SCALAR-GRAVITON SCATTERING IN NONCOMMUTATIVE SPACE AND DEFORMED NEWTON GRAVITY

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The isotropy of Newton potential in noncommutative space disappears as it deforms to a momentum dependent one. We generalize the earlier derivation of such a deformed potential to the relativistic regime by calculating the 2 scalar-1 graviton scattering amplitude by taking into account the noncommutativity of space.

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1. INTRODUCTION

There are several serious motivations, arising from the different arenas of the Planck scale physics [1-3], to consider a noncommutative (NC) algebra for the coordinates x^μ spanning the space-time manifold, *via*

$$[\widehat{x}^\mu, \widehat{x}^\nu] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ assumed to be a constant antisymmetric matrix. As a result of "θ-deformation" of the algebra of space-time coordinates one must replace the usual product among the fields with Weyl-Moyal product or $\star$-product [4], *i.e.*

$$\phi_1(\widehat{x}) \star \phi_2(\widehat{x}) \equiv \lim_{x\to y} e^{\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu \partial^y_\nu} \phi_1(x)\phi_2(y). \tag{2}$$

From the Feynman's rules point of view the only effect of the $\star$-product is to modify the n-point interaction vertices ($3 \leq n$) by the phase factor [4]

$$\tau(p_1, \ldots, p_n) = e^{-\frac{i}{2}\sum_{a<b}^n p_a \wedge p_b}, \tag{3}$$

where $p_a \wedge p_b = \theta_{\mu\nu} p_a^\mu p_b^\nu$. Here the momentum flow of the a-th field into the vertex is denoted by p_a. In the case of noncommutative QED (QED living on a NC space-time), by analyzing the electron-photon interaction vertex one can demonstrate that at the quantum mechanical level the θ-deformation of spatial coordinates gives rise to a deformed Coulomb potential [5]

$$V_\theta = -\frac{Ze^2}{|\vec{x}|} - \frac{Ze^2}{4|\vec{x}|^3}\vec{L}\cdot\vec{\theta} + O(\theta^2), \tag{4}$$

with $\theta_{ij} = \frac{1}{2}\epsilon_{ijk}\theta_k$ and $\vec{L} = \vec{x} \times \vec{p}$. However one may defer the field theoretic considerations and follow a more economical way to achieve (4) by redefining the coordinates

$$\widehat{x}_i \to x_i - \frac{\theta_{ij}}{2} p_j. \tag{5}$$

in the Coulomb potential $\widehat{V} = -\frac{Ze^2}{\sqrt{\widehat{x}_i \widehat{x}_i}}$ defined over a noncommutative space. The prescription which has been also followed to consider the θ-deformation of the Newton potential and its possible classical phenomenological consequences [6, 7].
So, as the aim of this letter, it seems to be a logical step to deduce the deformed Newton potential by looking at the mater-graviton interaction vertex in a NC background space. In the next section we write down the explicit form of the momentum space vertex factor for a Klein-Gordon field scattering from a graviton in NC space. In section 3, the two-body scattering problem in NC flat background is considered and the deformed Newton potential is re-derived by calculating the Fourier transform of the scattering amplitude. The scalar particle scattering off a static source of the graviton is also considered in section 4. Our result is the relativistic generalization of the previous calculations [6, 7]. In this work we assume $c = \hbar = 1$ and $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$.

II. MATTER-GRAVITON COUPLING

The Klein-Gordon Lagrangian density in curved space-times is given by

$$\mathcal{L}_{KG} = \frac{\sqrt{-g}}{2}(\nabla_\alpha \phi \nabla^\alpha \phi - m^2 \phi^2), \tag{6}$$

where ∇_α is the covariant derivative. By expanding the metric tensor and its determinants up to the first order in parameter $\kappa = \sqrt{32\pi G}$ as [8, 9, 10]

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + O(\kappa^2), \tag{7}$$

$$g = 1 + \frac{\kappa}{2} h + O(\kappa^2), \tag{8}$$

where $g = \det g_{\mu\nu}$ and $h = \eta^{\mu\nu} h_{\mu\nu}$, and then substituting them in (6) we find the Lagrangian density

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\alpha \phi \partial^\alpha \phi - m^2 \phi^2) - \frac{\kappa}{2} h^{\mu\nu}\Big[\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}(\partial_\alpha \phi \partial^\alpha \phi - m^2 \phi^2)\Big] + O(\kappa^2). \tag{9}$$

Now the second term of (9), first order in κ, clearly describes the 2 scalar-1 graviton interaction in a flat background. Thus we have the interaction term as

$$\mathcal{L}_{int} = -\frac{\kappa}{2} h^{\mu\nu}\Big[\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\big(\partial_\alpha \phi \partial^\alpha \phi - m^2 \phi^2\big)\Big]. \tag{10}$$

from which one obtains the corresponding vertex factor [8, 9, 10]

$$\tau_{\alpha\beta}(p,p') = -\frac{i\kappa}{2}\left[p_\alpha p'_\beta + p'_\alpha p_\beta - \eta_{\alpha\beta}(p\cdot p' - m^2)\right] \tag{11}$$

In de Donder gauge (harmonic gauge) $\partial^\alpha h_{\alpha\beta} - \frac{1}{2}\partial_\beta h = 0$ the Einstein-Hilbert action takes the form

$$S_{EH} = \frac{1}{2}\int d^4x h_{\mu\nu} Q^{\mu\nu,\alpha\beta} h_{\alpha\beta} \tag{12}$$

with

$$Q^{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta})\partial^2 \tag{13}$$

So the momentum-space graviton propagator is found to be

$$D_{\mu\nu,\alpha\beta}(q) = -\frac{i}{2q^2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}) \tag{14}$$

2. TWO BODY SCATTERING IN NC SPACE

In a NC flat background the interaction term must be changed to

$$\mathcal{L}_{int} = -\frac{\kappa}{2}h^{\mu\nu}\star\Big[\partial_\mu\phi\star\partial_\nu\phi - \frac{1}{2}\big(\partial_\alpha\phi\star\partial^\alpha\phi - m^2\phi\star\phi\big)\Big]. \tag{15}$$

Therefore, with the aid of formula (3) we obtain the deformed momentum space 2 scalar-1 graviton vertex factor as

$$\tau^\theta_{\alpha\beta}(p,p') = -\frac{i\kappa}{2}(p_\alpha p'_\beta + p'_\alpha p_\beta - \eta_{\alpha\beta}p\cdot p')e^{\frac{i}{2}\vec{p}\wedge\vec{p}'}. \tag{16}$$

where we have assumed $\theta_{\mu 0} = 0$ that implies $p\wedge q \to \theta_{ij}p_i q_j$ to avoid the problematic features of the NC models [4]. Now let us look at a typical two-body scattering mediated by a graviton. For the scalar particles with masses m_1 and m_2 the scattering amplitude is

$$\begin{aligned}\mathcal{M}_\theta &= \tau_\theta^{\mu\nu}(p_1,p'_1)D_{\mu\nu,\alpha\beta}(p_1-p'_1)\tau_\theta^{\alpha\beta}(p_2,p'_2) \qquad (17)\\ &= \frac{4\pi G}{(p_1-p'_1)^2}\Bigg\{\Big[(p_1+p_2)^2 - m_1^2 - m_2^2\Big]^2 + \Big[(p_1-p'_2)^2 - m_1^2 - m_2^2\Big]^2 \\ &- \Big[(p'_1-p_1)^2 + 4m_1^2m_2^2\Big]^2\Bigg\}e^{i\vec{p}\wedge(\vec{p}-\vec{p}')}\end{aligned}$$

In the non-relativistic limit we have

$$(p'_1 - p_1)^2 \approx -\vec{q}^2 \tag{18}$$

$$(p_1+p_2)^2 \approx (m_1+m_2)^2 \tag{19}$$

$$(p_1-p'_2)^2 \approx (m_1-m_2)^2 + \vec{q}^2 \tag{20}$$

By substituting (18)-(20) in (17) we find the deformed gravitational potential

$$\begin{aligned} U_\theta(\vec{x}) &= -\frac{1}{4m_1m_2}\int \frac{d^3q}{(2\pi)^3}\mathcal{M}_\theta(\vec{q}) \\ &= -G\frac{m_1m_2}{\sqrt{(x_i-\frac{1}{2}\theta_{ij}p_j)(x_i-\frac{1}{2}\theta_{ik}p_k)}}+4\pi G\delta(\vec{\widetilde{x}}) \end{aligned} \tag{21}$$

where

$$\widetilde{x}_i = x_i - \frac{\theta_{ij}}{2}p_j \tag{22}$$

One must note that the second term of (21) is repulsive and can only be measured for bound s-states and thus does not appear for gravitational field as a spin zero massless field [10].

3. SCATTERING OFF A STATIC GRAVITON SOURCE

The linearized Einstein equation in de Donder gauge reads

$$\Box\Big[h_{\mu\nu}(x) - \frac{1}{2}\eta_{\mu\nu}h(x)\Big] = -\kappa T_{\mu\nu}(x). \tag{23}$$

On the other hand for a static source of graviton with mass M we have the energy-momentum tensor as $T_{\mu\nu}(x) = M\delta_{\mu 0}\delta_{\nu 0}\delta(\vec{x})$. Hence in momentum space one finds

$$h_{\mu\nu}(q) = \frac{\kappa M}{4\vec{q}^2}(\eta_{\mu\nu} - 2\eta_{\mu 0}\eta_{\nu 0})2\pi\delta(q_0). \tag{24}$$

Therefore, having at hand the ingredients needed for calculation of the scattering amplitude $\mathcal{M}$ we get [10]

$$\begin{aligned} i\mathcal{M}_\theta &= ih_{\mu\nu}(\vec{q})\tau_\theta^{\mu\nu}(p,q), \\ &= -\frac{4\pi GM}{\vec{q}^2}(2m^2+4\vec{p}^2)e^{\frac{i}{2}\vec{p}\wedge\vec{q}}. \end{aligned} \tag{25}$$

The Fourier transform of scattering amplitude leaves us with the Newton potential as

$$\begin{aligned} U_\theta &= -4\pi GM\frac{m^2+2\vec{p}^2}{E}\int\frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{x}\cdot\vec{q}+\frac{i}{2}\theta_{ij}p_iq_j}}{\vec{q}^2}, \\ &= -\frac{m^2+2\vec{p}^2}{E}\frac{GM}{\sqrt{(x_i-\frac{1}{2}\theta_{ij}p_j)(x_i-\frac{1}{2}\theta_{ik}p_k)}}, \\ &= -\frac{m^2+2\vec{p}^2}{E}\left(\frac{GM}{|\vec{x}|}+\frac{GM}{4|\vec{x}|^3}\vec{L}\cdot\vec{\theta}+O(\theta^2)\right). \end{aligned} \tag{26}$$

In non-relativistic limit, *i.e.* $|\vec{p}| << m$ and $E \approx m$ the above result coincides with that of [6, 7]

$$U_\theta^{non-rel} = -\frac{GMm}{|\vec{x}|} - \frac{GMm}{4|\vec{x}|^3}\vec{L}\cdot\vec{\theta} + O(\theta^2) \tag{27}$$

which was gained by implementing (5) in $\widehat{U} = -G\frac{mM}{\sqrt{\widehat{x}_i\widehat{x}_i}}$. For a relativistic particle with $E \simeq |\vec{p}|$ we have

$$U_\theta^{rel} = -\frac{2GME}{|\vec{x}|} - \frac{GME}{2|\vec{x}|^3}\vec{L}\cdot\vec{\theta} + O(\theta^2). \tag{28}$$

where the first term of (17) is the well-known result arising from the scalar-graviton scattering in absence of the noncommutativity [10].

4. CONCLUSION

We followed a field theoretic approach to re-derive the θ-deformed Newton potential by calculating the matter-graviton scattering amplitude in a NC flat background. In the case of static graviton source our result confirms the earlier derivation and generalizes it to the relativistic regime.

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