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COSMOLOGICAL STRING MODEL IN THE PRESENCE OF A MAGNETIC FIELD: SPINOR APPROACH AND QUANTUM LOOP EFFECTS

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A Bianchi type-I string cosmological model in the presence of a magnetic field is investigated. A nonlinear spinor field simulation of the massive cosmic strings is presented. In the spinor approach the model is nonsingular at the end of the evolution and does not allow the anisotropic Universe to turn into an isotropic one. Quantum effects of a cosmic string are examined in the framework of loop quantum cosmology.

Key words: Bianchi type-I model, cosmological string, magnetic field, nonlinear spinor field, loop quantum cosmology.

1. INTRODUCTION

The observed Universe is almost homogeneous and isotropic at large scales being usually described by a Friedman-Lemaitre-Robertson-Walker cosmology. However the observational data revealed some anisotropies in the structure of the Universe like the anomalies found in the cosmic microwave background (CMB).

Among the models proposed to describe the anisotropies of the Universe, string cosmological models have generated a lot of interest in recent times because of their possible role in describing different interesting phenomena.

Cosmic strings are linear topological defects which could have formed during the phase transitions in the early stages of the evolution of the Universe. Massive closed loops of strings serve as seeds for the formation of large structures of the Universe like galaxies and even cluster of galaxies [1]. After the pioneer works of Stachel [2] and Letelier [3] the general relativistic cosmologies have been widely studied (see *e. g*. [4, 5, 6]).

On the other hand, the magnetic field has an important role at the cosmological scale and is present in galactic and intergalactic spaces (see *e. g*. the reviews [7, 8] and reference therein). Any theoretical study of cosmological models which contain a magnetic field must take into account that the corresponding Universes are necessarily anisotropic. Among the anisotropic spacetimes, Bianchi type-I (BI) metric seems to be the most convenient for testing different cosmological models.

In what follows we intend to study a spatially homogeneous and anisotropic BI Universe with massive strings in the presence of magnetic fields. In a recent paper [9] we investigated the Einstein's equations for this cosmological model using a few tractable assumptions usually accepted in the literature. In the present paper we shall use a nonlinear spinor field to simulate the cosmic strings.

The use of spinor fields in construction of effective theories to explain different phenomena is not new. Assuming that the affine and metric properties of the spacetime are independent, in 1950 Weyl [10] has shown that the spinor field obeys either a linear equation in a space with torsion, or a nonlinear one in a Riemannian space. Soon after that nonlinear quantum Dirac fields were used by Heisenberg [11] in an ambitious project to construct an unified theory of elementary particles.

In recent times the nonlinear fields have been a matter of large interest in many domains, mentioning here their application in cosmology and astrophysics (see *e. g*. [12, 13, 14, 15, 16, 17]). Moreover, it is shown that a nonlinear spinor field can be used to simulate a perfect fluid from ekpyrotic matter to phantom matter [18, 19, 20].

In the final part of the paper we shall investigate the quantum effects of a cosmological string in the framework of loop quantum cosmology [21].

The paper is organized as follows. In the next Section we describe the model and present the general features of the solutions of Einstein's equations. In Section 3 we simulate the cosmological strings with a nonlinear spinor field which offer the possibility to solve the Einstein's equations without any additional assumptions. We comment on the asymptotic behavior of the evolution of the Universe and present some numerical simulations. In Section 4 we estimate the quantum effects in the present model using the quantum loop cosmology approach. At the end we summarize the results and outline future prospects.

2. BASIC EQUATIONS

In what follows we shall study the evolution of a model of Universe in the presence of a cosmic string and a magnetic flux. For this purpose we shall use a BI spacetime represented by a line element of the form

$$ds^2 = (dt)^2 - a_1^2(t)\left(dx^1\right)^2 - a_2^2(t)\left(dx^2\right)^2 - a_3^2(t)\left(dx^3\right)^2 . \tag{1}$$

There are three scale factors a_i (i = 1, 2, 3) which are functions of time t only and consequently three expansion rates. In the vacuum case, the solutions of Einstein's equations describe a Kasner Universe with the scale factors a_i having a simple power like dependence of time

$$a_i = t^{\alpha_i} \ , \ i = 1,\ 2,\ 3 \tag{2}$$

with α_i constants, satisfying the constraints

$$\sum \alpha_i = \sum \alpha_i^2 = 1 . \tag{3}$$

There exist only two different solutions, the non-flat Kasner solution with α_i = (-1/3, 2/3, 2/3) and the Taub solution with α_i = (1, 0, 0).

In the presence of matter, Einstein's equations are more involved and some additional assumptions are required to solve them. In principle all these scale factors could be different and it is useful to express the mean expansion rate in termes of the average Hubble rate:

$$H = \frac{1}{3}\left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right), \tag{4}$$

where over-dot means differentiation with respect to t.

The energy momentum tensor for a system of cosmic string and magnetic field in comoving coordinates is given by

$$T_\mu^\nu = \rho u_\mu u^\nu - \lambda x_\mu x^\nu + E_\mu^\nu \ , \tag{5}$$

where ρ is the rest energy density of strings with massive particles attached to them and can be expressed as $\rho = \rho_p + \lambda$, where ρ_p is the rest energy density of the particles attached to the strings and λ is the tension density of the system of strings [3] which may be positive or negative. Here u_μ is the four velocity and x_μ is the direction of the string, obeying the relations

$$u_\mu u^\mu = -x_\mu x^\mu = 1 , \qquad u_\mu x^\mu = 0 . \tag{6}$$

In (5) $E_{\mu\nu}$ is the electromagnetic field and in what follows we shall choose the string and also the magnetic field along x^1 direction. Using comoving coordinates we have the following components of energy momentum tensor [22]:

$$T_0^0 - \rho = T_1^1 - \lambda = -T_2^2 = -T_3^3 = \frac{\beta^2}{2}\frac{a_1^2}{\tau^2} \ . \tag{7}$$

The constant β includes the value of the magnetic field which is assumed in the present model to be constant and the magnetic permeability characterizing the medium. Also we introduce the volume scale of the BI spacetime [13]

$$\tau = \sqrt{-g} = a_1 a_2 a_3 \ , \tag{8}$$

which is connected with the Hubble rate (4), namely $\dfrac{\dot{\tau}}{\tau} = 3H$.

In view of $T_2^2 = T_3^3$ from (7) one finds

$$a_2 = a_3 \ D \, exp\left(X \int \frac{dt'}{\tau} \right) \ , \tag{9}$$

with D and X some integration constants.

On the other hand, from Einstein's equations we have the following time evolution of the volume of the Universe:

$$\frac{\ddot{\tau}}{\tau} = \frac{1}{2} \ \kappa \ \left(3\rho + \lambda + \beta^2 \frac{a_1^2}{\tau^2} \right) \ . \tag{10}$$

Taking into account the conservation of the enrgy-momentum tensor, *i.e.*, $T^{\nu}_{\mu;\nu} = 0$, after a little manipulation of (7) one obtains [23, 9]:

$$\dot{\rho} + \frac{\dot{\tau}}{\tau} \rho - \frac{\dot{a}_1}{a_1} \lambda = 0 \, . \tag{11}$$

It is customary to assume a relation between ρ and λ in accordance with the state equations for strings. The simplest one is a proportionality relation [3]:

$$\rho = \alpha \lambda \, . \tag{12}$$

In this case from equations (11) we get

$$\rho = R a_1^{\frac{1-\alpha}{\alpha}} a_2^{-1} a_3^{-1} \ , \tag{13}$$

with R a constant of integration.

It is possible to consider a more general barotropic relation, $\rho = \rho\left(\lambda\right)$, than the linear relation (12), subject to the restrictions imposed by the energy conditions. The weak energy condition as well as the strong one require $\rho \geq \lambda$ with $\lambda \geq 0$ or $\rho \geq 0$ with $\lambda < 0$ and the dominant energy condition implies $\rho \geq 0$ and $\rho^2 \geq \lambda^2$ [3].

3. SPINOR FIELD APPROACH

Recently it was shown that a nonlinear spinor can be used to simulate different types of perfect fluids including those called ekpyrotic, phantom matter and dark energy [18, 19, 20]. Here we show that it is possible to describe cosmic strings in terms of spinor fields as well.

We shall simulate the cloud formed by massive cosmic strings with particles attached along their extensions with a nonlinear spinor field described by the Lagrangian:

$$L_{\text{sp}} = \frac{i}{2}\left[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi\right] - m\bar{\psi}\psi + F\,, \tag{14}$$

with F being some arbitrary function of the scalar $S = \bar{\psi}\psi$. The corresponding components of energy-momentum tensor take the form

$$T_0^0 = mS - F\,, \tag{15}$$

$$T_i^i = S\frac{dF}{dS} - F\,,\; i = 1, 2, 3\,. \tag{16}$$

In the present spinor field simulation of the cosmic string we shall describe the energy density ρ of the string by T_0^0 and the tension density λ and T_1^1 in agreement with (7). Inserting (15) and (16) into (12) we find

$$\text{S}\frac{dF}{dS} - \left(1 - \frac{1}{\alpha}\right)F - \frac{m}{\alpha}S = 0\,, \tag{17}$$

with the solution

$$F = -\nu S^{(\alpha-1)/\alpha} + mS\,, \tag{18}$$

where ν is a positive integration constant assuring the positivity of T_0^0.

With these preparatives we get

$$\rho = T_0^0 = \nu S^{(\alpha-1)/\alpha}\,, \tag{19}$$

$$\lambda = T_1^1 = \frac{\nu}{\alpha}S^{(\alpha-1)/\alpha}\,. \tag{20}$$

On the other hand from the spinor field equations for S one finds [13]

$$\dot{S} + \frac{\dot{\tau}}{\tau}S = 0\,, \tag{21}$$

with the solution

$$S = \frac{C_0}{\tau}, \tag{22}$$

C_0 being a constant.

Taking into account this simple behavior of *S* we have finally [24]

$$\rho = \nu C_0^{\frac{\alpha-1}{\alpha}} \tau^{-\frac{\alpha-1}{\alpha}}, \tag{23}$$

$$\lambda = \frac{\nu C_0^{\frac{\alpha-1}{\alpha}}}{\alpha} \tau^{-\frac{\alpha-1}{\alpha}}. \tag{24}$$

Using the above formulas for ρ and λ, from (11) we can determine the anisotropic factor a_1:

$$a_1 = A_1 \tau, \tag{25}$$

A_1 being a constant of integration. On the other hand, from equations (8) and (9) we obtain

$$a_2 = \sqrt{\frac{D}{A_1}} \exp\left(\frac{X}{2}\int \frac{dt'}{\tau}\right), \tag{26}$$

and

$$a_3 = \frac{1}{\sqrt{A_1 D}} \exp\left(-\frac{X}{2}\int \frac{dt'}{\tau}\right). \tag{27}$$

However these expressions for the anisotropic factors a_i depend on τ which satisfies the differential equation (10). We can evaluate the asymptotic behavior of the volume of the Universe at large time getting [24]

$$\tau \propto \exp t, \tag{28}$$

for $\alpha \geq 1$ or $\alpha < 0$. Consequently the anisotropic factor a_1 presents an exponential increase for $t \to \infty$, while the scale factors a_2 and a_3 tend to constants. We mention that for $0 < \alpha < 1$ there are no consistent solutions in this model, in agreement with the discussion from the final part of Section 2 regarding the general barotropic relation.

Concerning the asymptotic behavior of ρ and λ we infer from (23), (24) that they tend to zero as

$$\rho, \lambda \propto \frac{1}{\exp\left(\frac{\alpha-1}{\alpha}t\right)}. \tag{29}$$

Having in mind the complexity of the differential equation (10) we supplement the general discussion with some numerical estimations. In doing so for simplicity we set $A_1 = 1$, $D = 1$ and $X = 1$ and the initial value of τ to be unity. In Fig. 1 we plot the evolution of energy density ρ for different values of α, namely for $\alpha = -2$ and $\alpha = 2$. Evolution of τ corresponding to these values of parameters is shown in Fig. 2. Other numerical simulations could be found in [24].

The numerical simulations support the behavior described in (28), (29). From Fig. 2 one finds at first, in the case of a negative α the Universe expands slower than it does as for a positive α, though in both cases we have an exponential growth for large time.

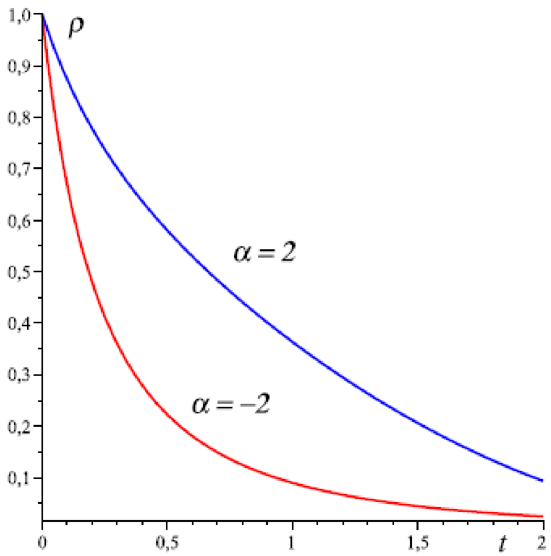

Fig. 1 – View of energy density of the cosmic string for different value of α.

The next task is to investigate the spacetime singularities in the present model. We study the singularity on the basis of Kretschmann scalar

$$K = R_{\alpha\beta\gamma\delta} \cdot R^{\alpha\beta\gamma\delta}\,, \tag{30}$$

written in terms of the Riemann tensor, which for the metric (1) reads

$$K = 4\left[\left(\frac{\ddot{a}_1}{a_1}\right)^2 + \left(\frac{\ddot{a}_2}{a_2}\right)^2 + \left(\frac{\ddot{a}_3}{a_3}\right)^2 + \left(\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2}\right)^2 + \left(\frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3}\right)^2 + \left(\frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}\right)^2\right]. \tag{31}$$

In the study of singularities we shall follow the criteria given in [25]. Essentially, the evolution of an Universe is considered free of singularities for

$t \to \infty$ if all scale factors a_i do not grow faster than exponentially, *i. e.* $a_i(t) >> \exp(k|t|)$, or vanish faster than exponentially, *i. e.* $a_i(t) << \mathcal{O}\left(\exp\left[-k|t|\right]\right)$, with k an arbitrary constant.

Looking at the asymptotic behaviors described above we conclude that in the spinor field approach of the cosmic strings the present model is nonsingular at the end of the evolution. No scale factor of the BI metric presents a growth faster than exponentially or vanishing faster than exponentially for $t \to \infty$.

Since the present-day Universe is surprisingly isotropic, it is important to see whether our anisotropic BI model evolves into an isotropic one. Isotropization means that at large physical times t, when the volume factor τ tends to infinity, the three scale factors $a_i(t)$ grow at the same rate.

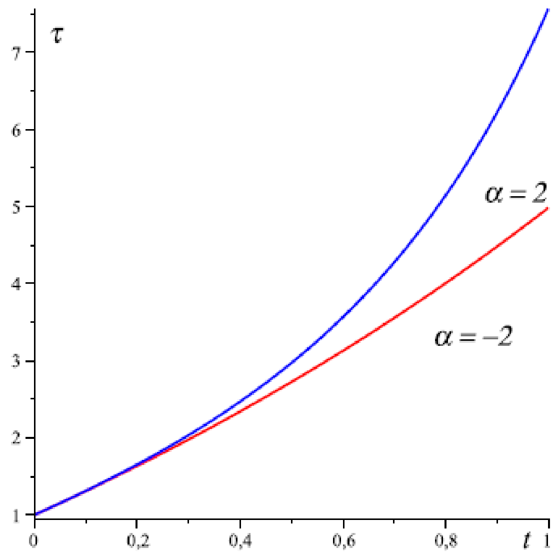

Fig. 2 – Evolution of the Universe corresponding to the energy densities given in Fig. 1.

There are many criteria of isotropization in the literature and here we shall investigate a simple one proposed in [25]

$$\frac{a_1}{a}\Big|_{t\to\infty} \to \text{const}. \tag{32}$$

where $a(t) = \tau^{1/3}$ is the average scale factor. For example, in the generic case with the exponential expansion of the volume of the Universe (28), $a_1 / a \to \infty$, $a_2 / a \to 0$ and $a_3 / a \to 0$. So in the case considered, no isotropization process takes place. Indeed, at the early stage of evolution, where $\tau \to 0$ we assume $a_1 \to \infty$, $a_2 \to 0$ and $a_3 \to 0$, that is at this stage the Universe looks like a one-dimensional string. In the asymptotic region where $t \to \infty$ and $\tau \to \infty$ we have $a_1 \to \infty$, while a_2 and a_3 evaluating to finite values.

A more detailed discussion about the lack of isotropization in the present model is given in [24].

4. EFFECTIVE LOOP QUANTUM DYNAMICS

In the loop quantum cosmology approach we shall use a Hamiltonian framework where the degrees of freedom of the BI model are encoded in triad components p_i and momentum components c_i as follows:

$$p_1 = a_2 a_3 \quad , \quad p_2 = a_1 a_3 \quad , \quad p_3 = a_1 a_2 \quad , \quad c_i = \gamma \dot{a}_i \, . \tag{33}$$

Here γ is the Barbero-Immirzi parameter and represent a quantum ambiguity of loop quantum gravity which is a non-negative real valued parameter.

In terms of these variables, the total Hamiltonian of the model is

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{\text{grav}} + \mathcal{H}_{\text{matter}} \\ &= \frac{-1}{\kappa\gamma^2}\left(c_2 c_3 p_2 p_3 + c_1 c_3 p_1 p_3 + c_1 c_2 p_1 p_2\right) + p_1 p_2 p_3 \rho_M \, , \end{aligned} \tag{34}$$

where ρ_M is the matter energy density [26]. In our model ρ_M comprises the contribution of cosmological string density ρ (13) and the energy density of the magnetic field (7)

$$\rho_{mag} = \frac{1}{2}\frac{\beta^2}{(a_2 a_3)^2} = \frac{1}{2}\frac{\beta^2}{p_1^2} \, . \tag{35}$$

In loop quantum cosmology the connection variables c_i do not have direct quantum analogues and are replaced by holonomies. The quantum effects are incorporated in the *effective* Hamiltonian constructed from the classical one by replacing the c_i terms with sine functions:

$$c_i \to \frac{\sin\left(\bar{\mu}_i c_i\right)}{\bar{\mu}_i} , \tag{36}$$

where $\bar{\mu}_i$ are real valued functions of the triad coefficients p_i.

The effective Hamiltonian is given by:

$$\mathcal{H}_{eff} = \frac{-1}{\kappa\gamma^2}\left\{\frac{\sin\left(\bar{\mu}_2 c_2\right)\sin\left(\bar{\mu}_3 c_3\right)}{\bar{\mu}_2\bar{\mu}_3} p_2 p_3 + cyclic\ terms\right\} + p_1 p_2 p_3 \rho_M \ . \quad (37)$$

It is quite evident that in the limit $\bar{\mu}_i \to 0$, the classical Hamiltonian $\mathcal{H}$ (34) is recovered. The expression of the parameters $\bar{\mu}_i$ as functions of the triad components p_i represent an ambiguity of the quantization. Two most preferable constructions are discussed in [27, 28] and more recently in [29].

The complexity of the equations of motion corresponding to Hamiltonian (37) imposes numerical simulations which will be reported elsewhere [30]. Here we limit ourselves to note that from the vanishing of the Hamiltonian we have the bound:

$$p_1 p_2 p_3 \rho_M \le \frac{1}{\kappa\gamma^2}\left\{\frac{p_2 p_3}{\bar{\mu}_2\bar{\mu}_3} + \frac{p_1 p_3}{\bar{\mu}_1\bar{\mu}_3} + \frac{p_1 p_2}{\bar{\mu}_1\bar{\mu}_2}\right\}. \quad (38)$$

In particular, in the so called $\bar{\mu}'$ scheme [29] the total density is bounded by [31]

$$\rho_{M\ crit} = 3\left(\kappa\gamma^2\Delta\right)^{-1}, \quad (39)$$

which is near the Planck density ρ_{Pl}.

The inequality (38) is remarkable showing that the matter energy density cannot increase unbounded as in the usual *big bang* scenario. Loop quantum cosmology offers a different picture of the early Universe, free of singularities.

4. CONCLUSIONS

In the first part of the paper we investigated an anisotropic spacetime of BI type describing a string cosmology model in the presence of a magnetic field. Using a nonlinear spinor field simulation, we studied the singularities, asymptotic behaviors, isotropization during the evolution of the Universe.

In the second part we considered the quantum effects in the frame of effective loop quantum cosmology. Loop quantum cosmology offers a different picture of the evolution of the spacetime, free of singularities.

In spite of its simplicity, BI models are suitable for investigation of the evolution of the Universe, dark matter, spacetime singularities, etc. The extension of the effective loop quantum cosmology approach to anisotropic models is very promising, deserving further studies.

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