ON THE FRACTIONAL-ORDER DIFFUSION-WAVE PROCESS

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One of the main applications of the fractional calculus, integration and differentiation of arbitrary orders is the modelling of the intermediate physical processes. Here we formulate a more general model which represents the diffusion wave process in all its cases, and give some examples discussing these different cases.

Key words: Evolution equation, fractional order derivative, Diffusion-Wave equation.

1. INTRODUCTION

Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. First there were almost no practical applications of this field, and it was considered by many as an abstract area containing only mathematical manipulations of little or no use. Nearly 30 years ago, the paradigm began to shift from pure mathematical formulations to applications in various fields. During the last decade Fractional Calculus has been applied to almost every field of science, engineering, and mathematics.

Several fields of application of fractional differentiation and fractional integration are already well established, some others have just started. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics [1–15]. In recent years, there has been a great deal of interest in fractional differential equations. Historical summaries of the developments of fractional calculus can be found in [1–4].

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One of the main applications of the fractional calculus is modelling of the intermediate physical process. A very important model is the fractional diffusion and wave equations. Many authors tried to model diffusion and wave equations from the classical diffusion or wave equation by replacing the first or second-order time derivative by a fractional derivative of order α with $0 < \alpha < 2$, (see [5, 6, 10, 16, 17, 18, 19, 20]). Mainardi (see [16]) defined the fractional diffusion equation by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\frac{\partial^\alpha u}{\partial x^2}, \quad 0<\alpha<1, \ D>0 \tag{1}$$

and the fractional wave equation by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\frac{\partial^\alpha u}{\partial x^2}, \quad 1<\alpha<2, \ D>0 \tag{2}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Riemman-Liouville fractional derivative. He discussed the two basic problems for both diffusion equation and wave equation

- The Cauchy problem is an initial value problem when the data are assigned at $t = 0^+$ on the space axis $-\infty < x < \infty$.

  $$u(x, 0) = g(x), \ -\infty < x < \infty \ \text{and} \ u(\pm\infty, t) = 0, t > 0$$

- The Signalling problem, considered in the domain $x, t \ge 0,$ is an initial boundary value problem when the data are assigned both at $t = 0^+$ on the semi-infinite time axis $x > 0$ (initial data) and $x = 0^+$ on the semi-infinite time axis $t > 0$ (boundary data).

  $$u(x, 0) = 0, x > 0, \text{and } u(0, t) = h(t), \ u(\infty, t) = 0, t > 0$$

If $1 < \alpha < 2$ the fractional wave problem we need to add an initial condition $u_t(x, 0) = 0$. El-Sayed (see [5]) define the abstract fractional order problem

$$D^\gamma u(t) = Au(t), \ t \in (0, T] \ \text{with } u(0) = u_o, \ \gamma \in (0,1) \tag{3}$$

and the abstract fractional order wave problem

$$D^\beta u(t) = Au(t), \ t \in (0, T] \ \text{with } u(0) = u_o, \ u_t(0) = 0, \ \beta \in (1,2) \tag{4}$$

with using the Caputo derivative and prove the existence and uniqueness of the solution under some conditions.

Gorenflo and Mainardi (see [17]) define the Feller space-fractional diffusion equation by

$$\frac{\partial u}{\partial t} = {}_x D_\theta^\alpha u, \ x \in R, \ t \ge 0, \ |\theta| \le \min\{\alpha, 2-\alpha\}, \ \alpha \in (0,2) \tag{5}$$

where ${}_x D_\theta^\alpha$ is the Riesz Feller space fractional derivative.

Mainardi (see [18]) replaced the Riemman Liouville fractional derivative in his work [16] by the Caputo derivative.

Mainardi, Luchko, Pagnini (see [10]) gave the definition of the space-time fractional diffusion equation

$$ {}_tD_*^{\beta}u(x,t) = {}_xD_{\theta}^{\alpha}u(x,t),\ x \in R,\ t \geq 0 \tag{6}$$

where α, β, θ are real parameters restricted as follows

$$0 < \alpha \leq 2,\ |\theta| \leq \min\{\alpha, 2-\alpha\},\ 0 < \beta \leq 2.$$

${}_xD_{\theta}^{\alpha}$ is the Riesz-Feller space fractional derivative of order α and skewness θ, and ${}_tD_*^{\beta}$ is the Caputo time fractional derivative of order β.

El-Sayed and M. Aly (see [6]) formulated a more accurate model of the abstract diffusion wave problem as

$$D^{\alpha}u(t) = \int_0^t h(t-s)Au(s)ds,\ t > 0,\ \alpha \in (0,1],\ u(0) = u_o \tag{7}$$

gave its solution and proved that it is a general model of diffusion-wave problem. W. Chen and S. Holm (see [19]) defined the fractional diffusion wave equation as

$$\frac{\partial^{\beta}u}{\partial t^{\beta}} = -k(-\Delta)^{\lambda/2}u,\ \ 0 < \lambda,\ \beta \leq 2 \tag{8}$$

where Δ is the Laplacian operator, and k denotes a physical constant. λ and β can be arbitrary real number.

Note that in each one of the previous fractional D-W process there are two equations one each for the fractional diffusion problem and the fractional wave problem.

The main purpose of this paper is to give a more general abstract model of homogeneous D-W equation which represents the D-W process in all cases.

The paper is organized as follows: In Section 2, we give the principal definitions and theorems used in this paper. In Section 3, we study our abstract fractional order D-W model

$$\frac{du(t)}{dt} = A\frac{d}{dt}I^{\gamma}u(t),\ u(0) = u_o,\ \gamma \in (0,2] \tag{9}$$

where A is a closed linear operator with dense domain $D(A) = X_A \subset X$, X is a Banach space, with giving some examples illustrate our model. Our conclusion is given in Section 4.

2. PRELIMINARIES

Let $f \in L(J,R)$ and let α be a positive real number.

Definition (fractional Riemman-Liouville integral) 2.1.

The fractional integral of order α of the function $f(t)$ is defined by (see [1–4])

$$I^{\alpha}f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds = f(t) * \phi_{\alpha}(t)$$

where $\phi_\alpha(t)=\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t>0$ and $\phi_\alpha(t)=0$ for $t\le 0$, and $\phi_\alpha(t)\to\delta(t)$ (the delta function) as $\alpha\to 0$.

For the fractional order derivative

Definition (Riemman-Liouville derivative) 2.2

The (Riemman-Liouville) fractional derivative of the function $f(t)$ of order $\alpha\in(n-1,n)$ is defined by (see [1–4])

$$\frac{d^\alpha}{dt^\alpha}f(t)=D^n I_\alpha^{n-\alpha}f(t)=\frac{d^n}{dt^n}\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}f(s)ds.$$

Definition (Caputo derivative) 2.3

The (Caputo) fractional derivative of order $\alpha\in(0,1)$ of the function $g(t)$ is defined by (see [1, 3])

$$D^\alpha g(t)=I^{1-\alpha}Dg(t),\ D\frac{d}{dt}.$$

The fractional derivative of order $\beta\in(n-1,n)$ of $g(t)$ is defined by

$$D^\beta g(t)=I^{n-\beta}D^n g(t),\quad D^n=\frac{d^n}{dt^n}.$$

Consider now the fractional order evolution problem

$$D^\alpha u(t)=Au(t)+f(t),\quad \alpha\in(0,1),\ u(0)=u_o. \tag{10}$$

Using the results of [21] we have (see [22])

Definition 2.3. A function $u\in C(J;X)$ is called a strong solution of (10) on J if $u\in C(J;X_A)$ and (10) holds on J;

Theorem 2.4. Let $\alpha\in(0,1)$, $u_o\in D(A)$, $f\in W^{1,1}(J,X_A)$ and A is the infinitesimal generator of a bounded C_o-semigroup $\{T(t),t\ge 0\}$. Then there is a unique strong solution $u_\alpha\in C(J,X_A)$ of (10) given by

$$u_\alpha(t)=e^tS_\alpha(t)u_o-\left(e^tS_\alpha(t)\right)*u_o+\left(e^tS_\alpha(t)\right)*\phi_\alpha(t)*\left(f(0)\delta(t)+\dot f(t)-f(t)\right) \tag{11}$$

where $S_\alpha\in C^{k-1}(J;B(X))$ is the resolvent operator with the resolvent equation

$$e^tS_\alpha(t)x=e^tx+\phi_\alpha(t)*\left(Ae^tS_\alpha(t)\right)x,\quad x\in D(A). \tag{12}$$

This solution satisfies the continuation property

$$\lim_{\alpha\to 1^-}u_\alpha(t)=u_1(t).$$

where $u_1(t)$ is the solution of the evolution equation

$$\frac{d}{dt}u(t) = Au(t) + f(t), \qquad u(0) = u_o. \tag{13}$$

Consider now the fractional order evolution problem

$$D^{\beta}u(t) = Au(t) + f(t),\ \alpha \in (0,1),\ u(0) = u_o,\ u_t(0) = 0 \tag{14}$$

we have (see [22]).

Definition 2.5. A function $u \in C(J;X)$ is called a strong solution of (14) on J if $u \in C(J;X_A) \bigcap C^1(J;x) \bigcap C^2((0,T];X)$ and (14) holds on J.

Theorem 2.6. Let $\beta \in (1,2), u_o \in D(A), f \in W^{1,1}(J, X_A)$ and A is the infinitesimal generator of a bounded C_o-semigroup $\{T(t), t \geq 0\}$. If $\left|\arg\left(\frac{1}{(\lambda+1)^{\beta}}\right)\right| \leq \theta < \frac{\pi}{2}$ then there is a unique strong solution $u_\beta \in C(J, X_A) \bigcap C^1(J;X) \bigcap C^2((0,T];X)$ of (14) given by

$$u_\beta(t) = e^t S_\beta(t)u_o - e^t S_\beta(t) * u_o + e^t S_\beta(t) * \phi_\beta(t) * (f(0)\delta(t) + f'(t) - f(t)) \tag{15}$$

where $S_\beta \in C^1(J;B(X)) \bigcap C^2((0,T];B(X))$ is the resolvent operator with the resolvent equation

$$e^t S_\beta(t)z = e^t z + \phi_\beta(t) * Ae^t S_\beta(t)z, \quad z \in D(A)$$

this solution satisfies the continuation properties

$$\lim_{\beta \to 1^+} u_\alpha(t) = u_1(t), \quad \lim_{\beta \to 2^-} u_\alpha(t) = u_2(t)$$

where $u_1(t)$ is the solution of the evolution equation (13), and $u_2(t)$ is the solution of the Cauchy problem (wave equation)

$$\frac{d^2}{dt^2}u(t) = Au(t) + f(t),\ u(0) = u_o,\ u_t(0) = 0. \tag{16}$$

3. ABSTRACT DIFFUSION-WAVE PROBLEM

Consider the Cauchy problem (9) where A is a closed linear operator with dense domain $D(A) = X_A \subset X$.

Definition 3.1. A function $u \in C(J;X)$ is called a strong solution of (9) on J if $u \in C(J;X_A) \bigcap C^1(J;X)$ and (9) holds on J.

Theorem 3.2. Let $\gamma \in (0,2), u_o \in D(A),$ and A is the infinitesimal generator of a bounded C_o semigroup $\{T(t); t \geq 0\}$. Then there is a unique strong solution $u_\gamma \in C(J;X_A) \cap C^1(J;X)$ of (9) given by

$$u_\gamma(t) = e^t S_\gamma(t) u_o - e^t S_\gamma(t) * u_o \tag{17}$$

where $S_\gamma \in C^1(J;B(X))$ is the resolvent operator with the resolvent equation

$$S_\gamma(t)x = x + \left(e^{-t}\phi_\gamma(t)\right) * AS_\gamma(t)x, \quad x \in D(A) \tag{18}$$

Proof.

(i) For $\gamma \in (0,1)$ we find that

$$\frac{du}{dt} = AD\left(\phi_\gamma(t) * u(t)\right) = A\left(\phi_\gamma(t)u_o + \phi_\gamma(t) * \frac{du}{dt}\right)$$

Operating on both sides by the convolution of $\phi_{1-\gamma}$ we get

$$D^\gamma u(t) = Au(t), \quad u(0) = u_o$$

which by Theorem 2.4 with $f(t) = 0$ has the solution given by (17) with the resolvent operator $S_\gamma \in C^1(J;B(X))$ given by (18). This solution converges, as $\gamma \to 1^-$ to the solution of the homogeneous diffusion problem

$$\frac{du}{dt} = Au(t), \ u(0) = u_o. \tag{19}$$

(ii) For $\gamma = (1, 2)$ we have

$$\frac{du}{dt} = ADI^\gamma u(t) = AI^{\gamma-1}u(t) = A\left(\phi_{\gamma-1}(t) * u(t)\right)$$

Differentiating both sides we get

$$\frac{d^2u}{dt^2} = A\left(\phi_{\gamma-1}(t)u_o + \phi_{\gamma-1} * Du(t)\right).$$

Operating by the convolution of $\phi_{2-\gamma}(t)$ on both sides we get

$$\phi_{2-\gamma}(t) * D^2u(t) = \phi_{2-\gamma}(t) * A\left(\phi_{\gamma-1}(t)u_o + \phi_{\gamma-1} * Du(t)\right)$$

thus we get

$$D^\gamma u(t) = Au(t)$$

and we note that

$$\left.\frac{du}{dt}\right|_{t=0} = A\left(\phi_{\gamma-1}(t) * u(t)\right)\Big|_{t=0} = 0$$

Thus we get

$$D^{\gamma}u(t)=Au(t),\ u(0)=u_o,\ u_t(0)=0$$

which by Theorem 2.6 with $f(t)=0$ has the solution given by (17) with the resolvent operator $S_{\gamma}\in C^1(J;B(X))$ given by (18), which converges, as $\gamma\to 1^+$, to the solution of the homogeneous diffusion problem (19) and converges, as $\gamma\to 2^-$, to the solution of the homogeneous wave problem

$$\frac{d^2u}{dt^2}=Au(t),\ u(0)=u_o,\ u_t(0)=0. \qquad (20)$$

Finally, we prove that the function $u_{\gamma}(t)$ given by (17) is the solution of our problem (9). Using (18) we get:

$$\begin{aligned}\frac{d}{dt}u_{\gamma}(t)&=\frac{d}{dt}\left(e^tS_{\gamma}(t)u_o-e^tS_{\gamma}(t)*u_o\right)\\&=\frac{d}{dt}\left(e^tu_o+\phi_{\gamma}(t)*e^tS_{\gamma}(t)Au_o-\left(e^tu_o+\phi_{\gamma}(t)*e^tS_{\gamma}(t)Au_o\right)*\phi_1(t)\right)\\&=\frac{d}{dt}\left(\phi_{\gamma}(t)*e^tS_{\gamma}(t)Au_o+u_o+\phi_{\gamma}(t)*e^tS_{\gamma}(t)Au_o*\phi_1(t)\right)\\&=\frac{d}{dt}\left(\phi_{\gamma}(t)*A\left(e^tS_{\gamma}(t)u_o+e^tS_{\gamma}(t)*u_o\right)+u_o\right)\\&=\frac{d}{dt}I^{\gamma}Au_{\gamma}(t).\end{aligned}$$

And we have

$$u_{\gamma}(0)=\left(e^tS_{\gamma}(t)u_o-e^tS_{\gamma}(t)*u_o\right)\big|_{t=0}=u_o$$

which completes the proof.

Now we prove that our model represents the fractional order diffusion wave process in all cases.

Theorem 3.3. If, for $\gamma\in(0,1)$, the solution of our model (9) and the fractional order diffusion problem (3) exist then they are equivalent.

Proof. We proved in Theorem 3.2 that if $u_{\gamma}(t)$ is the solution of (9) then it is the solution of (3). Conversely, let $u_{\gamma}(t)$ be the solution of (3) then get

$$D^{\gamma}u(t)=\phi_{1-\gamma}(t)*Du_{\gamma}(t)=Au_{\gamma}(t)$$

Operating by the convolution of $u_{\gamma}(t)$ on the both sides we get

$$u_{\gamma}(t)-u_o=\phi_{\gamma}(t)*Au_{\gamma}(t)$$

Differentiate both sides we get

$$\frac{d}{dt}u_\gamma(t) = A\frac{d}{dt}\left(\phi_\gamma(t) * u_\gamma(t)\right) = A\frac{d}{dt}I^\gamma u_\gamma(t),$$

which completes the proof.

Theorem 3.4. If, for $\gamma \in (1,2)$, the solution of our model (9) and the fractional order wave problem (4) exist then they are equivalent.

Proof. We prove in Theorem 3.2 that if $u_\gamma(t)$ is the solution of (9) then it is the solution of (4). Conversely, let $u_\gamma(t)$ be the solution of (4) then we get

$$\phi_{2-\gamma}(t) * D^2 u_\gamma(t) = Au_\gamma(t)$$

Operating by the convolution of $\phi_\gamma(t)$ on both sides we get

$$\phi_2(t) * D^2 u(t) = \phi_\gamma(t) * Au(t)$$

$$\phi_1(t) * Du_\gamma(t) - Du(t)\big|_{t=0} = \phi_\gamma(t) * Au_\gamma(t)$$

Differentiate both sides we get

$$\frac{d}{dt}u_\gamma(t) = A\frac{d}{dt}I^\gamma u_\gamma(t), \quad u(0) = u_o,$$

which completes the proof.

We finish this section with giving some examples of our model.

Example 3.5. Let the operator A be defined by

$$D(A) = \left\{u(x,t) \in C^2(-\infty,\infty),\ \lim_{x\to\pm\infty} u(x,t) = 0\right\},\ Au(x,t) = \frac{\partial^2}{\partial x^2}u(x,t).$$

Applying Theorem 3.2 then (9) has a unique strong solution. Taking Fourier transform for the variable x, with the parameter ν, and Laplace transform for the variable t, with the parameter λ, give

$$\hat{U}(\nu,\lambda) = \frac{\lambda^{\gamma-1}}{\lambda^\gamma + \nu^2}U_o(\nu) \tag{21}$$

where $U(\nu, t)$ is the Fourier transform of $u(x, t)$, and $\hat{U}(\nu,\lambda)$ is the Laplace transform of $U(\nu, t)$.

Now we discuss the different cases of γ as follows:

Case 1. For $\gamma \in (0,1) \cup (1,2)$, and taking the Laplace inverse transform and the Fourier inverse transform to (21) we get the solution in the form

$$u(x,t) = G(x,t) * u_o(x)$$

where

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\nu x} E_{\gamma,1}\left(-\nu^2 t^\gamma\right) d\nu$$

where $E_{\gamma,1}\left(-\nu^2 t^\gamma\right)$ is the Mittag-Leffler function (see [3]).

Case 2. For $\gamma = 1$, we get the solution in the form

$$u(x,t) = G(x,t) * u_o(x)$$

where

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\nu^2 t} e^{-i\nu x} d\nu = \frac{1}{2\sqrt{\pi t}} e^{\frac{-x^2}{4t}}.$$

Which is the known solution to the diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad with \quad u(x,0) = u_o(x),\ \lim_{x\to\pm\infty} u(x,t) = 0.$$

Case 3. For $\gamma = 2$, Laplace inverse transformation transform (21) to

$$U(\nu,t) = E_{2,1}\left(-\nu^2 t^2\right) U_o(\nu) = \cos(\nu t) U_o(\nu)$$

taking the Fourier inverse transform, we get

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\nu x}\cos(\nu t) U_o(\nu) d\nu \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\nu(x+t)} U_o(\nu) d\nu + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\nu(x-t)} U_o(\nu) d\nu\right) \\ &= \frac{1}{2}\left(u_o(x+t) + u_o(x-t)\right) \end{aligned}$$

which is the known D'Alembert's solution to the wave problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad with \quad u(x,0) = u_o(x),\ u_t(x,0) = 0,\ \lim_{x\to\pm\infty} u(x,t) = 0.$$

Example 3.6. Let the operator A be defined by

$$D(A) = \left\{u(x,t) \in C^2(0,\infty), u(0,t) = h(t),\ \lim_{x\to\infty} u(x,t) = 0\right\},\ Au(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$$

which gives the Signalling problem with $u(x, 0) = u_o$.

Taking Laplace transform with respect to t with parameter λ, we get the ordinary differential equation

$$\frac{d^2}{dx^2}\hat{u}(x,\lambda) - \lambda^\gamma \hat{u}(x,\lambda) = -\lambda^{\gamma-1} u_o,\ \hat{u}(0,\lambda) = \hat{h}(\lambda), \hat{u}(\infty,\lambda) = 0.$$

Putting $\gamma = 2\mu$, we get the general solution of this O.D.E. in the form

$$\hat{u}(x,\lambda) = \left(\hat{h}(\lambda) - \frac{1}{\lambda} u_o\right) e^{-\lambda^\mu x} + \frac{1}{\lambda} u_o.$$

Taking the inverse Laplace transform, we get

$$u(x,t)=\left(h(t)-u_o\right)*\left(\frac{x}{t^{\mu+1}}M\left(\frac{x}{t^{\mu}},\mu\right)\right)+u_o \quad \text{for } 0<\mu<1$$

where $M(z, \mu)$ is given by (see [17])

$$M(z,\mu)=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma\left(-\mu n+(1-\mu)\right)}=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^n}{(n-1)!}\Gamma(\mu n)\sin(\pi\mu n),\ z\in C,\ 0<\mu<1.$$

Example 3.7. Let the operator A be defined by

$$D(A)=\left\{u(x,t)\in C^2(0,L),\ u(0,t)=u(L,t)=0\right\},\ Au(x,t)=\frac{\partial^2}{\partial x^2}u(x,t).$$

Using separation of variable method, we get our solution in the form

$$u(x,t)=\sum_{m=1}^{\infty}c_m E_{\gamma,1}\left(-\frac{m^2\pi^2}{L^2}t^{\gamma}\right)\sin\left(\frac{m\pi x}{L}\right),\ 0<\gamma\le 2$$

where

$$c_m=\frac{2}{L}\int_0^L u_o(x)\sin\left(\frac{m\pi x}{L}\right)dx.$$

4. CONCLUSION

In this paper we give a fractional order diffusion-wave model which is more accurate proving the existence, uniqueness and continuation of the solution and get the solution some special cases which give the ordinary solution of the ordinary diffusion problem D'Alembert's solution to the ordinary wave problem.

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