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PERIODIC AND STATIONARY WAVE SOLUTIONS OF COUPLED NLS EQUATIONS

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A system of coupled NLS equations (integrable and non-integrable) is discussed using a Madelung fluid description. The problem is equivalent with a two-component fluid of densities ρ_1 and ρ_2 and velocities v_1 and v_2 , which satisfy equations of continuity and equations of motion. Provided that the nonlinear coupling coefficients are identical, several periodic solutions, expressed through Jacobi elliptic functions, and localized solutions in the form of bright, dark and grey solitons were obtained in different simplifying conditions (motion with constant but equal velocities, i.e. $v_1 = v_2 = v$, and equal "energies", i.e. $E_1 = E_2 = E$; motion with stationary profile of the current velocity). For different "energies" ($E_1 \neq E_2$) a direct method is used, which can be easily extended to more complex situations (different nonlinear coupling coefficients, i.e. β and γ).

1. INTRODUCTION

Coupled nonlinear Schrödinger equations (CNLS), such as

$$\begin{aligned} i\alpha \frac{\partial \psi_1}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \psi_1}{\partial x^2} + (\beta |\psi_1|^2 + \gamma |\psi_2|^2) \psi_1 &= 0, \\ i\alpha \frac{\partial \psi_2}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \psi_2}{\partial x^2} + (\gamma |\psi_1|^2 + \beta |\psi_2|^2) \psi_2 &= 0, \end{aligned} \quad (1)$$

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were used for a long time to describe several physical phenomena. We mention briefly: (a) nonlinear dynamics of deep-water gravity waves [1], [2]; (b) soliton transmission in optical fiber [3]-[5]; (c) dilute gas Bose-Einstein condensate in quasi-1-D regime [6], [7]. Here β and γ are the coefficients of the self-phase and cross modulation contributions to the nonlinear effects. When $\beta = \gamma$ the system (1) becomes

$$\begin{aligned} i\alpha \frac{\partial \psi_1}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \psi_1}{\partial x^2} + \beta (|\psi_1|^2 + |\psi_2|^2) \psi_1 &= 0, \\ i\alpha \frac{\partial \psi_2}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \psi_2}{\partial x^2} + \beta (|\psi_1|^2 + |\psi_2|^2) \psi_2 &= 0, \end{aligned} \quad (2)$$

which is the well known completely integrable Manakov's system [8]. In what follows we shall consider mostly this case.

A number of methods were developed to find solutions of the system of equations (1) as well as of the completely integrable case, i.e. (2), such as inverse scattering method for eq. (2) [8]-[11] and several direct methods for eqs. (1) and (2) [12]-[17].

Few years ago a new approach to solve nonlinear Schrödinger equations, based on Madelung's fluid representation [18], was discussed by Fedele and coworkers [19]. Let us consider a complex function, say Ψ , as the solution of the following NLS-type equation

$$i\alpha \frac{\partial \Psi}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - U(|\Psi|^2) \Psi = 0, \quad (3)$$

where $U(|\Psi|^2)$ is a functional of $|\Psi|^2$. Then it is well known that, under the Madelung's transformation [18], i.e.

$$\Psi(x, t) = \sqrt{\rho(x, t)} \exp\left[\frac{i}{\alpha} \theta(x, t)\right], \quad (4)$$

the above NLS type equation is transformed into the following pair of coupled fluid equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\ \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x}\right)v &= \frac{\alpha^2}{2} \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} \right) - \frac{\partial U}{\partial x}. \end{aligned} \quad (5)$$

Here

$$v(x, t) = \frac{\partial \theta(x, t)}{\partial x} \quad (6)$$

is the "velocity" of the fluid of "density" $\rho(x, t)$. It can be proven that the second of equations (5) can be transformed into the following Korteweg-de Vries (KdV)-type equation (see Ref. [19])

$$\begin{aligned} -\frac{\partial v}{\partial t} \rho + v \frac{\partial \rho}{\partial t} + 2 \left[c_0(t) - \int \left(\frac{\partial v}{\partial t} \right) dx \right] \frac{\partial \rho}{\partial x} - \\ - \left(\rho \frac{dU}{d\rho} + 2U \right) \frac{\partial \rho}{\partial x} + \frac{\alpha^2}{4} \frac{\partial^3 \rho}{\partial x^3} = 0, \end{aligned} \quad (7)$$

where $c_0(t)$ is an arbitrary real function of t . The eq. (7) was the starting point to obtain a number of solitary wave solutions of (4) and also to establish a connection between NLS equation (4) and a wide class of KdV equations for the density ρ . Later on this approach was used for derivative NLS equation [20], and a general connection between extended NLS and KdV equations, at least in the class of traveling wave solutions, was emphasized in [21].

In the following the system of coupled NLS equation (1) and (2) will be discussed using the same Madelung's fluid approach.

2. BASIC EQUATIONS

Following Fedele et al. [19] we consider

$$\psi_i = \sqrt{\rho_i} \exp \left[\frac{i}{\alpha} \theta_i \right]. \quad (8)$$

Introducing into (1) and separating the real and the imaginary part one obtains

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x} (v_i \rho_i) = 0, \quad i = 1, 2 \quad (9)$$

and

$$\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x} \right) v_1 = \frac{\alpha^2}{2} \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{\rho_1}} \frac{\partial^2 \sqrt{\rho_1}}{\partial x^2} \right) + \frac{\partial}{\partial x} (\beta \rho_1 + \gamma \rho_2) = 0 \quad (10)$$

and a similar equation for v_2 . Here

$$v_i(x, t) = \frac{\partial \theta_i(x, t)}{\partial x} \quad (11)$$

is the "velocity" of the fluid of "density" ρ_i . The first equation is the equation of continuity of the fluid and the second the equation of motion for the velocities v_1 and v_2 . Following Fedele's transformation [19] the second equation can be written in the following form

$$-\rho_1 \frac{\partial v_1}{\partial t} + v_1 \frac{\partial \rho_1}{\partial t} + 2 \left[c_1(t) - \int \frac{\partial v_1}{\partial t} dx \right] \frac{\partial \rho_1}{\partial x} + \frac{\alpha^2}{4} \frac{\partial^3 \rho_1}{\partial x^3} + \rho_1 \frac{\partial}{\partial x} (\beta \rho_1 + \gamma \rho_2) + 2(\beta \rho_1 + \gamma \rho_2) \frac{\partial \rho_1}{\partial x} = 0. \quad (12)$$

and a similar equation for v_2 . Here $c_1(t)$, $c_2(t)$ are two arbitrary real functions of t . As mentioned before in the following we shall consider mostly the integrable case $\beta = \gamma$, when the equation of motion writes

$$-\rho_i \frac{\partial v_i}{\partial t} + v_i \frac{\partial \rho_i}{\partial t} + 2 \left[c_i(t) - \int \frac{\partial v_i}{\partial t} dx \right] \frac{\partial \rho_i}{\partial x} + \frac{\alpha^2}{4} \frac{\partial^3 \rho_i}{\partial x^3} + \rho_i \frac{\partial}{\partial x} \beta (\rho_1 + \rho_2) + 2\beta (\rho_1 + \rho_2) \frac{\partial \rho_i}{\partial x} = 0. \quad (13)$$

3. SPECIAL SOLUTIONS

3.1. MOTION WITH CONSTANT VELOCITY

The first case to be discussed is the motion with constant velocity

$$v_1 = v_2 = v_0. \quad (14)$$

From the equations of continuity (9) follows that both $\rho_1(x, t)$ and $\rho_2(x, t)$ are depending on $\xi = x - v_0 t$. Then (13) becomes

$$\frac{\alpha^2}{4} \frac{d^3 \rho_i}{d\xi^3} - E_i \frac{d\rho_i}{d\xi} + \beta \rho_i \frac{d}{d\xi} (\rho_1 + \rho_2) + 2\beta (\rho_1 + \rho_2) \frac{d\rho_i}{d\xi} = 0, \quad (15)$$

where we denoted $E_i = -(2c_i - v_0^2)$.

We shall discuss firstly the case

$$E_1 = E_2 = E, \quad (16)$$

when a very simple solution of equations (15) is easily obtained. Indeed introducing the quantity $z_+ = \rho_1 + \rho_2$ and adding the equations (15) we obtain

$$\frac{\alpha^2}{4} \frac{d^3 z_+}{d\xi^3} - E \frac{dz_+}{d\xi} + \frac{3}{2} \beta \frac{d}{d\xi} z_+^2 = 0. \quad (17)$$

Integrating (17) two times we find

$$\frac{\alpha^2}{4} \left(\frac{dz_+}{d\xi} \right)^2 = -\beta z_+^3 + Ez_+^2 + Az_+ + B = P_3(z_+), \quad (18)$$

where A and B are two integration constants.

By subtracting the two equations (15) the following equation for $z_- = \rho_1 - \rho_2$ is obtained

$$\frac{\alpha^2}{4} \frac{d^3 z_-}{d\xi^3} - Ez_- + \beta z_- \frac{dz_+}{d\xi} + 2\beta z_+ \frac{dz_-}{d\xi} = 0, \quad (19)$$

a linear differential equation in z_- , once $z_+(\xi)$ is known. It is easily seen that $z_- \sim z_+$ is a solution of (19) (eq. (19) transforms into (17)). This leads to the following special solution of our problem; writing

$$z_- = (\rho_1^2 - \rho_2^2) z_+ \quad \rho_1^2 + \rho_2^2 = 1 \quad (20)$$

one obtains

$$\rho_1 = \rho_1^2 z_+, \quad \rho_2 = \rho_2^2 z_+. \quad (21)$$

Here p_1 and p_2 will play the role of polarization factors.

However, for the motion with constant velocity the densities ρ_i have to satisfy a supplementary condition. Indeed for the solutions (21) the following condition is obtained

$$\frac{\alpha^2}{2} \frac{1}{\sqrt{z_+}} \frac{d^2 \sqrt{z_+}}{dx^2} \beta z_+ = \lambda \quad (22)$$

which transforms into

$$\frac{\alpha^2}{4} \left[\frac{d^2 z_+}{d\xi^2} - \frac{1}{2z_+} \left(\frac{dz_+}{d\xi} \right)^2 \right] + \beta z_+^2 - \lambda z_+ = 0. \quad (23)$$

But both $\frac{d^2 z_+}{d\xi^2}$ and $\frac{dz_+}{d\xi}$ can be expressed as polynomials in z_+ (see (18)) and

it is easily seen that (23) can be satisfied if

$$\lambda = \frac{E}{2}, \quad B = 0. \quad (24)$$

Furthermore, we shall discuss separately the cases $\beta > 0$ and $\beta < 0$. Let us assume that the polynomial $P_3(z_+)$ has three distinct roots $z_1 > z_2 > z_3$. The

condition $B = 0$ implies $z_2 = 0$, or $z_3 = 0$. The solutions of (18) are given in terms of Jacobi elliptic functions [22]. In the case of $\beta > 0$ we have two acceptable situations ($z_+ > 0$), one corresponding to $z_1 > 0, z_2 = 0, z_3 < 0$

$$\begin{aligned} z_+ &= z_1 \operatorname{cn}^2 u \\ u &= \frac{2\sqrt{\beta}}{g|\alpha|} \xi, \quad k^2 = \frac{z_1}{z_1 + |z_3|}, \quad g = \frac{2}{\sqrt{z_1 + |z_3|}} \end{aligned} \quad (25)$$

and the other to $z_3 = 0, 0 < z_2 < z_1$

$$\begin{aligned} z_+ &= z_1 - (z_1 - z_2) \operatorname{sn}^2 u \\ k^2 &= \frac{z_1 - z_2}{z_1}, \quad g = \frac{2}{\sqrt{z_1}}. \end{aligned} \quad (26)$$

In the "degenerate" case, when $k^2 = 1$ both solutions (25) and (26) transform into

$$\begin{aligned} z_+ &= z_1 \frac{1}{\cosh^2 u} \\ u &= \frac{2\sqrt{\beta}}{g|\alpha|}, \quad g = \frac{2}{\sqrt{z_1}}, \end{aligned} \quad (27)$$

which represent a bright soliton. If $\beta < 0$ the only acceptable situation is

$$\begin{aligned} z_3 &= 0, \quad z_1 > z_2 > 0 \\ z_+ &= z_2 \operatorname{sn}^2 u \\ u &= \frac{2\sqrt{|\beta|}}{g|\alpha|} \xi, \quad k^2 = \frac{z_2}{z_1}, \quad g = \frac{2}{\sqrt{z_1}}, \end{aligned} \quad (28)$$

which for $k^2 = 1$ transforms in the dark soliton

$$z_+ = z_2 \tanh^2 u. \quad (29)$$

The phase $\theta_i(x, t)$ is easily calculated from (6) and the time dependent equation satisfied by θ_i , namely

$$-\frac{\partial \theta_i}{\partial t} - \frac{1}{2} v_0^2 + \left(\frac{\alpha^2}{2} \frac{1}{\sqrt{\rho_i}} \frac{\partial^2 \sqrt{\rho_i}}{\partial \xi^2} + \beta z_+ \right) = 0. \quad (30)$$

Using (22) and (24) we find

$$\theta_i = v_0 x - \left(\frac{1}{2} v_0^2 - \frac{E}{2} \right) t + \delta_i, \quad (31)$$

with δ_1, δ_2 arbitrary phases.

3.1.1. Higher nonlinearity

All the previous considerations can be easily extended to higher nonlinearity of equation (2) of the form $U(|\Psi_1|^2 + |\Psi_2|^2)$, where U is a certain functional of $(|\Psi_1|^2 + |\Psi_2|^2)$. As an example we shall consider

$$U(z_+) = \beta z_+ - \frac{3}{2} \gamma z_+^2. \quad (32)$$

For motion with constant velocity the equation (15) is replaced by

$$\frac{\alpha^2}{4} \frac{d^3 \rho_i}{d\xi^3} - E_i \frac{d\rho_i}{d\xi} + \rho_i \frac{d}{d\xi} U(\rho_1 + \rho_2) + 2U(\rho_1 + \rho_2) \frac{d\rho_i}{d\xi} = 0 \quad (33)$$

and for equal E_i the following equation is satisfied by z_+

$$\frac{\alpha^2}{4} \frac{d^3 z_+}{d\xi^3} - E \frac{dz_+}{d\xi} + \left(2U + z_+ \frac{dU}{dz_+} \right) \frac{dz_+}{d\xi} = 0. \quad (34)$$

With $U(z_+)$ of the form (32) after two integrations one obtains

$$\frac{\alpha^2}{4} \left(\frac{dz_+}{d\xi} \right)^2 = P_4(z_+) \quad (35)$$

$$P_4(z) = \gamma z^4 - \beta z^3 + E z^2 + A z + B.$$

The equation satisfied by z_- is given by

$$\frac{\alpha^2}{4} \frac{d^3 z_-}{d\xi^3} - E \frac{dz_-}{d\xi} + 2U(z_+) \frac{dz_-}{d\xi} + \frac{dU(z_+)}{d\xi} z_- = 0, \quad (36)$$

which has a particular solution of the form (20). The supplementary condition (23) becomes

$$\frac{\alpha^2}{4} \left[\frac{d^2 z_+}{d\xi^2} - \frac{1}{2z_+} \left(\frac{dz_+}{d\xi} \right)^2 \right] + \beta z_+^2 - \frac{3}{2} \gamma z_+^3 - \lambda z_+ = 0,$$

and the conditions (24) are re-obtained. Then the problem is reduced to solving (35) with the condition $B = 0$. Again we have to discuss separately the case $\gamma > 0$ and $\gamma < 0$.

We shall discuss only the case $\gamma > 0$. Two distinct situations are possible:

(i) $z_4 = 0 < z_3 < z_2 < z_1$

(ii) $z_4 < z_3 = 0 < z_2 < z_1$

In the first case (i) the solution of (35) is [22]

$$\begin{aligned} z_+ &= \frac{z_3}{1 - \alpha^2 \operatorname{sn}^2 u}, & u &= \frac{2\sqrt{\gamma}}{|\alpha|g} \xi \\ k^2 &= \frac{z_1(z_2 - z_3)}{z_2(z_1 - z_3)}, & \alpha^2 &= \frac{z_2 - z_3}{z_2}, & g &= \frac{2}{\sqrt{z_2(z_1 - z_3)}} \end{aligned} \quad (37)$$

In the degenerate case $z_1 = z_2$, $k^2 = 1$ and the solution (37) becomes

$$z_+ = \frac{z_3}{1 - \alpha^2 \tanh^2 u}, \quad (38)$$

describing a grey soliton ($u = 0$, $z_+(0) = z_3$; $|u| \rightarrow \infty$, $z_+(\infty) = z_2$).

In the case (ii) the solution is

$$\begin{aligned} z_+ &= \frac{|z_4| \alpha^2 \operatorname{sn}^2 u}{1 - \alpha^2 \operatorname{sn}^2 u}. \\ k^2 &= \frac{z_2(z_1 + |z_4|)}{z_1(z_2 + |z_4|)}, & \alpha^2 &= \frac{z_2}{z_2 + |z_4|}, & g &= \frac{2}{\sqrt{z_1 + |z_4|}}, \end{aligned} \quad (39)$$

which in the degenerate case $z_1 = z_2$ becomes

$$z_+ = \frac{|z_4| \alpha^2 \tanh^2 u}{1 - \alpha^2 \tanh^2 u}, \quad (40)$$

describing a dark soliton ($u = 0$, $z_+(0) = 0$; $|u| \rightarrow \infty$, $z_+(\infty) = z_2$).

3.2. MOTION WITH STATIONARY PROFILE CURRENT VELOCITY

The next case we are discussing is the motion with stationary-profile current velocity, when both $\rho_i(x, t)$ and $v_i(x, t)$ are functions of the combined variable $\xi = x - u_0 t$. Then the equation of continuity (9) can be integrated giving

$$v_i(\xi) = u_0 + \frac{A_i}{\rho_i(\xi)}, \quad (41)$$

with A_i an integration constant. It is easily seen that the equation of motion keeps the same form (15) with $E_i = -(2c_i + u_0^2)$. Restricting to the case $E_1 = E_2 = E$ and introducing the variables z_+ and z_- as before they satisfy the same equations (18) and (19), and the particular solutions (20) and (21) are still valid. The only

difference from the previous discussion is that no restriction are now imposed, and consequently a larger class of solutions exists.

In the case $\beta > 0$ if the polynomial $P_3(z_+)$ has three distinct roots, with at least two positive, the solution is given by [22]

$$\begin{aligned} z_+ &= z_1 - (z_1 - z_2)\text{sn}^2 u \\ u &= \frac{2\sqrt{\beta}}{g|\alpha|} \xi, \quad k^2 = \frac{z_1 - z_2}{z_1 - z_3}, \quad g = \frac{2}{\sqrt{z_1 - z_2}}. \end{aligned} \quad (42)$$

which in degenerate case $z_3 = z_2 > 0$ becomes

$$z_+ = z_1 - (z_1 - z_2)\tanh^2 u, \quad (43)$$

describing a bright type soliton with nonvanishing value at infinity ($u = 0, z_+(0) = z_1; |u| \rightarrow \infty, z_+(\infty) = z_2$).

In the case $\beta < 0$ if the polynomial $P_3(z_+)$ has three distinct roots, all positive, the solution is given by

$$\begin{aligned} z_+ &= z_3 + (z_2 - z_3)\text{sn}^2 u \\ u &= \frac{2\sqrt{|\beta|}}{g|\alpha|} \xi, \quad k^2 = \frac{z_2 - z_3}{z_1 - z_3}, \quad g = \frac{2}{\sqrt{z_1 - z_3}} \end{aligned} \quad (44)$$

which in the degenerate case $z_2 = z_3$ becomes

$$z_+ = z_3 + (z_2 - z_3)\tanh^2 u \quad (45)$$

describing a dark-grey soliton ($u = 0, z_+(0) = z_3; |u| \rightarrow \infty, z_+(\infty) = z_1$).

The phase $\theta_i(\xi)$ is calculated from (41) namely

$$\theta_i(\xi) = u_0 \xi + A_i \int_0^\xi \frac{dt}{\rho_i(t)} + \delta_i.$$

For $\beta > 0$, using (21) and (42) we have

$$\theta_i(\xi) = u_0 \xi + \bar{A}_i \int_0^u \frac{dt}{1 - \alpha^2 \text{sn}^2 t} + \delta_i, \quad (46)$$

where

$$\bar{A}_i = \frac{g|\alpha|}{2\sqrt{\beta}} \frac{A_i}{p_i^2} \frac{1}{z_1}, \quad a^2 = \frac{z_1 - z_2}{z_1} < 1$$

and the integral is exactly the definition of the incomplete elliptic integral of third kind [22], $\Pi(\varphi, a^2, k)$, ($\sin \varphi = \text{sn} u$).

$$\theta_i(x, t) = u_0(x - u_0 t) + \bar{A}_i \Pi(\varphi, a^2, k) + \delta_i. \quad (47)$$

In the degenerate case the integral appearing in (46) becomes

$$\int_0^u \frac{dt}{1 - a^2 \tanh^2 t} = \frac{1}{2(1 - a^2)} \left[u - a \operatorname{arg} \tanh(a \tanh u) \right] \quad (48)$$

3.3. DIRECT METHOD

When $E_1 \neq E_2$ the method used in the previous sections can not be applied. For motion with stationary-profile current velocity we shall try a solution of the form

$$\rho_i = A_i + B_i \operatorname{sn}^2 u, \quad (49)$$

where $\operatorname{sn}(u, k)$ is the Jacobi sine-elliptic function of argument $u = \lambda \zeta$ and modulus k . Using well known formula from the theory of Jacobi elliptic functions we get

$$\frac{d\rho_i}{d\xi} = B_i (2\lambda \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u) \quad (50)$$

$$\frac{d^3 \rho_i}{d\xi^3} = 4\lambda^2 B_i \left[3k^2 \operatorname{sn}^2 u - (1 + k^2) \right] (2\lambda \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u),$$

which introduced into (15) gives (the coefficient $\frac{\alpha^2}{4}$ was included in the definition of ζ)

$$\begin{aligned} 4\lambda^2 \left[3k^2 \operatorname{sn}^2 u - (1 + k^2) \right] B_i - E_i B_i + \beta (B_1 + B_2) (A_i + B_i \operatorname{sn}^2 u) \\ + 2\beta \left[(A_1 + A_2) + (B_1 + B_2) \operatorname{sn}^2 u \right] B_i = 0. \end{aligned} \quad (51)$$

This relation has to be satisfied for any value of $\operatorname{sn}^2 u$. Equating with zero the coefficient of $\operatorname{sn}^2 u$ one obtains

$$B_1 + B_2 = -\frac{4\lambda^2 k^2}{\beta}. \quad (52)$$

It is convenient to write

$$B_i = -\frac{4\lambda^2 k^2}{\beta} b_i \quad (53)$$

with b_i satisfying

$$b_1 + b_2 = 1. \quad (54)$$

The vanishing of the free term of (51) gives

$$-\left[4\lambda^2(1+k^2)+E_i\right]B_i+\beta(B_1+B_2)A_i+2\beta(A_1+A_2)B_i=0. \quad (55)$$

It is convenient to write

$$E_1=E+\Delta, \quad E_2=E-\Delta \quad (56)$$

and without any loss of generality we can assume $\Delta > 0$. Adding and subtracting the two equations (55) we obtain

$$\begin{aligned} A_1+A_2 &= \frac{1}{3\beta}\left[4\lambda^2(1+k^2)+E+\Delta(b_1-b_2)\right] \\ A_1-A_2 &= \frac{\Delta}{\beta}+\frac{1}{3\beta}\left[4\lambda^2(1+k^2)+E\right](b_1-b_2)-\frac{2\Delta}{3\beta}(b_1-b_2)^2 \end{aligned} \quad (57)$$

It is convenient to scale E and Δ

$$E=4\lambda^2k^2e, \quad \Delta=4\lambda^2k^2\delta, \quad \delta>0 \quad (58)$$

and to write

$$A_i=\frac{4\lambda^2k^2}{|\beta|}a_i. \quad (59)$$

Then, for $\beta > 0$, the system (57) becomes

$$\begin{aligned} a_1+a_2 &= \frac{1}{3}\left[e+\frac{1+k^2}{k^2}+\delta(b_1-b_2)\right] \\ a_1-a_2 &= \delta+\frac{1}{3}\left(e+\frac{1+k^2}{k^2}\right)(b_1-b_2)-\frac{2}{3}\delta(b_1-b_2)^2 \end{aligned} \quad (60)$$

which can be solved for a_1 and a_2 . Using (54) we can eliminate b_2 from the expression of a_1 and b_1 from the expression of a_2 . Finally one obtains

$$\begin{aligned} a_1 &= \frac{1}{3}\left[e+\frac{1+k^2}{k^2}+\delta+4\delta(1-b_1)\right]b_1 \\ a_2 &= \frac{1}{3}\left[e+\frac{1+k^2}{k^2}-\delta-4\delta(1-b_2)\right]b_2. \end{aligned} \quad (61)$$

As it is expected these results verify the symmetry condition $1 \leftrightarrow 2$ if $\delta \leftrightarrow -\delta$. The relation (54) is satisfied in two distinct situations:

- (i) both b_1, b_2 positive quantities, less than unity,
- (ii) $b_1 = b > 1, b_2 = -(b-1) < 0$. Several restrictions come from the positiveness of ρ_i . In the case (i) this requirement implies

$$a_i \geq b_i \geq 0 \quad (62)$$

while in the case (ii) we have

$$a_1 \geq b_1 > 1, \quad a_2 > 0. \quad (63)$$

In both cases these conditions transform into restrictions of the allowed values of e . In the first case, denoting

$$\mu = \frac{1}{3} \left[e + \frac{1+k^2}{k^2} - \delta(1+4b) \right] \quad (64)$$

the condition (62) is satisfied if

$$\mu \geq 1 \quad (65)$$

for any value of $b \in [0, 1]$. Using this notations the coefficients a_1, a_2 write

$$a_1 = (\mu + 2\delta)b, \quad a_2 = \mu(1-b). \quad (66)$$

In the degenerate case ($k^2 = 1$) the solutions are

$$\begin{aligned} \rho_1 &= \frac{4\lambda^2}{\beta} b (\mu + 2\delta - \tanh^2 u) \\ \rho_2 &= \frac{4\lambda^2}{\beta} (1-b) (\mu - \tanh^2 u), \end{aligned} \quad (67)$$

both corresponding to bright-type soliton solutions with nonvanishing values of infinity. For the limiting case $\mu = 1$ the solution ρ_2 transforms into a pure bright soliton.

In the second case (ii), with the same definition of μ , the conditions (63) becomes

$$-(2\delta - 1) < \mu < 0 \quad (68)$$

and this can be satisfied only if $\delta \geq 2$. The solutions are

$$\begin{aligned} \rho_1 &= \frac{4\lambda^2 k^2}{\beta} b (a_1 - b \operatorname{sn}^2 u) \\ \rho_2 &= \frac{4\lambda^2 k^2}{\beta} (a_2 + (b-1) \operatorname{sn}^2 u). \end{aligned} \quad (69)$$

In the degenerate case they become

$$\begin{aligned}\rho_1 &= \frac{4\lambda^2}{\beta} (a_1 - b \tanh^2 u) \\ \rho_2 &= \frac{4\lambda^2}{\beta} [a_2 + (b-1) \tanh^2 u],\end{aligned}\tag{70}$$

describing ρ_1 a bright type solution with nonvanishing value at infinity, and ρ_2 a grey solution. For $\delta = \frac{1}{2}$ and $\mu = 0$, $a_1 = b$, $a_2 = 0$, ρ_1 transforms into the bright

soliton $\rho_1 = \frac{4\lambda^2}{\beta} b \operatorname{sech}^2 \mu$, while ρ_2 into the dark soliton $\rho_2 = \frac{4\lambda^2}{\beta} (b-1) \tanh^2 u$.

If $\beta < 0$ it is convenient to scale E as

$$E = -4\lambda^2 k^2 e_1$$

and as in the previous case, we have two possibilities for b_1 and b_2 . The case

- (i) is the same as before and
(ii) $b_1 = -(b-1) < 0$, $b_2 = b > 1$.

The corresponding system for a_1 and a_2 is easily solved, giving

$$\begin{aligned}a_1 &= \frac{1}{3} \left[e_1 - \frac{1+k^2}{k^2} - \delta - 4\delta(1-b_1) \right] b_1 \\ a_2 &= \frac{1}{3} \left[e_1 - \frac{1+k^2}{k^2} + \delta + 4\delta(1-b_2) \right] b_2.\end{aligned}\tag{71}$$

In the case (i) both B_1, B_2 are positive and the positiveness requirement for ρ_1, ρ_2 implies $a_1 \geq 0$. Writing $b_1 = 1 - b$, $b_2 = b$ this is satisfied if

$$\mu_1 = \frac{1}{3} \left[e_1 - \frac{1+k^2}{k^2} - \delta(1+4b) \right] \geq 0\tag{72}$$

for any value of $b \in [0, 1]$. In the degenerate case ($k^2 = 1$) both ρ_1 and ρ_2 describe grey type solitons. In the second case (ii) we have $B_1 > 0$ and $B_2 < 0$ and the positivity conditions become

$$a_1 > 0, a_2 > b.\tag{73}$$

With the same definition of μ_1 this is realized if

$$-(2\delta - 1) \leq -\mu_1 \leq 0\tag{74}$$

and as in the previous case δ has to be greater than $\frac{1}{2}$. In the degenerate case the solutions become

$$\begin{aligned}\rho_1 &= \frac{4\lambda^2}{|\beta|} (b-1) (|\mu_1| - \tanh^2 u) \\ \rho_2 &= \frac{4\lambda^2}{|\beta|} b (2\delta - |\mu_1| + \tanh^2 u)\end{aligned}\tag{75}$$

describing, ρ_1 a bright type solution with nonvanishing condition at infinity,

and ρ_2 a grey type solution. If $\delta = \frac{1}{2}$, $\mu_1 = 0$, ρ_1 becomes a bright soliton and ρ_2 a dark one.

The phases $\theta_i(\xi)$ can be calculated as in paragraph (3.2) and the result is expressed in terms of incomplete elliptic integral of third kind. Indeed noting

$$\rho_i(u) = \Gamma_i (1 - \nu \operatorname{sn}^2 u)$$

with $\nu \in (-\infty, 1]$ one obtains

$$\theta_i(\xi) = u_0 \xi + \frac{A_i}{\lambda \Gamma_i} \Pi(u, \nu, k^2) + \delta_i\tag{76}$$

and depending on the value of ν we distinguish different cases [22]

$$-\infty < \nu < 0, \quad k^2 < \nu < 1 \text{ circular case}$$

and

$$0 < \nu < k^2 \quad \text{hyperbolic case}$$

with specific expressions in each case.

4. CONCLUSIONS

Using the Madelung approach the system of coupled cubic NLS equations (1) (and the integrable case (2)) was described as a two-component fluid of densities ρ_1 and ρ_2 and velocities v_1 and v_2 . In a natural way these quantities are satisfying continuity equations (9) and equations of motion (12) (or (13) for the integrable case). Several periodic and travelling solutions were found in different simplifying conditions. The case of equal velocities ($v_1 = v_2 = v$) and equal "energies" ($E_1 = E_2$)

$= E$) for the integrable case was integrated in section 3.1. The calculations were extended easily to higher nonlinearities (the system is no more completely integrable). In the same case of equal "energies" ($E_1 = E_2 = E$) the problem of motion with stationary-profile current velocities was investigated in section 3.2. Periodic solutions expressed through Jacobi elliptic functions were found, and in the "degenerate" case $k^2 = 1$ they transform in stationary states of bright, dark or grey solutions. For different "energies" ($E_1 \neq E_2$) a direct method was used in section 3.3 and a larger class of solutions were found (solutions with possible energy transfer between the components are found). Several extensions are possible. The most interesting refers to use the direct method including $\text{sn}^4 u$ terms. Investigations in this direction are under way.

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