Dedicated to Professor Mihai Gavrilă's 80th Anniversary

EFFECTIVE DESCRIPTION OF BILOCAL FIELD THEORIES

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We discuss a specific representation of scalar noncommutative field theory, in which space noncommutativity is traded for bilocality of the fundamental degrees of freedom. The theory is causal and unitary, but somewhat tricky to interpret in a first quantized-context. The form of this effective description, and in particular the nature of the interaction potential, is discussed in the present paper.

1. INTRODUCTION

Consider a scalar field ϕ depending on the space-time coordinates x^1, x^2, x^3 and $t \equiv x^{\text{o}}$. The field theory is called noncommutative (NC) if some of these four independent variables have a nontrivial operatorial character, namely

$$\phi = \phi\left(\hat{x}^{\mu}\right), \quad \left[\hat{x}^{\mu}, \hat{x}^{\nu}\right] = i\theta^{\mu\nu}\left(\hat{x}\right) \neq 0, \quad \mu, \nu = 0, 1, 2, 3 \tag{1}$$

We will consider here the case in which only the x and y coordinates do not commute,

$$\left[\hat{x}, \hat{y}\right] = i\theta I, \tag{2}$$

I being the identity operator and θ a constant with the dimension of an area. Although the NC field theory literature [1] is quite rich, some important structural issues are still open.

One problem appearing in NC field theory is that the nature of the corresponding first-quantized formulation is not clear. In particular, the so-called NC mechanics (for a nice review see [2]) is not the one-particle sector of NC field theory, contrary to quite common beliefs. Our first objective in that paper is to clarify the "particle" aspect of NC field theory.

Another problem is related to the almost exclusive use of the Weyl-Moyal quantization procedure. Through it, NC space is mapped to a continuum of the same dimensionality (four, in our case) parametrized by the so-called Weyl symbols, whereas the usual product of functions depending on the Weyl symbols is mapped to the so-called Moyal star-product

$$f(x)\cdot g(x) \;\to\; f(x)*g(x)\equiv \lim_{y\to x}\exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y\right)f(x)g(y). \qquad (3)$$

The correspondence to physical space (assumed NC) is at best statistical, since a ”point” in Weyl symbol space has no precise correspondent in the physical (NC) space. This renders the usual, commutative space, micro-causality condition

$$\left[\phi(x),\phi(0)\right]=0, \quad t^2-\vec{x}^2\le 0 \qquad (4)$$

difficult to generalize to NC space, as it is not clear anymore what a space-like interval means. An additional ambiguity is that it is not clear whether one should consider the commutator or the star-commutator of two fields [3] in attempting to generalize the causality condition (4).

We first review and develop our previous formulation [4] of NC fields in terms of bilocal but commuting objects. The peculiar nature of the theory describing such fundamental degrees of freedom appears clearly in this representation of NC space. Subsequently we apply it to discuss a first-quantized description and causality.

2. BILOCALITY

Consider a (2+1)-dimensional scalar $\Phi(t,\hat{x},\hat{y})$, defined over a commuting time t and a pair of NC coordinates satisfying (2). The operators $\hat{x}$ and $\hat{y}$ act on a harmonic oscillator Hilbert space $\mathcal{H}$ in the usual way. Out of an infinite number of possibilities, $\mathcal{H}$ may be given a discrete basis {|n>} formed by eigenstates of $\hat{x}^2+\hat{y}^2$ [5], or a continuous one {|x>}, composed of eigenstates of, say, $\hat{x}$ [4]. We will use this last basis.

To quantize Φ [4], transform the classical normal modes coefficients a and a^* into the usual creation $\hat{a}$ and annihilation $\hat{a}^\dagger$ operators acting on the standard Fock space $\mathcal{F}$. To make the underlying space noncommutative, introduce (2) and apply the Weyl (not Weyl-Moyal!) quantization procedure to the exponentials $e^{i(k_x x+k_y y)}$. The result is

$$\Phi=\int\int\frac{dk_x dk_y}{2\pi\sqrt{2\omega_{\vec{k}}}}\left[\hat{a}_{k_x k_y}e^{i(\omega_{\vec{k}}t-k_x\hat{x}-k_y\hat{y})}+\hat{a}^\dagger_{k_x k_y}e^{-i(\omega_{\vec{k}}t-k_x\hat{x}-k_y\hat{y})}\right]. \qquad (5)$$

Φ is thus a ‘doubly’-quantum field operator, acting on a direct product of two Hilbert spaces, Φ : $\mathcal{F}\otimes\mathcal{H}\rightarrow\mathcal{F}\otimes\mathcal{H}$. Φ creates (destroys), *via* $\hat{a}^{\dagger}_{k_x k_y}$ $\left(\hat{a}_{k_x k_y}\right)$, an excitation represented by an ”operatorial plane wave” $e^{i\left(\omega_{\vec{k}}t-k_x\hat{x}-k_y\hat{y}\right)}$.

One may work with Φ as an operator ready to act on both $\mathcal{F}$ and $\mathcal{H}$. It is however simpler to saturate its action on $\mathcal{H}$, working with expectation values $<x'|\Phi|x>$: $\mathcal{F}\rightarrow\mathcal{F}$. At this point *bilocality* appears. For, consider the family {|x>} of eigenstates of $\hat{x}$: $\hat{x}|x>=x|x>, \hat{y}|x>=-i\theta\frac{\partial}{\partial x}|x>$. A simple but key equation involving the ”operatorial plane wave” is

$$<x'|e^{i\left(k_x\hat{x}+k_y\hat{y}\right)}|x>=e^{ik_x\left(x+k_y\theta/2\right)}\delta\left(x'-x-k_y\theta\right)=e^{ik_x\frac{x+x'}{2}}\delta\left(x'-x-k_y\theta\right). \tag{6}$$

This is a bilocal expression, and we already see that its span along the x axis, $(x'-x)$, is proportional to the momentum along the conjugate y direction, *i.e.* $(x'-x)=\theta k_y$. The energy follows from the modified dispersion relation [4]

$$\omega_{\vec{k}}=\sqrt{k_x^2+\frac{\Delta x^2}{\theta^2}+m^2}\,, \tag{7}$$

m being the mass appearing in the field Lagrangian. In general, given n independent pairs of NC directions, one can keep only one coordinate out of every pair; commutativity is gained on the reduced space, at the expense of locality. Using Eqs. (5) and (6) one sees that

$$<x'|\Phi|x>=\int\frac{dk_x}{2\pi\sqrt{2\omega_{k_x,k_y}}}\left[\hat{a}_{k_x,k_y}e^{i\left(\omega_{\vec{k}}t-k_x\frac{x+x'}{2}\right)}+a^{\dagger}_{k_x,-k_y}e^{-i\left(\omega_{\vec{k}}t+k_x\frac{x+x'}{2}\right)}\right], \tag{8}$$

where $k_y=(x'-x)/\theta$. Thus, Φ annihilates a linear combination of rods of (arbitrary) momentum k_x and (fixed) length θk_y, and creates rods of momentum k_x and length $-\theta k_y$. It is not anymore a local operator, in contrast to usual field theory. Failure to recognize that feature explicitely in the Moyal formulation may lead to erroneous conclusions, e.g. about causality.

3. EFFECTIVE DESCRIPTION

Let us introduce the representation of the previous section in the equations of motion, and reintroduce the commutative z direction.

We use the notation $< x'|\phi|x > \equiv \phi(x', x) \equiv \phi(\bar{x}, \Delta x)$, with $\bar{x} \equiv \frac{x + x'}{2}$ being the center of the dipole and $\Delta x \equiv (x' - x)$ its length. The free equations of motion are written operatorially for $\phi(t, \hat{x}, \hat{y}, z)$ as

$$\left(\partial_t^2 - \partial_z^2 + m^2\right)\phi + \frac{1}{\theta^2}\left[\hat{y}, \left[\hat{y}, \phi\right]\right] + \frac{1}{\theta^2}\left[\hat{x}, \left[\hat{x}, \phi\right]\right] = 0, \tag{9}$$

which is nothing else than an operatorial wave equation, given that (2) implies

$$\left[\hat{x}, \phi\left(\hat{x}, \hat{y}\right)\right] = i\theta\frac{\partial\phi}{\partial\hat{y}} \qquad \left[\hat{y}, \phi\left(\hat{x}, \hat{y}\right)\right] = -i\theta\frac{\partial\phi}{\partial\hat{x}}. \tag{10}$$

In fact, these equations follow from the free-field action

$$S = Tr_H \int dt \int dz \left(\left(\dot{\phi}\right)^2 + \frac{1}{\theta^2}\left[\hat{x}, \phi\right]^2 + \frac{1}{\theta^2}\left[\hat{y}, \phi\right]^2 - \left(\partial_z\phi\right)^2 + m^2\phi^2 \right), \tag{11}$$

where the usual integral $\int dx \int dy$, which does not make sense in a NC space, is replaced by the trace Tr_H over the (harmonic oscillator) Hilbert space on which $\hat{x}$ and $\hat{y}$ act. Sandwiching the operatorial equation between $|x>$ states, one gets rid of noncommutativity and obtains the wave equation

$$\left(\partial_t^2 - \partial_{\bar{x}}^2 - \partial_z^2 + \frac{\left(x' - x\right)^2}{\theta^2} + m^2\right)\phi\left(x, x'\right) = 0. \tag{12}$$

for a dipole living in (2+1) commutative dimensions at $t, \bar{x}, z$ and having extension Δx. We notice the full agreement with the dispersion relation (7). Eq. (12) is proved as follows. First, it is trivial to show that

$$< x'\left|\left[\hat{x}, \left[\hat{x}, \phi\right]\right]\right|x > = \left(x' - x\right)^2 \phi .$$

Second, by operating with the commutator in $\phi(\hat{x}, \hat{y})$, Fourier expanding, then sandwiching between |x> bras and kets, one can rewrite

$$< x'\left|\left[\hat{y}, \left[\hat{y}, \phi\right]\right]\right|x > = -\partial_{\bar{x}}\phi\left(\bar{x}, \Delta x\right)$$

and Eq. (12) is proved. In the interacting case, the relevant Lagrangian is

$$2L = \left(\partial_t\phi\right)^2 - \left(\partial_{\bar{x}}\phi\right)^2 + \left[\left(\theta^{-1}\Delta x\right)^2 + m^2\right]\phi^2 - 2V\left(\phi\right) \tag{13}$$

and includes the potential $V(\phi)$ for the fields, for instance $V = \lambda\phi^4$. The Lagrangian L has the property of being invariant under Lorentz boosts along the $\overline{x}$ -axis, as well as along the z-axis, independently. The only think to prove in this respect is the invariance of the third term in the RHS. This immediately follows from the tensorial character of $\theta = \theta_{xy} \sim xy$ and the usual Lorentz transformation of Δx. These symmetries, which we found in the bilocal representation for NC space, are at variance with the claims usually made within the Moyal approach, that the symmetry group is the product between the rotation group O(2) in the $x - y$ NC space and the Lorentz group O(1, 1) in $t - z$ space.

Now, the question is how would a Schrödinger equation look-like for a system of two dipoles (at low energies, say). For two dipoles with end-points labelled by x_1, x_2 and x_3, x_4 respectively, one should see what is the wave-function, and what is the potential, *e.g.* one may try

$$\psi = \psi\left(x_1, x_2; x_3, x_4\right), \qquad V = V\left(x_1, x_2; x_3, x_4\right) \tag{14}$$

As we will see, in the NC case the effective potential between two fixed dipole sources depends in fact on the separation between the dipole centers and on their common length.

In the commutative case, one approach is as follows (see e.g. Ref. [6], pp. 39-42). Consider two sources J_1 and J_2 coupled to a scalar field, and the free scalar field propagator

$$G(x) = \int d^4k \frac{e^{-ik\cdot x}}{k^2 + m^2} \tag{15}$$

where x and k are four-dimensional vectors. The $i\epsilon$ prescription is tacitly understood; writing it explicitly is not needed in this section. The affective potential between J_1 and J_2 results from $S_{eff} = \int dt V_{eff}$, with

$$S_{eff}\left(J_1, J_2\right) = \int d^4x \int d^4y J_1(x) G(x-y) J_2(y). \tag{16}$$

For static sources $J_1(x) = g_1\delta(\vec{x} - \vec{x}_0)$, $J_1(y) = g_1\delta(\vec{y} - \vec{y}_0)$, one obtains the Yukawa potential $V = \pi g_1 g_2 \frac{e^{-m|\vec{x}_0 - \vec{y}_0|}}{|\vec{x}_0 - \vec{y}_0|}$. We will begin by obtaining this result in a slightly unusual way, which is however easy to adapt to the NC situation. Since in the NC case the coordinate y ”disappears”, it is natural to Fourier transform along the y-coordinate (what actually labels the NC state is k_y). For the free propagator we have

$$G(t, x, y, z) = \int dk_y e^{+ik_y y} \tilde{G}\left(t, x, k_y, z\right), \tag{17}$$

with

$$\tilde{G}\left(t, x, k_y, z\right) = \int dk_0 \int dk_x \int dk_z \, \frac{e^{-ik_0 t + ik_x x + ik_z z}}{k_0^2 - k_x^2 - k_z^2 - \left(k_y^2 + m^2\right)}. \tag{18}$$

For the sources, we use the following notation. A source given in configuration space *x, y, z* and which is localized at x_0, y_0, z_0 is written

$$J_{[x_0, y_0, z_0]}^{(x,y,z)} = \delta\left(x - x_0\right)\delta\left(y - y_0\right)\delta\left(z - z_0\right) \tag{19}$$

whereas its Fourier transform, defined in x, k_y, z space, is

$$\tilde{J}_{[x_0, y_0, z_0]}^{(x,k_y,z)} = \delta\left(x - x_0\right)\delta\left(z - z_0\right) e^{-ik_y y_o}. \tag{20}$$

Indeed, $\int dk_y e^{ik_y y} \tilde{J}_{[x_0, y_0, z_0]}^{(x,k_y,z)} = J_{[x_0, y_0, z_0]}^{(x,y,z)}$. No time dependence will appear inside the sources in this paper.

The effective action describing the interaction of two sources J_1 and J_2 localized at x_1, y_1, z_1, respectively x_2, y_2, z_2 reads, cf. (16),

$$S_{eff}\left[J_1, J_2\right] = \int d^4 x \int d^4 \tilde{J}_{[x_1, y_1, z_1]}^{(x,y,z)} G\left(x_\mu - \tilde{x}_\mu\right) J_{[x_2, y_2, z_2]}^{(\tilde{x},\tilde{y},\tilde{z})}. \tag{21}$$

Fourier transforming along y and $\tilde{y}$ we get

$$S_{eff} = \int_{k_y} \int_t \int_{\tilde{t}} \int_x \int_{\tilde{x}} \int_z \int_{\tilde{z}} \tilde{J}_{[x_1, y_1, z_1]}^{(x,k_y,z)} \tilde{G}\left(t - \tilde{t}, x - \tilde{x}, k_y, z - \tilde{z}\right) \tilde{J}_{[x_2, y_2, z_2]}^{(\tilde{x},k_y,\tilde{z})}. \tag{22}$$

Expression (22) is important for us: since the *y* directions do not appear anymore in it, it applies equally well into the NC case.

Let us stay for the moment in the commutative case, and introduce in (22) the propagator (18) and sources of the type (20). Denoting the components of the distance between the two sources by $\vec{R} = \left(x, y, z\right)$, $x_1 - x_2 \equiv x$, $y_1 - y_2 \equiv$ $\equiv y$, $z_1 - z_2 \equiv z$, we obtain

$$S_{eff} \equiv \int dt V_{eff}\left(x, y, z\right) \equiv \int dt \int dk_y e^{ik_y y} V\left(x, z; k_y\right) \tag{23}$$

with $V(x, z; k_y)$, the *y*-Fourier transform of $V_{eff}\left(x, y, z\right)$, being

$$V\left(x, z; k_y\right) = -\int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_z \, \frac{e^{ik_x x} e^{ik_z z}}{k_x^2 + k_z^2 + \left(k_y^2 + m^2\right)}. \tag{24}$$

For simplicity in writing, we introduce the two-dimensional vectors $\vec{r} = \left(x, z\right)$ and $\vec{k} = \left(k_x, k_z\right)$, the first being of modulus $r \equiv \sqrt{x^2 + z^2}$. By passing to polar coordinates the integral in (24) becomes (cf. [7], p.336 or p.902)

$$V\left(x, z; k_y\right) = -2\pi \int_{-\infty}^{+\infty} dk \frac{kJ_0\left(kr\right)}{k^2 + M^2}. \tag{25}$$

J_0 denotes the Bessel function of order zero, and $M^2 \equiv m^2 + k_y^2$. This integral can also be evaluated ([7], p.663) leading to

$$V\left(x, z; k_y\right) = -i\pi^2 H_0^{(1)}\left(ir\, M\right) \to -\sqrt{\frac{2\pi^3}{rM}} e^{-rM}. \tag{26}$$

The $rM \to \infty$ expansion above (cf. [7], p. 910) of the Hankel function $H_0^{(1)}$ shows that the effective potential is actually real and that the modified dispersion relation (7) applies, the effective mass being indeed $M^2 = m^2 + k_y^2$. The potential (26) can be interpreted as describing the interaction of two (commutative) particles localized in the plane $x - z$, but completely delocalized along y (actually in momentum eigenstates along this direction).

Finally, Fourier transform to obtain (cf. [7], p.690)

$$V_{eff}\left(x, y, z\right) = -\int_{-\infty}^{+\infty} dk_y e^{ik_y y} H_0^{(0)}\left(ir\, M\right) = -\left(2\pi^2\right)\frac{e^{-mR}}{R}, \tag{27}$$

which is the Yukawa potential. We have verified that this approach gives sound results, and performed all the integrals we shall need in the NC case.

In the NC case, we again apply Eq. (22), with the important modification that the sources, although still defined in the space x, k_y, z, are now localized in k_y (recall the length of the dipole $\Delta x = k_y\theta$), not in y. They thus take the form

$$J_{\left[x_0, k_y^0, z_0\right]}^{\left(x, k_y, z\right)} = \delta\left(x - x_0\right)\delta\left(k_y - k_y^0\right)\delta\left(z - z_0\right); \tag{28}$$

however only minor changes occur with respect to previous manipulations. We finally obtain, for two sources localized in y-momentum at k_y and $\tilde{k}_y$,

$$V_{eff}^{NC} = -i\pi^2 H_0^{(0)}\left(i\sqrt{x^2 + z^2} \times \sqrt{m^2 + k_y^2}\right) \times \delta\left(k_y - \tilde{k}_y\right). \tag{29}$$

At infinite rM, we then have

$$V_{eff}^{NC}\left(x, z; k_y, \tilde{k}_y\right) \to -\sqrt{\frac{2\pi^3}{rM}} e^{-rM} \delta\left(k_y - \tilde{k}_y\right). \tag{30}$$

x, z describe the interdipole distance, $k_y\theta$ and $\tilde{k}_y\theta$ the extensions of the dipoles, and $H_0^{(1)}(x)$ is again the zero order Hankel function of the first kind, which decays as $\frac{1}{\sqrt{x}}$ as $x \to \infty$. We notice that the interaction potential depends only on the interdipole distance in the reduced space and on their (common) length. Several remarks are in order about this potential: 1. The effective mass is $M = k_y^2 + m^2$; the dependence of the potential on the dipoles' lengths enters only through it. 2. The decay at large distances in the plane *x*–*z* is like $1/\sqrt{r}$, not the 2D commutative log *r*. This is due to the additional momentum k_y entering. 3. The fact that the two sources are completely spread out along "*y*" leads also to the singularity $\delta\left(k_y - \tilde{k}_y\right)$. If we discretize the momentum k_y this is no longer an issue. 4. For inequal lenghts, the dipoles do not interact! This can be understood in the dual *y*-momentum view as no recoil due to the fixed nature of the sources.

4. CAUSALITY

It has been shown in [4] that *free* NC fields behave exactly like commutative fields living in a lower-dimensional space. In fact a free (1+1)-dimensional dipole [resulting from the 2 + 1 NC theory we were discussing] with endpoints situated at x and x' behaves like a commutative (1+1) point particle centered at $\frac{x+x'}{2}$, but with a modified dispersion relation $\omega^2 = k_x^2 + \frac{\left(x-x'\right)^2}{\theta^2}$. In conclusion all usual manipulations performed on propagators in (1 + 1)-dimensions can be carried over, including those used to demonstrate causality. This immediately shows that at the free level NC field theories are causal, contrary to previous claims [3].

For interacting fields, one expects the dipolar character of the degrees of freedom to manifest, as e.g. in perturbation theory [4]. It is however remarkable that as far as causality issues are concerned, bilocality has little influence, and a proof of causality can be given like for commutative theories. For, consider the vanishing of the commutator to hold,

$$\left[\phi\left(t_1, \bar{x}_1\right), \phi\left(t_2, \bar{x}_2\right)\right] = 0 \tag{31}$$

with $\bar{x}_1 = \frac{x_1 + x'_1}{2}$, $\bar{x}_2 = \frac{x_2 + x'_2}{2}$ being the average positions (centers of mass) of the two dipoles considered. We want (31) to be true when the interval, defined with respect to the average dipole positions, is space-like,

$$\left(t_1 - t_2\right)^2 - \left(\overline{x}_1 - \overline{x}_2\right)^2 \leq 0. \tag{32}$$

Equations (31, 32) are however generically equivalent to

$$\left[\phi\left(t, \overline{x}\right), \phi\left(t, \overline{y}\right)\right] = 0 \qquad \vec{x} \neq \vec{y}, \tag{33}$$

provided one can apply a boost along x to render equal the two times appearing in Eq. (31). Now, the (1 + 1)-dimensional dipole theory *is* invariant under boosts in the x-direction, as proved in the precedent section. In consequence, Eqs. (31, 32) are tantamount, *via* a boost, to

$$e^{iH't}\left[\phi\left(0, \overline{x}\right), \phi\left(0, \overline{y}\right)\right] e^{-iH't} = 0 \tag{34}$$

H' denotes the interacting part of the Hamiltonian - including V - in the interaction representation, or the full Hamiltonian in the Heisenberg representation. Now, Eq.(34) is true either at $t = -\infty$, when the fields are assumed to be free, or at some time t_0 if a causal configuration is assumed, for instance through the initial conditions. We stress that the above causality argument works for a fully interacting theory.

Adding now the (passive) commutative coordinate z, we conclude that the correct criterion for causality is

$$\left[\phi\left(t_1, \overline{x}_1, z_1\right), \phi\left(t_2, \overline{x}_2, z_2\right)\right] = 0, \;\; \left(t_1 - t_2\right)^2 - \left(\overline{x}_1 - \overline{x}_2\right)^2 - \left(z_1 - z_2\right)^2 \leq 0, \tag{35}$$

and that it *is* satisfied in NC field theory. Unitarity is self-evident once this representation is constructed. Finally, we notice that in formulating the causality criterion we had to drop only one spatial coordinate, in contradistinction to [8], where both NC coordinates were dropped.

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