

*Dedicated to Professor Mihai Gavrilă's 80<sup>th</sup> Anniversary*

## THE COMPLETENESS OF VOLKOV SPINORS

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*Received February 23, 2010*

We present an elementary proof of the completeness property of Volkov solutions of the Dirac equation, based on a direct calculation. A summary of other properties of Volkov spinors and their elementary justification is included.

### 1. INTRODUCTION

The solutions of the Dirac equation for the electron interacting with an electromagnetic plane wave, derived by Volkov [1] and known as the Volkov spinors, are much used in the theoretical description of various radiative processes in the presence of intense fields. The first bunch of calculations were performed soon after the invention of the laser and concern nonlinear Compton scattering, pair creation and other at that time (and even now) exotic atomic processes [2]. With the spectacular progress in laser performances in terms of intensity, these processes have become the subject of new calculations and even of experimental investigation [3].

Along the years the properties of the Volkov solutions have been investigated by the theorists at various levels of rigor. Some of them are easy to be derived, some need a more careful treatment, this is the case of their orthogonality and normalization properties and of the completeness. The orthogonality property even lead to some controversy [4, 5] in the literature. The most refined proof of the orthonormalization relation was recently given in a mathematical paper [6].

To our knowledge there is no published proof of the completeness relation for the set of Volkov solutions, in the form [Eq. (42) in the following] physicists tacitly use in several studies on the electron interaction with an electromagnetic plane wave. The need for a rigorous proof was recently claimed in Ref. [6]. Ritus [7] has written a completeness relation, that we reproduce in our Eq. (40), but it has not the form used in atomic physics. Later, Bergou and Varro [8] have proved a

completeness relation on the null plane. So we give in this paper a proof of the completeness relation used by physicists. Our derivation is an elementary one, based on a direct calculation. We include also the proof of the orthogonality property.

We underline that the completeness property is essential for the work with Volkov spinors. Based on it one describes the dynamics of a free electron by a packet of Volkov solutions (see [9, 10] and the references therein). This property is also needed for proving that a relativistic generalization [11, 12] of the Kramers-Henneberger transformation can be performed with an unitary operator.

We start in Sect. 2 by briefly describing several methods by which Volkov solutions were derived. The analytic expression of Volkov solutions of the Dirac equation is reproduced in Sect. 3. The presentation of orthonormalization properties, reviewed in Sect. 4, is an extension of the proof that Filipowicz [13] has given for the positive energy solutions only. Section 5 contains our contribution concerning the completeness property of the Volkov spinors.

## 2. DERIVATIONS FOR VOLKOV SOLUTIONS IN THE LITERATURE

In 1934 Volkov [1] has derived exact solutions of the temporal Dirac equation for an electron (mass  $m$ , charge  $e < 0$ ) subject to an electromagnetic plane wave,

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \left\{ c \boldsymbol{\alpha} \cdot [\mathbf{P} - e\mathbf{A}(\tau)] + mc^2 \beta \right\} \psi(\mathbf{r}, t), \quad (1)$$

where  $\boldsymbol{\alpha}$  and  $\beta$  are Dirac matrices [14],  $\mathbf{P}$  the momentum operator and  $c$  is the velocity of light. The electromagnetic wave, described by the vector potential  $\mathbf{A}$ , has a fixed direction of propagation, characterized by an unity vector  $\mathbf{n}$ , and its temporal and spatial dependence is contained in one variable

$$\tau = t - \frac{\mathbf{r} \cdot \mathbf{n}}{c}. \quad (2)$$

The two vectors  $\mathbf{A}$  and  $\mathbf{n}$  are orthogonal.

The remarkable fact about the solutions is that the vector potential  $\mathbf{A}$  can be any function of the variable  $\tau$ . The monochromatic plane wave is only a particular case.

The original derivation by Volkov [1] was based on the integration of the *second-order Dirac equation in the presence of the external field*. This approach is presented by several textbooks as, for instance, the book of Berestetski *al.* [15]. We mention several other papers in which Volkov solutions are reobtained with different methods. The oldest is Alperin's paper [16] in which Dirac equation is integrated directly using the null-coordinates and a complex vector potential. The two solutions found by Alperin correspond to positive energies in the absence of

the field. As Volkov paper is not quoted, one can assume that this was an independent calculation.

Other calculations appeared many years after Volkov's paper. Beers and Nickle [17] have written Dirac equation as an eigenvalue equation for a time dependent Hamiltonian (see Eq. (5) in Sect. 2 here) and have constructed a set of operators that commute with this Hamiltonian and obey the commutation relations of the generators of the Poincaré group. The Volkov solutions are found as eigenvectors of the momentum operators in this set. This set of operators coincides with the one introduced before by Chakrabarti [18] by transforming the usual generators of the Poincaré group. Chakrabarti has also derived solutions of the Dirac equation for the case of a charged particle with an anomalous magnetic moment or an electric moment interacting with a restricted class of electromagnetic fields. Solutions for the case of an anomalous magnetic moment have been presented also by Becker and Mitter [19] who solved directly the Dirac equation. Another solution to Volkov's problem appears in the paper of Bergou and Varro [8]. They rederive the solutions given by Alperin in 1944 [16], using Majorana representation. These solutions are connected by an unitary transformation to the Volkov solutions. New classes of solutions are presented recently by Bagrov *et al.* [20].

### 3. THE VOLKOV SPINORS

We follow the conventions of Bjorken and Drell [14] for the metric tensor and Dirac matrices. Using the four-vectors  $n = (1, \mathbf{n})$ ,  $A = (0, \mathbf{A})$ ,  $x = (ct, \mathbf{r})$  we transcribe the variable  $\tau$  in (2) and the transversality condition as

$$\tau = \frac{n \cdot x}{c}, \quad n \cdot A = 0. \quad (3)$$

We introduce the dimensionless vector  $\mathbf{a}$  and the associated four-vector  $a$ , given respectively by

$$\mathbf{a} = \frac{e \mathbf{A}}{mc}, \quad a = (0, \mathbf{a}), \quad (4)$$

and write the Dirac equation (1) as an eigen value equation

$$\left( \hat{P} - mc\hat{a} \right) \psi = mc\psi, \quad (5)$$

where  $\hat{P} = i\hbar(\gamma_0 \partial/\partial ct + \boldsymbol{\gamma} \cdot \nabla)$ ,  $\boldsymbol{\gamma}$  are four Dirac matrices that anticommute and have the properties

$$\gamma_0^2 = -\gamma_k^2 = 1, \quad \gamma_0^\dagger = \gamma_0, \quad \gamma_k^\dagger = -\gamma_k, \quad k = 1, 2, 3 \quad (6)$$

and we have introduced the notation  $\hat{v} = \gamma_0 v_0 - \boldsymbol{\gamma} \cdot \mathbf{v}$ , for any four-vector  $v$ .

We give here a brief description of the Volkov spinors, needed in the following.

To any real vector  $\mathbf{p}$ , having the dimension of a momentum, one associates the four-vector  $p = (E/c, \mathbf{p})$  with

$$E \equiv \sqrt{m^2 c^4 + c^2 \mathbf{p}^2} > 0. \quad (7)$$

For each vector  $\mathbf{p}$ , Volkov has found four linearly independent solutions of the Dirac equation, which can be written compactly as

$$\Psi_{\mathbf{p}\sigma}(\mathbf{r}, t) = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left\{\frac{i}{\hbar}[\mathbf{p} \cdot \mathbf{r} - E_\sigma t + \Lambda_{\mathbf{p}\sigma}(\tau)]\right\} \Omega_{\mathbf{p}\sigma}(\tau) \xi_{\mathbf{p}\sigma}, \quad \sigma = 1, \dots, 4, \quad (8)$$

with

$$E_\sigma = E \quad \text{for } \sigma = 1, 2, \quad E_\sigma = -E \quad \text{for } \sigma = 3, 4 \quad (9)$$

and

$$\Lambda_{\mathbf{p}\sigma}(\tau) = \frac{(mc^2)^2}{2(E_\sigma - c\mathbf{p} \cdot \mathbf{n})} \int_{-\infty}^{\tau} \left[ 2 \frac{\mathbf{p}}{mc} \cdot \mathbf{a}(\chi) - \mathbf{a}^2(\chi) \right] d\chi, \quad (10)$$

$$\Omega_{\mathbf{p}\sigma}(\tau) = I + \frac{mc^2}{2(E_\sigma - c\mathbf{p} \cdot \mathbf{n})} \hat{n} \hat{a}. \quad (11)$$

We note that in (10) the choice  $-\infty$  for the lower limit of the integral makes sense for the case of a finite pulse only, when one has  $\lim_{\tau \rightarrow \pm\infty} \mathbf{A}(\tau) = 0$ ; for a monochromatic field the indefinite integral is used. For  $\mathbf{A} = 0$  the solutions reduce to the free spinors  $u_{\mathbf{p}\sigma}(\mathbf{r}, \tau)$  given by

$$u_{\mathbf{p}\sigma}(\mathbf{r}, t) = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left[\frac{i}{\hbar}(\mathbf{p} \cdot \mathbf{r} - E_\sigma t)\right] \xi_{\mathbf{p}\sigma}. \quad (12)$$

These are solutions of the free particle Dirac equation, corresponding to positive energies for  $\sigma = 1, 2$  and for negative energies for  $\sigma = 3, 4$ . The constant spinors  $\xi_{\mathbf{p}\sigma}$  satisfy the equation

$$(c\boldsymbol{\gamma} \cdot \mathbf{p} + mc^2) \xi_{\mathbf{p}\sigma} = E_\sigma \gamma_0 \xi_{\mathbf{p}\sigma} \quad (13)$$

and are normalized according to

$$\xi_{\mathbf{p}\sigma}^\dagger \xi_{\mathbf{p}\sigma} = 1, \quad \sigma = 1, \dots, 4. \quad (14)$$

Also, if in the limit  $\tau \rightarrow -\infty$  one has  $\mathbf{A}(\tau) \rightarrow 0$ , then for  $t \rightarrow -\infty$  the Volkov solutions reduce to free solutions.

Alternatively, we shall use the functions  $\Lambda_{\mathbf{p}\pm}$

$$\Lambda_{\mathbf{p}\pm}(\tau) = \pm \frac{(mc^2)^2}{2(E \mp c\mathbf{p} \cdot \mathbf{n})} \int_{-\infty}^{\tau} \left[ 2 \frac{\mathbf{p}}{mc} \cdot \mathbf{a}(\chi) - \mathbf{a}^2(\chi) \right] d\chi, \quad (15)$$

connected with the functions in (10) by  $\Lambda_{\mathbf{p}\sigma} = \Lambda_{\mathbf{p}+}$  for  $\sigma = 1, 2$  and  $\Lambda_{\mathbf{p}\sigma} = \Lambda_{\mathbf{p}-}$  for  $\sigma = 3, 4$ , and the matrices  $\Omega_{\mathbf{p}\pm}$

$$\Omega_{\mathbf{p}\pm}(\tau) = I \pm \frac{mc^2}{2(E \mp c\mathbf{p} \cdot \mathbf{n})} \hat{n} \hat{a} \quad (16)$$

connected with the matrices in (11) by  $\Omega_{\mathbf{p}\sigma} = \Omega_{\mathbf{p}+}$  for  $\sigma = 1, 2$  and  $\Omega_{\mathbf{p}\sigma} = \Omega_{\mathbf{p}-}$  for  $\sigma = 3, 4$ .

The simplest properties of the Volkov solutions are expressed by *three first order differential equations each of their components satisfy*. In order to write these equations, we use the decomposition of any vector in its transversal and longitudinal parts as

$$\mathbf{q} = \mathbf{q}_{\perp} + (\mathbf{q} \cdot \mathbf{n}) \mathbf{n}. \quad (17)$$

Using such a decomposition for the momentum operator  $\mathbf{P} = -i\hbar \nabla$  and for the vector  $\mathbf{p}$  that characterizes a Volkov solution, we have two equations expressed by

$$\mathbf{P}_{\perp} \psi_{\mathbf{p}\sigma}(\mathbf{r}, t) = \mathbf{p}_{\perp} \psi_{\mathbf{p}\sigma}(\mathbf{r}, t). \quad (18)$$

The third equation implies the operator

$$n \cdot P = P_0 - \mathbf{n} \cdot \mathbf{P} = i\hbar \left( \mathbf{n} \cdot \nabla + \frac{\partial}{\partial ct} \right), \quad (19)$$

and it reads

$$n \cdot P \psi_{\mathbf{p}\sigma}(\mathbf{r}, t) = (E_{\sigma}/c - \mathbf{n} \cdot \mathbf{p}) \psi_{\mathbf{p}\sigma}(\mathbf{r}, t). \quad (20)$$

Eq. (18) shows that Volkov solutions are characterized by a well-defined transversal momentum with respect to the direction of propagation of the electromagnetic plane wave.

The calculations implied in the next two sections become more compact if we notice that the following relation is valid:

$$\Omega_{\mathbf{p}\sigma} \xi_{\mathbf{p}\sigma} = \frac{\gamma_0 E_{\sigma} - c\boldsymbol{\gamma} \cdot (\mathbf{p} - mc\mathbf{a}) + mc^2}{2(E_{\sigma} - c\mathbf{p} \cdot \mathbf{n})} \hat{n} \xi_{\mathbf{p}\sigma}. \quad (21)$$

This is a generalization [21] of the corresponding relation given by Filipowicz [13] for the case of positive energy spinors. The proof of this relation implies only the equation (13) the spinors  $\xi_{\mathbf{p}\sigma}$  satisfy. In fact, this relation shows that the solutions given by Filipowicz in his Eq. (10) for the case of positive frequencies are identical with Volkov solutions, a property not obvious at the first sight.

#### 4. ORTHONORMALIZATION

We write the orthonormalization relation for Volkov spinors as:

$$\lim_{R \rightarrow \infty} \int_R \Psi_{\mathbf{p}\sigma}^\dagger(\mathbf{r}, t) \Psi_{\mathbf{p}'\sigma'}(\mathbf{r}, t) d\mathbf{r} = \delta(\mathbf{p} - \mathbf{p}') \delta_{\sigma\sigma'} \quad (22)$$

The limiting process implied by the  $\delta$  function is rendered here by the extension of the finite integration volume  $R$  to cover the whole space.

We sketch here a proof of the relation (22). We first decompose the position vector  $\mathbf{r}$  as in (17) and we use systematically the notations  $x_3 = \mathbf{r} \cdot \mathbf{n}$  and  $p_3 = \mathbf{p} \cdot \mathbf{n}$ .

The procedure we follow is somewhat similar to that used by Ritus [22] or, later, by Filipowicz [13], and is based on a change of variable in the integral over  $x_3$ , after the integration over  $\mathbf{r}_\perp$  has been performed.

We denote by  $P_{\mathbf{p}\sigma; \mathbf{p}'\sigma'}$  the scalar product on the left side of Eq. (22) and take the volume  $R$  as a cube of side  $L$ . The integral can be handled with elementary methods: for  $L \rightarrow \infty$ , the integral on the transversal variables  $\mathbf{r}_\perp$  leads directly to  $\delta$ -functions,

$$P_{\mathbf{p}\sigma; \mathbf{p}'\sigma'} = \frac{1}{2\pi\hbar} \delta(\mathbf{p}_\perp - \mathbf{p}'_\perp) \exp\left[\frac{i}{\hbar}(E_\sigma - E'_{\sigma'})t\right] J \quad (23)$$

with

$$J = \lim_{L \rightarrow \infty} \int_{-L}^L dx_3 S_{\mathbf{p}\sigma; \mathbf{p}'\sigma'}(\tau) \exp\left[\frac{i}{\hbar} Q_{\mathbf{p}\sigma; \mathbf{p}'\sigma'}(\tau; x_3)\right]. \quad (24)$$

In the integral  $J$  both the exponent

$$Q_{\mathbf{p}\sigma; \mathbf{p}'\sigma'}(\tau; x_3) \equiv (p'_3 - p_3)x_3 + \Lambda_{\mathbf{p}'\sigma'} - \Lambda_{\mathbf{p}\sigma} \quad (25)$$

and the matrix

$$S_{\mathbf{p}\sigma; \mathbf{p}'\sigma'}(\tau) \equiv \bar{\xi}_{\mathbf{p}\sigma} \gamma_0 \Omega_{\mathbf{p}\sigma}^\dagger(\tau) \Omega_{\mathbf{p}'\sigma'}(\tau) \xi_{\mathbf{p}'\sigma'}, \quad \bar{\xi}_{\mathbf{p}\sigma} \equiv \xi_{\mathbf{p}\sigma}^\dagger \gamma_0 \quad (26)$$

become simpler for equal transversal components of the vectors  $\mathbf{p}$  and  $\mathbf{p}'$ . In the following we use a *tilde* above any quantity in which we have  $\mathbf{p}'_\perp = \mathbf{p}_\perp$ . By doing this and using the notation

$$E_{\perp} = c\sqrt{\mathbf{p}_{\perp}^2 + m^2 c^2} \quad (27)$$

the expressions of  $Q$  and  $S$  become

$$\begin{aligned} \tilde{Q} \equiv Q_{\mathbf{p}\sigma; \mathbf{p}'\sigma'}(\tau; x_3) \Big|_{\mathbf{p}'_{\perp} = \mathbf{p}_{\perp}} &= (p'_3 - p_3) x_3 \\ &+ \frac{(mc^2)^2}{2E_{\perp}^2} [\tilde{E}'_{\sigma'} - E_{\sigma} + c(p'_3 - p_3)] \int_{-\infty}^{\tau} \left( 2 \frac{\mathbf{p}_{\perp}}{mc} \cdot \mathbf{a}(\tau') - \mathbf{a}^2(\tau') \right) d\tau', \end{aligned}$$

and

$$\tilde{S} \equiv S_{\mathbf{p}\sigma; \mathbf{p}'\sigma'}(\tau) \Big|_{\mathbf{p}'_{\perp} = \mathbf{p}_{\perp}} = \frac{1}{2} \left\{ 1 + c^2 \frac{(\mathbf{p}_{\perp} - m\mathbf{c}\mathbf{a})^2 + m^2 c^2}{(E_{\sigma} - cp_3)(\tilde{E}'_{\sigma'} - cp'_3)} \right\} \bar{\xi}_{\mathbf{p}\sigma} \hat{n} \xi_{\mathbf{p}'\sigma'}. \quad (28)$$

Then, the calculation proceeds differently in the cases in which the two spinors are attached to energies with the same sign or to energies with different signs.

In the first case, of energies  $E_{\sigma}$  and  $E'_{\sigma'}$  with the same sign, one writes

$$E_{\sigma} - \tilde{E}'_{\sigma'} = \frac{c^2(p_3^2 - p_3'^2)}{E_{\sigma} + \tilde{E}'_{\sigma'}}, \quad (29)$$

which makes possible the factorization

$$\tilde{Q} = (p'_3 - p_3) X_3 \quad (30)$$

with

$$X_3 = x_3 + \frac{m^2 c^5}{2E_{\perp}^2} \frac{\tilde{E}'_{\sigma'} + E_{\sigma} + c(p'_3 + p_3)}{E_{\sigma} + \tilde{E}'_{\sigma'}} \int_{-\infty}^{\tau} \left( 2 \frac{\mathbf{p}_{\perp}}{mc} \cdot \mathbf{a}(\tau') - \mathbf{a}^2(\tau') \right) d\tau'. \quad (31)$$

Then the change of variable from  $x_3$  to  $X_3$  appears as natural if one notices that  $\tilde{S}$  and  $\partial X_3 / \partial x_3$  are proportional. Indeed, the use of the identity

$$E_{\perp}^2 [\tilde{E}'_{\sigma'} + E_{\sigma} - c(p_3 + p'_3)] = (E_{\sigma} - cp_3)(\tilde{E}'_{\sigma'} - cp'_3) [\tilde{E}'_{\sigma'} + E_{\sigma} + c(p_3 + p'_3)] \quad (32)$$

leads to

$$\tilde{S} = F_1 \frac{\partial X_3}{\partial x_3}, \quad F_1 = \frac{\tilde{E}'_{\sigma'} + E_{\sigma}}{\tilde{E}'_{\sigma'} + E_{\sigma} - c(p_3 + p'_3)} \bar{\xi}_{\mathbf{p}\sigma} \hat{n} \xi_{\mathbf{p}'\sigma'}. \quad (33)$$

and then, going back to (24), we have

$$J \Big|_{\mathbf{p}_\perp = \mathbf{p}'_\perp} = F_1 \lim_{L \rightarrow \infty} \int_{-L}^L e^{\frac{i}{\hbar}(p'_3 - p_3) X_3} dX_3 = 2 \pi \hbar F_1 \delta(p_3 - p'_3) \quad (34)$$

and finally we get for the scalar product in (23)

$$P_{\mathbf{p}, \sigma; \mathbf{p}', \sigma'} = \delta(\mathbf{p} - \mathbf{p}') \delta_{\sigma \sigma'} \quad (35)$$

as for  $\mathbf{p} = \mathbf{p}'$  the exponential function becomes equal to 1 and the factor  $F_1$  becomes  $\delta_{\sigma, \sigma'}$ .

In the case of *energies*  $E_\sigma$  and  $E'_{\sigma'}$  with *different signs* one takes directly as the integration variable the exponent  $\tilde{Q}$ . Again the dependence on the integration variable is absorbed in the Jacobian of the transformation. The identity

$$E_\perp^2 [E_\sigma - \tilde{E}'_{\sigma'} + c(p'_3 - p_3)] = (E_\sigma - cp_3)(\tilde{E}'_{\sigma'} - cp'_3)[\tilde{E}'_{\sigma'} - E_\sigma + c(p'_3 - cp_3)] \quad (36)$$

allows to prove that

$$\tilde{S} = F_2 \frac{\partial \tilde{Q}}{\partial x_3}. \quad (37)$$

The expression of the constant factor  $F_2$  is irrelevant here. This way one gets

$$J \Big|_{\mathbf{p}'_\perp = \mathbf{p}_\perp} = F_2 \lim_{L \rightarrow \infty} \int_{-L}^L e^{\frac{i}{\hbar} \tilde{Q}} d\tilde{Q} = 0. \quad (38)$$

## 5. THE COMPLETENESS

As mentioned in the Introduction, to our knowledge a justification of the completeness property of the Volkov solutions was not given in the literature. Ritus [7] refers to the matrices

$$E_{\mathbf{p}\pm}(\mathbf{r}, t) \equiv \Omega_{\mathbf{p}\pm}(\tau) \exp \left\{ \frac{i}{\hbar} [\mathbf{p} \cdot \mathbf{r} \mp E t + \Lambda_{\mathbf{p}\pm}(\tau)] \right\} \quad (39)$$

present in the Volkov solutions (8), that we have transcribed here with the notations (10) and (11). The relation (6) of Ritus paper [7],

$$\int \frac{d^4 p}{(2\pi \hbar)^4} E_{\mathbf{p}\pm}(\mathbf{r}, t) \gamma_0 E_{\mathbf{p}'\pm}^\dagger(\mathbf{r}', t') \gamma_0 = \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (40)$$

implies a four-dimensional integration. This relation can be found also in [8].



Later, Bergou and Varro [8] have proved a completeness relation [expressed by their Eq.(2.12)] valid on the plane  $n \cdot x = \text{constant}$ . They have worked with the null-coordinates,

$$\zeta = x_3 - ct, \quad \eta = x_3 + ct, \quad (41)$$

and have shown that the Volkov solutions have an orthonormalization property with respect to a different scalar product defined using the variable  $\eta$  instead of  $x_3$  as the integration variable, inserting the matrix  $\gamma_3 - \gamma_0$  between  $\psi_{\mathbf{p}\sigma}^\dagger(\mathbf{r}, t)$  and  $\psi_{\mathbf{p}\sigma}(\mathbf{r}, t)$  and keeping the variable  $\xi$  constant instead of  $t$ .

We are interested here in *the completeness relation at constant time*:

$$\lim_{\Pi \rightarrow \infty} \int_{\Pi} d\mathbf{p} \sum_{\sigma=1}^4 \psi_{\mathbf{p}\sigma}(\mathbf{r}, t) \psi_{\mathbf{p}\sigma}^\dagger(\mathbf{r}', t) d\mathbf{p} = \delta(\mathbf{r} - \mathbf{r}') I, \quad (42)$$

where  $I$  is the four by four unit matrix. We have expressed the limiting process implied by the  $\delta$ -function by the expansion of the finite volume of integration  $\Pi$  to the whole momentum space, including eventually all Volkov solutions. Our proof is based on a direct evaluation of the integral on the left hand side of Eq. (42) under these conditions.

We remark that the limiting procedure considered here (which maintains unmodified the form of Volkov spinors) is not the only one possible. E.g. it would be possible to introduce a convergence factor in the Volkov spinors, but this would result in a different calculation.

We first separate the contribution of the positive and negative frequencies to the matrix under the integral sign

$$\begin{aligned} C(\mathbf{p}; \mathbf{r}, \mathbf{r}', t) &\equiv \sum_{\sigma=1}^4 \psi_{\mathbf{p}\sigma}(\mathbf{r}, t) \psi_{\mathbf{p}\sigma}^\dagger(\mathbf{r}', t) \\ &= \frac{1}{(2\pi\hbar)^{3/2}} \left[ C_+(\mathbf{p}; \tau, \tau') \exp(i D_+) + C_-(\mathbf{p}; \tau, \tau') \exp(i D_-) \right], \end{aligned} \quad (43)$$

where the matrices  $C_+$  and  $C_-$  are defined by

$$C_{\pm}(\mathbf{p}; \tau, \tau') = \Omega_{\mathbf{p}\pm}(\tau) P_{\pm} \Omega_{\mathbf{p}\pm}^\dagger(\tau') \quad (44)$$

and depend on  $\tau = t - \mathbf{r} \cdot \mathbf{n}/c$  and  $\tau' = t - \mathbf{r}' \cdot \mathbf{n}/c$ . By  $P_+$  and  $P_-$  we have denoted the projectors

$$P_+ = \sum_{\sigma=1,2} \xi_{\mathbf{p}\sigma} \xi_{\mathbf{p}\sigma}^\dagger, \quad P_- = \sum_{\sigma=3,4} \xi_{\mathbf{p}\sigma} \xi_{\mathbf{p}\sigma}^\dagger, \quad (45)$$

and the matrices  $\Omega_{\mathbf{p}\sigma\pm}$  are given by (16). The functions  $D_{\pm}$  in (43) are

$$\hbar D_{\pm}(\mathbf{p}; \mathbf{r}, \mathbf{r}', t) = \mathbf{p} \cdot (\mathbf{r} - \mathbf{r}') \pm \frac{1}{2} \frac{(mc^2)^2}{E \mp cp_3} \int_{t'}^t \left[ 2 \frac{\mathbf{p}}{mc} \cdot \mathbf{a}(\chi) - \mathbf{a}^2(\chi) \right] d\chi. \quad (46)$$

Concerning the integration limits in Eq. (42), we note that there is no difficulty in extending the integration limits over  $\mathbf{p}_{\perp}$  to the whole  $\mathbf{p}_{\perp}$  plane. The  $p_3$  integration will then carry the limiting process implied in Eq. (42) by restricting it to  $-L < p_3 < L$  and then allowing  $L \rightarrow \infty$ .

We then change the integration variables in Eqs. (42) and (43) from  $\mathbf{p}$  to  $(\mathbf{p}_{\perp}, v_{\pm})$ , depending on the term  $C_{\pm}(\mathbf{p}, \tau, \tau')$  considered, where

$$v_{\pm} = \frac{E \mp cp_3}{mc^2}. \quad (47)$$

The transformations map the interval  $-\infty < p_3 < +\infty$  onto  $0 < v_+ < \infty$ , respectively  $\infty > v_- > 0$ . But we have to refer to  $-L < p_3 < L$ , which for the  $v_{\pm}$  variables implies:  $\varepsilon(L) < v_+ < \varepsilon(L)$  and  $\varepsilon(L) > v_- > \varepsilon(L)$ , where  $\varepsilon(L)$  is a small number and  $\varepsilon(L)$  is a large number, i.e.  $\lim_{L \rightarrow \infty} \varepsilon(L) = 0$  and  $\lim_{L \rightarrow \infty} \varepsilon(L) = \infty$ .

One has further:

$$p_3 = \pm \frac{E_{\perp}^2 - m^2 c^4 v_{\pm}^2}{2 m c^3 v_{\pm}}, \quad E = \frac{E_{\perp}^2 + m^2 c^4 v_{\pm}^2}{2 m c^2 v_{\pm}} \quad (48)$$

with  $E_{\perp}$  given by Eq. (27). We also have:

$$\left| \frac{dp_3}{dv_{\pm}} \right| = \frac{E}{c v_{\pm}} \quad (49)$$

The functions  $D_{\pm}$  in (46) expressed in terms of the variables  $\mathbf{p}_{\perp}, v_{\pm}$  are:

$$D_{\pm}(\mathbf{r}, \mathbf{r}', t) = \frac{\mathbf{p}_{\perp}}{\hbar} \cdot (\mathbf{r} - \mathbf{r}') \mp b v_{\pm} \pm \frac{d}{v_{\pm}}, \quad (50)$$

where

$$b = \frac{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}')}{2 \lambda_e}, \quad d = \frac{c}{2 \lambda_e} \int_{t'}^t \left[ \left( \frac{\mathbf{p}_{\perp}}{mc} - \mathbf{a} \right)^2 + 1 \right] d\chi, \quad (51)$$

with  $\lambda_e$  the Compton wavelength. It is important to notice in the following that the functions  $b$  and  $d$  have always the same sign, as  $x_3 - x'_3$  and  $\tau' - \tau$  have the same sign.

The matrices  $C_+$  and  $C_-$  in (43) are evaluated using the identity (21) and the expression of the projectors

$$P_{\pm} = \frac{E \mp mc^2 \gamma_0 \mp c \boldsymbol{\gamma} \cdot \mathbf{p}}{2E} \quad (52)$$

After noticing that  $\hat{n} P_{\pm} \gamma_0 \hat{n} = (mc^2 / E v_{\pm}) \hat{n}$ , the variables  $v_+$  and  $v_-$  systematically used in the expression of the matrices  $C_+$  and  $C_-$ , respectively, and after elementary transformations one obtains

$$C_{\pm}(\mathbf{p}; \tau, \tau') = \frac{mc^2 v_{\pm}}{4E} \left[ \Gamma_0 \pm i \frac{\Gamma_{-1}}{v_{\pm}} + \frac{\Gamma_{-2}}{v_{\pm}^2} \right], \quad (53)$$

where  $\Gamma_0, \Gamma_{-1}$  and  $\Gamma_{-2}$  are four by four matrices. The matrix  $\Gamma_0$  is simply

$$\Gamma_0 = mc (1 - \gamma_0 \boldsymbol{\gamma} \cdot \mathbf{n}) \quad (54)$$

while the other matrices  $\Gamma_{-1}$  and  $\Gamma_{-2}$  have the expressions

$$\begin{aligned} \Gamma_{-1}(p_{\perp}; \tau, \tau') &= \\ &= -i mc \left\{ \left[ 2 \left( \frac{\hat{p}_{\perp}}{mc} + 1 \right) - \hat{a}(\tau) - \hat{a}(\tau') \right] \gamma_0 + [\hat{a}(\tau) - \hat{a}(\tau')] \boldsymbol{\gamma} \cdot \mathbf{n} \right\} \end{aligned} \quad (55)$$

and

$$\begin{aligned} \Gamma_{-2}(p_{\perp}; \tau, \tau') &= \\ &= mc \left[ \frac{E_{\perp}^2}{m^2 c^4} + \hat{a}(\tau) \frac{\hat{p}_{\perp}}{mc} + \frac{\hat{p}_{\perp}}{mc} \hat{a}(\tau') - \hat{a}(\tau) \hat{a}(\tau') + \hat{a}(\tau') - \hat{a}(\tau) \right] \times (1 + \gamma_0 \boldsymbol{\gamma} \cdot \mathbf{n}). \end{aligned} \quad (56)$$

Finally, using the notation  $v$  for both the integration variables  $v_+$  and  $v_-$  and using the integration limits mentioned before, we write Eq. (42) as

$$\begin{aligned} \lim_{\Pi \rightarrow \infty} \int_{\Pi} d\mathbf{p} C(\mathbf{p}; \mathbf{r}, \mathbf{r}') &= \int_{\infty} d\mathbf{p}_{\perp} \lim_{L \rightarrow \infty} \int_{-L}^{+L} dp_3 C(\mathbf{p}; \mathbf{r}, \mathbf{r}') \\ &= \int_{\infty} d\mathbf{p}_{\perp} \lim_{\epsilon \rightarrow 0, \epsilon \rightarrow \infty} \int_{\epsilon}^{\epsilon} dv C(\mathbf{p}_{\perp}, v; \mathbf{r}, \mathbf{r}'). \end{aligned} \quad (57)$$

In the integral over  $\mathbf{p}_{\perp}$  the positive and negative energy contributions combine and give three distinct integrals in the variable  $v$ , denoted by  $S_{-1}$ ,  $C_0$  and  $C_{-2}$ :

$$\begin{aligned}
& \int_{\infty} d\mathbf{p}_{\perp} \lim_{\varepsilon \rightarrow 0, \epsilon \rightarrow \infty} \int_{\varepsilon}^{\epsilon} dv C(\mathbf{p}_{\perp}, v; \mathbf{r}, \mathbf{r}') = \\
& = \frac{1}{(2\pi\hbar)^3} \int_{\infty} d\mathbf{p}_{\perp} e^{\frac{i}{\hbar} \mathbf{p}_{\perp} \cdot (\mathbf{r} - \mathbf{r}')} \times \frac{1}{2} \lim_{\varepsilon \rightarrow 0, \epsilon \rightarrow \infty} [\Gamma_0 C_0(b, d; \varepsilon, \epsilon) + \\
& + \Gamma_{-1}(\mathbf{p}_{\perp}; \tau, \tau') S_{-1}(b, d; \varepsilon, \epsilon) + \Gamma_{-2}(\mathbf{p}_{\perp}; \tau, \tau') C_{-2}(b, d; \varepsilon, \epsilon)]. \quad (58)
\end{aligned}$$

We have defined here the integrals:

$$S_{-1}(b, d; \varepsilon, \epsilon) \equiv \int_{\varepsilon}^{\epsilon} \frac{\sin\left(bv - \frac{d}{v}\right) dv}{v}, \quad (59)$$

$$C_n(b, d; \varepsilon, \epsilon) \equiv \int_{\varepsilon}^{\epsilon} v^n \cos\left(bv - \frac{d}{v}\right) dv, \text{ for } n=0, -2. \quad (60)$$

They are calculated in the Appendix for the case  $b, d > 0$ .

According to (74), (68) and (75) and taking into account that in our case the parameters  $b$  and  $d$  have the same sign, in the limit  $\varepsilon \rightarrow 0$  and  $\epsilon \rightarrow \infty$ , we replace the integral  $S_{-1}$  by 0, the integral  $C_0$  by  $\pi \delta(b)$ , and the integral  $C_{-2}$  by  $\pi \delta(d)$ . We then notice that  $b$  and  $d$  vanish for  $x_3 = x'_3$ , so we have

$$\delta(b) = 2 \lambda_e \delta(x_3 - x'_3), \quad \delta(d) = \frac{2 \lambda_e}{\left(\frac{\mathbf{p}_{\perp}}{mc} - \mathbf{a}\right)^2 + 1} \delta(x_3 - x'_3). \quad (61)$$

Once we have performed the integral over the variable  $v$ , the presence of  $\delta(x_3 - x'_3)$  leads to a simplified expression for  $\Gamma_{-2}$ :

$$\Gamma_{-2}|_{x_3=x'_3} = mc \left[ \left( \frac{\mathbf{p}_{\perp}}{mc} - \mathbf{a} \right)^2 + 1 \right] (1 + \gamma_0 \boldsymbol{\gamma} \cdot \mathbf{n}). \quad (62)$$

Replacing the integrals  $C_0$ ,  $S_{-1}$  and  $C_{-2}$  and the previous form of  $\Gamma_{-2}$  in (58) we get

$$\begin{aligned}
& \int_{\infty} d\mathbf{p}_{\perp} \lim_{\varepsilon \rightarrow 0, \epsilon \rightarrow \infty} \int_{\varepsilon}^{\epsilon} dv C(\mathbf{p}_{\perp}, v; \mathbf{r}, \mathbf{r}') = \\
& \frac{1}{(2\pi\hbar)^2} \int_{\infty} d\mathbf{p}_{\perp} e^{\frac{i}{\hbar} \mathbf{p}_{\perp} \cdot (\mathbf{r} - \mathbf{r}')} \left( \frac{1}{2} (1 + \gamma_0 \boldsymbol{\gamma} \cdot \mathbf{n}) + \frac{1}{2} (1 - \gamma_0 \boldsymbol{\gamma} \cdot \mathbf{n}) \right) \delta(x_3 - x'_3), \quad (63)
\end{aligned}$$

i.e., we have obtained

$$\lim_{\Pi \rightarrow \infty} \int_{\Pi} d\mathbf{p} C(\mathbf{p}; \mathbf{r}, \mathbf{r}', t) = \delta(\mathbf{r} - \mathbf{r}') I. \quad (64)$$

This way the relation (42) is justified.

*Acknowledgements.* The authors are grateful to Mihai Gavrilă for his invaluable guidance and encouragements. This work was supported by ANCS project number CEEEX-D11-56/2005 and by CNCSIS-UEFISCSU, project number 488 PNII – IDEI code 1909/2008.

## 6. APPENDIX: THE INTEGRALS $S_{-1}$ , $C_0$ , $C_{-2}$

Our proof of the completeness relation in Sect. 4 implies the integrals defined in (59) and (60). We first consider the integral (59),

$$S_{-1}(b, d; \varepsilon, \epsilon) \equiv \int_{\varepsilon}^{\epsilon} \frac{\sin\left(bv - \frac{d}{v}\right)}{v} dv. \quad (65)$$

We refer to the case  $bd > 0$  only, the case we are interested in, and make the change of variable

$$v = \frac{d}{b} \frac{1}{u} \quad (66)$$

which leads to

$$S_{-1}(b, d; \varepsilon, \epsilon) = -S_{-1}\left(b, d; \frac{d}{b\epsilon}, \frac{d}{b\varepsilon}\right). \quad (67)$$

It follows that

$$\lim_{\varepsilon \rightarrow 0, \epsilon \rightarrow \infty} S_{-1}(b, d; \varepsilon, \epsilon) = 0 \quad \text{for } bd > 0. \quad (68)$$

We notice that this result is in agreement with Eq. (3.863.3) in [23].

Now we consider integrals defined in (60)

$$C_0(b, d; \varepsilon, \epsilon) \equiv \int_{\varepsilon}^{\epsilon} \cos\left(bv - \frac{d}{v}\right) dv, \quad C_{-2}(b, d; \varepsilon, \epsilon) \equiv \int_{\varepsilon}^{\epsilon} \frac{\cos\left(bv - \frac{d}{v}\right)}{v^2} dv. \quad (69)$$

The two integrals are connected by

$$C_{-2}(b, d; \varepsilon, \epsilon) = C_0\left(d, b; \frac{1}{\epsilon}, \frac{1}{\varepsilon}\right). \quad (70)$$

In order to analyze the integral  $C_0$  we write it as

$$C_0(b, d; \varepsilon, \epsilon) = \frac{1}{2} \left( \int_{-\epsilon}^{-\varepsilon} + \int_{\varepsilon}^{\epsilon} \right) \exp \left[ i \left( b v - \frac{d}{v} \right) \right] dv . \quad (71)$$

In the case  $b > 0, d > 0$ , we form a closed integration path by adding two semicircles, one of radius  $\epsilon$  and the other of radius  $\varepsilon$ , in the upper semiplane of the complex plane. Cauchy theorem gives

$$C_0(b, d; \varepsilon, \epsilon) + C_{\varepsilon} + C_{\epsilon} = 0 , \quad (72)$$

where  $C_{\varepsilon}$  and  $C_{\epsilon}$  are the contributions of the two semicircles. For large values of  $\epsilon$  one has

$$C_{\epsilon} = - \frac{\exp(ibz)}{ib} \Big|_{z=-\epsilon}^{z=\epsilon} + O\left(\frac{1}{\epsilon}\right) = -2 \frac{\sin(b\epsilon)}{b} + O\left(\frac{1}{\epsilon}\right) . \quad (73)$$

The integral  $C_{\varepsilon}$  goes to zero for  $\varepsilon \rightarrow 0$ .

In the case  $b < 0, d < 0$  we proceed in a similar way closing the integration path by two semicircles of radii  $\varepsilon$  and  $\epsilon$  located in the lower semiplane of the complex plane. The contribution of the two semicircles are equal to those derived before. This way we reach the result

$$\lim_{\varepsilon \rightarrow 0, \epsilon \rightarrow \infty} C_0(b, d; \varepsilon, \epsilon) = \pi \delta(b) , \quad b, d \geq 0 \quad (74)$$

Finally, using (70), we get

$$\lim_{\varepsilon \rightarrow 0, \epsilon \rightarrow \infty} C_{-2}(b, d; \varepsilon, \epsilon) = \pi \delta(d) , \quad b, d \geq 0 \quad (75)$$

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