + \frac{(2n+1)^2}{4}r^4(1+|w|^2)^2 + + (2n+1)r^2[(1+|w|^2)(|\alpha|^2-\frac{1}{2}) + |\alpha\bar{w}+\bar{\alpha}|^2] + (|\alpha|^2-\frac{1}{2})^2,$$

$$< n > = (n+\frac{1}{2})\frac{1+|w|^2}{1-|w|^2} + |\alpha|^2 - \frac{1}{2},$$

$$< (\Delta n)^2 > = 2(n^2+n+1)r^4|w|^2 + (2n+1)r^2|\alpha\bar{w}+\bar{\alpha}|^2,$$

$$\frac{< (\Delta n)^2 > - < n >}{r^2} = 2(n^2+n+1)\frac{|w|^2}{1-|w|^2} + (\frac{1}{2}-|\alpha|^2)(1-|w|^2) + (2n+1)[(1+|w|^2)(|\alpha|^2-\frac{1}{2}) + \bar{\alpha}^2 w + \alpha^2\bar{w}.$$

Finally, we get for the Mandel parameter Q the expression

$$\frac{Q}{2r^2} = \frac{2(n^2+n+1)r^2|w|^2 + (2|\alpha|^2-1)[(n+1)|w|^2+n] + 2(2n+1)\,\mathrm{Re}(\alpha^2\bar{w})}{(2n+1)(1+|w|^2) + (2|\alpha|^2-1)(1-|w|^2)}.$$

Now we briefly comment the above formula of the Mandel parameter Q as function of $\alpha,\ w$.

1) Firstly, if $w=0$, then we have $Q = \frac{n\left(2|\alpha|^2-1\right)}{n+|\alpha|^2}$ and $Q=0$ if and only if $n=0$ or $|\alpha| = 1/\sqrt{2}$. In figure 2 we represent $Q = f(|\alpha|^2)$ for $n = 1,2,3$.

2) If $\alpha = 0$, then the Mandel parameter is

$$Q = r^4\frac{(n+1)|w|^4 + (2n^2+2n+1)|w|^2 - n}{n+(n+1)|w|^2}$$

and we get $Q=0$ for the value

$$|w|^2 = \frac{1}{2(n+1)}\left(\sqrt{4n^4+8n^3+12n^2+8n+1} - 2n^2 - 2n - 1\right).$$

Also we have $Q=-1$ for $w=0$ and $n>0$. In figure 3 we represent $Q = f(|w|^2)$, $0 \le |w| < 1$, for $n = 0,1,2$. We see that if $w=0$ and $n=0$, then $Q=0$, *i.e.* a Poisson distribution.

The preceding formulas are compatible with some formulas in [7, 43, 46].

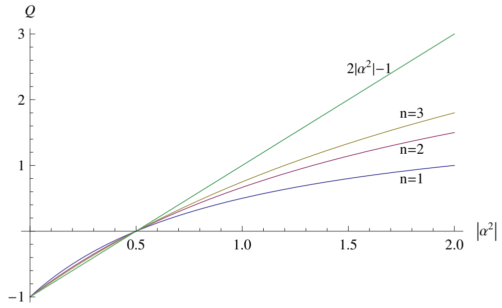

Fig. 2 – Mandel's function, $w = 0$.

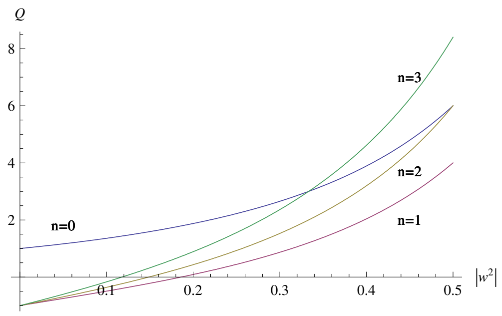

Fig. 3 – Mandel's function for α=0.

Acknowledgements. This investigation was partially supported by the CNCSIS-UEFISCSU project PNII- IDEI 454/2009, CNCSIS Cod ID-44.

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