A POSSIBLE INTERPRETATION OF THE MULTIPLETS 0^+ AND 2^+ IN ^{168}Er

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A sixth-order quadrupole boson Hamiltonian is used to describe 26 states 0^+ and 79 states 2^+ which have been recently identified in ^{168}Er. Two alternative descriptions of energy levels are proposed. One corresponds to a semi-classical approach of a model Hamiltonian, while the other one provides the exact eigenvalues. The first procedure involves four parameters, while the second one involves a compact formula with five parameters. In each case the parameters are fixed by a least-square fit procedure. Both methods yield results which are in a surprisingly good agreement with the experimental data. We give also our predicted reduced transition probabilities, although the corresponding experimental data are not available.

The collective states of deformed nuclei are usually classified in rotational bands distinguished by a quantum number K, which is the angular momentum projection on the z axis of the intrinsic reference frame. The collective character of the states is diminished by increasing the value of K [1–4]. In Ref. [5] one of us (A.A.R.) suggested a possible method of developing bands in a *horizontal* fashion, *i.e.* bands of states with the same angular momenta. Indeed, therein on the top of each state in the ground band a full band of monopole multi-phonon states has been constructed. This idea has been recently considered in a phenomenological context trying to organize the states, describing the motion of the intrinsic degrees of freedom, in bands. Thus, two intrinsic collective coordinates, similar to the nuclear deformations β and γ, are described by the irreducible representations of a SU(2) group acting in a fictitious space (*i.e.* not in ordinary space). Compact formulas for excitation energies have been obtained [6, 7].

Recently, about 26 states 0^+ and 67 states 2^+ have been populated in ^{168}Er by means of a (p, t) reaction [8]. In the cited paper the excitation energies and the corresponding reaction strength have been provided. These data were described qualitatively by two microscopic models, called *projected shell model* (PSM) and *quasiparticle phonon model* (QPM), respectively.

Both models have some inherent drawbacks. PSM restricts the fermion space to four quasiparticle states. Moreover, even from the four qp space the states with four alike quasiparticles are excluded. This is not the case of QPM where the multi-quasiparticle components are taken into account by means of the QRPA approach. However, the final states contain at most two phonon states. These states violate the Pauli principle and, moreover, are not states of good angular momentum.

In Ref. [9] some of us made a first attempt to fit the data of Ref. [8] using a phenomenological model, namely a sixth order quadrupole boson Hamiltonian that was developed in Ref. [7]. Since then about 12 new 2^+ states have been identified by a more careful analysis of the data produced in the (t, p) experiment [10].

In this letter we show that the complete 0^+ and 2^+ data sets, presently available, are nicely described by the closed formulas provided by the model of Ref. [7]. Although more details on the model can be found in Ref. [7], we give here the essential information to make the presentation self-contained.

We use a sixth-order quadrupole boson Hamiltonian:

$$H = \epsilon\hat{N} + \sum_{J=0,2,4} C_J\left[\left(b_2^\dagger b_2^\dagger\right)_J\left(b_2 b_2\right)_J\right]_0 + F\left(b_2^\dagger b_2^\dagger\right)_0\hat{N}\left(b_2 b_2\right)_0, \tag{1}$$

where $b_{2\mu}^\dagger$, $b_{2\mu}$, with $-2 \le \mu \le 2$, are the quadrupole boson operators and $\hat{N}$ the boson number operator. This Hamiltonian has been first treated semi-classically. Averaging H on a coherent state for the bosons $b_{20}^\dagger$ and $\frac{1}{\sqrt{2}}(b_{22}^\dagger + b_{2,-2}^\dagger)$,

$$\left|\Psi\right\rangle = exp\left[z_0 b_{20}^\dagger + z_2(b_{22}^\dagger + b_{2,-2}^\dagger) - h.c.)\right]\left|0\right\rangle, \tag{2}$$

one obtains a classical Hamilton function, $\mathcal{H}$, depending on two coordinates, q_1 and q_2, and their corresponding conjugate momenta,

$$q_i = 2^{(k+2)/4}\,Re(z_k), \qquad p_i = \hbar 2^{(k+2)/4}\,Im(z_k), \qquad k = 0, 2, \qquad i = \frac{k+2}{2}. \tag{3}$$

$\mathcal{H}$ contains two distinct terms describing an anharmonic motion of a classical plane oscillator and a pseudo-rotation around an axis perpendicular to the oscillator plane, respectively. Taking into account that the third component of the pseudo-angular momentum is a constant of motion, the classical Hamiltonian in the reduced space can be easily quantized and the resulting energy is:

$$\epsilon_{n,M} = A(n+1) + B(n+1)^2 + \frac{C}{2}M^2 + \frac{F}{5}\left[(n+1)^3 - 4(n+1)M^2\right], \tag{4}$$

where the factors A, B and C have simple expressions in terms of the coefficients ϵ, C_J involved in the boson Hamiltonian:

$$A=\epsilon, \quad B=\frac{1}{5}C_0+\frac{2}{7\sqrt{5}}C_2+\frac{6}{35}C_4, \quad C=-\frac{8}{5}C_0+\frac{16}{7\sqrt{5}}C_2-\frac{8}{35}C_4. \tag{5}$$

The number of the plane oscillator quanta is denoted by n, while the value of the third component of the pseudo-angular momentum is M. Actually, Eq. (4) represents a semi-classical spectrum which describes the motion of the intrinsic degrees of freedom related to β and γ. Assuming that the rotational degrees of freedom are only weakly coupled to the motion of the intrinsic coordinates, the total energy associated to the motion in the laboratory frame can be written as a sum of two terms corresponding to the intrinsic and rotational motion, respectively:

$$\epsilon_{n,M,J}=A(n+1)+B(n+1)^2+\frac{C}{2}M^2+\frac{F}{5}\left[(n+1)^3-4(n+1)M^2\right]+\delta J(J+1). \tag{6}$$

According to Ref. [7], to the values $J=0$ and $J=2$ correspond different values of M, namely $M=0$ and $M=1$, respectively. Therefore, considering the above equation for the sets of states with angular momenta $J=0$, 2 and normalizing the results to the energy of the first 0^+, one obtains the following expressions for the excitation energies:

$$\begin{aligned} E_{n,0}&=\frac{1}{5}Fn^3+\left(\frac{3}{5}F+B\right)n^2+\left(A+2B+\frac{3}{5}F\right)n, \quad n\geq 0,\\ E_{n,1}&=\frac{1}{5}Fn^3+\left(\frac{3}{5}F+B\right)n^2+\left(A+2B-\frac{1}{5}F\right)n+C, \quad n\geq 1,\\ \mathcal{C}&=\frac{C}{2}+6\delta-\frac{4}{5}F. \end{aligned} \tag{7}$$

These equations are used to describe the available data for the states 0^+ and 2^+ in ^{162}Er. The coefficients involved in the mentioned equations were fixed by a least square procedure. The results are:

$$A=394.2\ \text{keV}, \quad B=-10.4\ \text{keV}, \quad F=0.3865\ \text{keV}, \quad C=-280\ \text{keV}. \tag{8}$$

In order to have a feeling about the fit quality, we compare the results of our calculations with the experimental data in Figs. 1 and 2, panels a).

Although the starting boson Hamiltonian comprises fourth and sixth-order boson anharmonicities, one can easily derive analytical expressions for its eigenvalues. Indeed, aiming to this goal, the boson Hamiltonian H is written in an equivalent form [7]:

$$\begin{aligned} H=(A+\gamma)\hat{N}+\left(B+\frac{C}{8}\right)\hat{N}^2-\frac{1}{6}\left(B+\frac{C}{8}+\gamma\right)\hat{J}^2-\\ -\frac{5}{8}C\left(b_2^\dagger b_2^\dagger\right)_0\left(b_2b_2\right)_0+F\left(b_2^\dagger b_2^\dagger\right)_0\hat{N}\left(b_2b_2\right)_0, \end{aligned} \tag{9}$$

where the coefficient γ has the expression:

$$\gamma = \frac{2}{7\sqrt{5}} C_2 - \frac{3}{7} C_4 \,. \tag{10}$$

The Hamiltonian is diagonal in the basis $|N\upsilon\alpha JM\rangle$, where the specified quantum numbers are the number of bosons, seniority, missing quantum number, angular momentum and its projection on the z axis, respectively. The corresponding eigenvalues are:

$$\begin{aligned} E_{N,\upsilon,J} = & \frac{1}{5} FN^3 + \left(B + \frac{1}{5} F\right) N^2 + \left(A + \gamma - 3\left(\frac{1}{8} C + \frac{2}{5} F\right)\right) N - \\ & - \frac{1}{6}\left(B + \frac{1}{8} C + \gamma\right) J(J+1) + \left(\frac{1}{8} C + \frac{2}{5} F\right) \upsilon^2 + \\ & + 3\left(\frac{1}{8} C + \frac{2}{5} F\right) \upsilon - \frac{1}{5} FN\upsilon^2 - \frac{3}{5} FN\upsilon \,. \end{aligned} \tag{11}$$

By contrast to the semi-classical energies given by Eq. (6), the eigenvalues of H are characterized by two quantum numbers, N and υ. Due to this feature one expects that the new expression for energies yields a better description of the data.

For each angular momentum, we obtain two distinct formulas, for even and odd values of N. The corresponding expressions are denoted by $E_{N,0,0}(N-even)$, $E_{N,3,0}(N-odd\ and\ N>1)$, $E_{N,1,2}(N-odd)$, $E_{N,2,2}(N-even\ and\ N>0)$, respectively. These energy expressions depend linearly on the parameters A, B, C, F, γ, defined before. These parameters together with the quantum number N are to be fixed by a least square procedure. It is noteworthy to remark that both procedures lead to a cubic expression in n and N respectively, although the two quantum numbers have different significance. Indeed, the quantum number n represents the number of the plane oscillator quanta associated to the intrinsic degrees of freedom q_1, q_2 related to the nuclear deformations β, γ. On the other hand, the quantum number N is the number of the quadrupole bosons which are describing the system in the laboratory frame.

Note that in principle both the boson number and seniority could be obtained by solving the least square equations, but the procedure would be quite tedious. For the sake of simplicity we kept only the boson number as a variable to be determined and chose, otherwise, the lowest seniorities. The reason is that for these seniorities the energy equation has a similar structure as in the semiclassical case. For the states 0^+ we started with the $\upsilon = 0$ expressions and tried to describe all energies as corresponding to $\upsilon = 0$. The result was that some of calculated energies exhibit large deviation from the experimental data. These states were considered to have $\upsilon = 3$. In the next step both expressions,

corresponding to $\upsilon = 0$ and $\upsilon = 3$, have been used with the assignments determined before, and new least square equations have been written for the five parameters and the N_s (number of states) values of N (boson number).

Considering first the data for 0^+ and 2^+ energies reported in Ref. [8], the least square procedure yields the results:

$$A = 83.2319 \text{ keV}, \quad B = -10.2454 \text{ keV}, \quad C = 24.00 \text{ keV}, \\ F = 0.3865 \text{ keV}, \quad \gamma = 299.8638 \text{ keV}. \tag{12}$$

The four sets of energies, $E_{N,0,0}$, $E_{N,3,0}$, $E_{N,1,2}$, $E_{N,2,2}$ with the restrictions for N mentioned above, are represented in Figs. 1 b), 1 c), 2 b), 2 c), respectively. The full line curve represents the energy as function of N, with N considered as a continuous variable. The integer number which lies closest to the experimental data is the assigned quantum number. We remark that the agreement with the experimental data is quite good. The remarkable feature of our approach is that by compact formulas we obtain a realistic description of a large number of excitation energies, despite the fact that the number of the fitting parameters is relatively small.

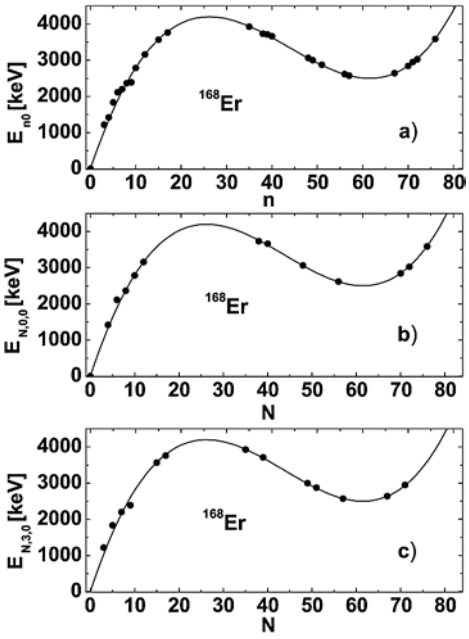

Fig. 1 – Excitation energies of the $J^\pi = 0^+$ states described semi-classically (panel a) and by eigenvalues of the model Hamiltonian, corresponding to the seniority $\upsilon = 0$ (panel b) and $\upsilon = 3$ (panel c), are compared with the experimental data.

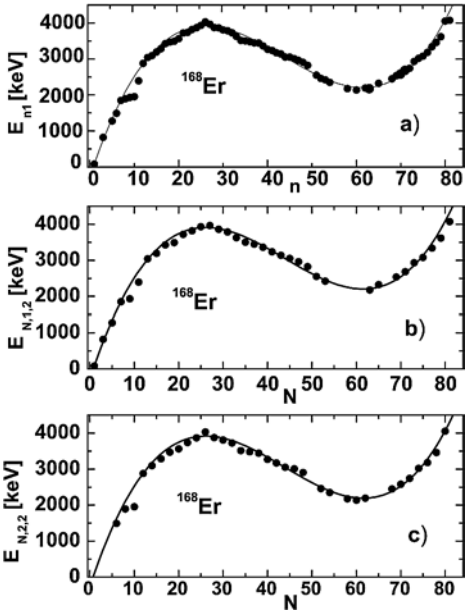

Fig. 2 – Excitation energies of the $J^\pi = 2^+$ states, described semi-classically (panel a) and by eigenvalues of the model Hamiltonian, corresponding to the seniority $\upsilon = 1$ (panel b) and $\upsilon = 2$ (panel c), are compared with the experimental data.

As in any other theoretical model, the number of predicted energy levels in our model is equal to the number of the considered basis states which, in general, is different from the number of the experimentally identified states. Clearly, in the plots shown here there are more predicted energy levels than experimental energies.

The predicted energies which do not have correspondent data may require higher resolution or a different type of experiment. For example, for ^{158}Gd, the authors of Ref. [10] found several new states through an (p, t) experiment, that could not be seen by the previous (n, n') experiment [11]. Moreover, after the publication of data in Ref. [8] and short after we provided a phenomenological interpretation in Ref. [9], Bucurescu and his collaborators analyzed more carefully the data and found another 12 energy levels with angular momentum 2 and positive parity [12]. Of course, this was a challenge for us since explaining the new data is indeed a severe test for the theoretical description we provided. As mentioned before, the aim of this letter is to see how the new data fall in our previous curves, keeping the fitted parameters unchanged. The new data for the 2^+ energies are given on the first column of Table 1. The first three values fill the vacancies in the curves of Ref. [9] and presented here in Fig. 2. The assigned quantum numbers N, υ, are those given in the second column of Table 1. The remaining data lies very close to the data which is already represented in Fig. 2. The later is given also in the fourth column of Table 1, together with the

Table 1

Twelve data for newly identified 2^+ states are given on the left column. The first three are falling on the graph representing the data from Ref. [8] and are interpreted as $\upsilon = 1$ and $\upsilon = 2$ states of large N. The remaining energies from the first column lie close to the old energies from Ref. [8] described as low seniority states

New energies [keV]	N	υ	Old energies [keV]
(2174.0)	63	1	
2580.4	70	2	
2683.2	71	1	
2969.3	47	1	2.961.2
3391.1	39	1	3361.9
3418.2	17	1	3429.2
3794.1	31	1	3789.5
3838.0	29	1	3861.9
(3923.4)	25	1	3933
(4009.6)	26	2	4033.5
4060.7	80	2	4055
4069.2	81	1	4075.6

correspondingly assigned N and υ values. The quasi-degeneracy for the energy levels between 2.9 and 4.1 MeV, shown in Table 1, may suggest that a symmetry exists. However, the set of states $|N\upsilon\alpha JM\rangle$ [13, 14] does not comprise any degeneracy for $J = 2$. Indeed, let us recall that the missing quantum number α labels the solutions of the double inequality for the integer number p [15]:

$$\upsilon - J \le 3p \le \upsilon - \frac{J}{2}, \quad \text{for } J = \text{even}, \quad p = \text{integer}. \tag{13}$$

It is clear that for $J = 2$ and fixed υ, the number of solutions of Eq. (13) is either 0 or 1. For example, there is no state 2^+ with $\upsilon = 0, 3, 6, 9, \ldots$. However, the new energies from the first column might correspond to different (N, υ) values than those given on the column 2 and 3. This suspicion is based on the nonlinear character of the equation in N and υ,

$$E_{N,\upsilon,J} = \mathcal{E}, \tag{14}$$

for a given value of $\mathcal{E}$. Keeping the same parameters as before, we obtained the theoretical values for energies given in Table 2. Concluding, the compact formula given by Eq. (11) may describe a large amount of data despite the fact that only few parameters are involved. Most of the data is described as low seniority states but there are also energy levels which correspond to high seniority states.

Table 2

The excitation energies for the newly identified 2^+ states, first column, are compared with the predictions of the energy expression (11, 12), given in second column, for N and υ from third and fourth column, respectively

New energies [keV]	Theory	N	υ
2969.3	2961.4	45	5
3391.1	3408.4	39	7
3418.2	3426.8	14	10
3794.1	3783.7	17	11
3838.0	3833.1	80	8
(3923.4)	3921.9	28	4
(4009.6)	4007.5	26	8
4060.7	4055.6	26	10
4069.2	4068.8	27	11

Although data concerning the E2 properties of these levels are lacking, in what follows we consider also the reduced E2 probabilities characterizing the

states. We assume that the leading term of the transition operator is linear in the quadrupole bosons:

$$T_{2\mu} = q_h(b^{\dagger}_{2\mu} + (-)^{\mu} b_{2,-\mu}), \tag{15}$$

where q_h denotes the effective charge of the transition operator. Within the semi-classical formalism the transition operator characterizing the intrinsic variables is obtained by averaging $T_{2\mu}$ on the coherent state:

$$\left\langle T_{2\mu} \right\rangle \equiv \mathcal{T}_{2\mu} = q_h\left[\delta_{\mu,0}\sqrt{2}q_1 + \left(\delta_{\mu,2} + \delta_{\mu,-2}\right)q_2\right]. \tag{16}$$

The classical spectrum has been obtained by quantizing the plane oscillator of coordinates q_1 and q_2. Thus, the energies are depending on the total number of quanta along the two axes. It is convenient to consider the polar coordinates corresponding to the Cartesian coordinates q_1 and q_2. The major and radial quantum numbers of the two dimensional harmonic oscillator, for the $M = 0$ and $M = 1$ states, are related by:

$$2n_r + \delta_{J,2} = n. \tag{17}$$

Therefore, n is odd for the states 2^+ and even for 0^+. The intrinsic transition operator has the matrix elements:

$$\left\langle 2_{n'}\left|\mathcal{T}_{2\mu}\right|0_n\right\rangle = \frac{q_h}{2}\left(\delta_{\mu,0} + \frac{\delta_{\mu,2} + \delta_{\mu,-2}}{i\sqrt{2}}\right)\left[\delta_{n'-1,n}\sqrt{n+2} - \delta_{n',n-1}\sqrt{n}\right], \quad n \geq 0. \tag{18}$$

In the laboratory frame the transition operator is

$$\mathbf{T}_{2M} = q_h\sqrt{2}\left(q_1 D^2_{M0} + \frac{q_2}{\sqrt{2}}(D^2_{M2} + D^2_{M,-2})\right), \tag{19}$$

where D^J_{MK} denotes the Wigner functions describing the rotation matrix. The states 2^+ are described by a product of the plane oscillator eigenstate, which describes the intrinsic motion, and a factor accounting for the rotation which, for simplicity, is considered to be

$$\Psi_{2M} = \sqrt{\frac{5}{3}}\frac{1}{2\pi\sqrt{2}}\sum_{K=even} D^2_{MK}. \tag{20}$$

Adopting the convention of Rose [16] for the reduced matrix elements, the final result for the reduced transition probability is:

$$B(E2; 2^+_{n'} \to 0^+_n) = \frac{q_h^2}{12}\left((n+2)\delta_{n'-1,n} + n\delta_{n',n-1}\right), \tag{21}$$

with $n \geq 1$. From this equation one obtains the ratios:

$$\begin{aligned} &\frac{B(E2; 2_n^+ \to 0_{n-1}^+)}{B(E2; 2_n^+ \to 0_{n+1}^+)} = 1, \\ &\frac{B(E2; 2_{n+1}^+ \to 0_n^+)}{B(E2; 2_{n-1}^+ \to 0_n^+)} = \frac{n+2}{n}, \qquad n \geq 1. \end{aligned} \tag{22}$$

For a transition operator having a harmonic structure, the E2 transition between any two states 2^+ is forbidden. Details about the contribution of anharmonic transition operators to the transition probabilities will be given elsewhere.

Within the boson picture the transition probabilities can be also presented in a compact analytical form. Indeed, using a harmonic structure for the quadrupole transition operator, one obtains:

$$\begin{aligned} &B\left(E2; \left|n+1\,102\right\rangle \to \left|n000\right\rangle\right) = q_h^2 \frac{n+5}{10}, \\ &B\left(E2; \left|n-1\,102\right\rangle \to \left|n000\right\rangle\right) = q_h^2 \frac{n}{10}, \\ &B\left(E2; \left|n+1\,202\right\rangle \to \left|n310\right\rangle\right) = q_h^2 \frac{1}{30}(n-1), \\ &B\left(E2; \left|n-1\,202\right\rangle \to \left|n310\right\rangle\right) = q_h^2 \frac{1}{30}(n+6), \\ &B\left(E2; \left|n+1\,202\right\rangle \to \left|n102\right\rangle\right) = q_h^2 \frac{n+6}{7}, \\ &B\left(E2; \left|n-1\,202\right\rangle \to \left|n102\right\rangle\right) = q_h^2 \frac{n-1}{7}, \\ &B\left(E2; \left|n+1\,102\right\rangle \to \left|n202\right\rangle\right) = q_h^2 \frac{n}{7}, \\ &B\left(E2; \left|n-1\,102\right\rangle \to \left|n202\right\rangle\right) = q_h^2 \frac{n+5}{7}. \end{aligned} \tag{23}$$

Anharmonic terms of the type $(b_2^\dagger b_2)_{2\mu}$, $(b_2^\dagger b_2^\dagger)_{2\mu}$ are however necessary in order to have non-vanishing quadrupole moments and cross-over transitions.

Summarizing, we proposed two phenomenological descriptions of the excitation energies of the states 0^+ and 2^+ experimentally identified in ^{168}Er. They correspond to two distinct ways of treating the same sixth-order quadrupole boson Hamiltonian. One is a semi-classical description while the second one uses the exact eigenvalues. While in the yrast band the highest seniority states are the best candidates for a realistic description, for the states of the same angular momentum, conventionally called horizontal bands, the lowest seniority states are used for most of the states. We found that some of the states are high seniority states. It is worth noticing that both 0^+ and 2^+ states exhibit a

cubic n dependence. We know that such a behavior for energy in the yrast bands is determining a back-bending [17] phenomenon for the moment of inertia as a function of the rotationalfrequency. Here a back-bending also shows up but the cause is different from that determining the bending in the moment of inertia in the yrast band.

The terms of the classical Hamiltonian which do not depend on momenta define the potential of the classical system. This has been plotted in Fig. 3 for two sets of parameters used for classical and exact descriptions, respectively. From the upper panel we notice that some semiclassical states may accommodate the second well of the potential [7]. The boson description yields similar spectrum as the semiclassical method but with different structure parameters, *i.e.* those from Eq. (12). The quoted parameters define a classical potential, given in the lower panel of Fig. 3, which is very different from the one used in the classical picture. The discrepancy is caused by the high anharmonicities involved. Actually the two pictures, semi-classical and quantal, agree with each other only in the harmonic limit.

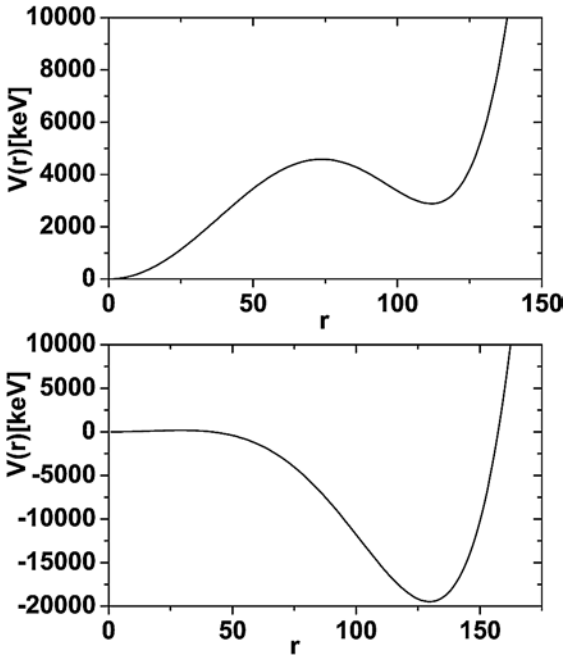

Fig. 3 – The classical potential corresponding to the sets of parameters specified by Eqs. (8) (upper panel) and (12) (lower panel) is plotted as a function of the polar coordinate r $(=10\sqrt{(q_1^2+q_2^2)})$.

It is worth noticing that for a long period of time theoretical works were focused on explaining the high spin states in the ground band, but not so much was done about bands aside the ground state band. Now we are confronted with a new situation. Indeed, to explain consistently very many excitation energies of states with low angular momenta is a real challenge for any theoretical approach.

One may argue that for many of the states considered here, the single particle degrees of freedom prevail. Actually we may share this opinion but, on the other hand, we think that the single particle behavior may be simulated by the anharmonicities involved in the present phenomenological model. Some of the considered states may have collective features. It is worth mentioning that the present model is able to account for these properties shown by a deformed nucleus such as ^{168}Er despite the fact that uses a boson number conserving Hamiltonian. Our attempt is not singular in this respect. Indeed, this is one of the signatures of the interacting boson approximation [18] which is successfully describing rotational bands in non-spherical nuclei. Moreover, the Hamiltonian given by Eq. (1) with $F = 0$, has been previously used to describe the yrast bands in transitional and deformed nuclei [19, 20]. The results of the quoted papers show that some properties determined by the nuclear deformation can be described by a suitable choice of the structure coefficients multiplying the anharmonic terms. Certainly, data concerning the electromagnetic transitions of these states are necessary in order to have an additional test and a more complete picture.

A very nice test of the predictive power of our simple formulas was obtained by applying them to the newly found data by keeping the numerical values for the structure coefficients as obtained in our previous calculations. We showed that the new data are surprisingly well described by the same parameter set.

Also, very simple formulas for the B(E2) values characterizing the transitions between the states are derived within the semi-classical approach. Details about the formalism, as well as an extensive study of similar data for other nuclei, will be published elsewhere.

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REFERENCES

1. A. Bohr, Mat. Fys. Medd. Dan. Vidensk. Selsk. **26** (14) (1952) 1; A. Bohr, B. Mottelson, *ibid* **27** (16) (1953) 1.
2. J. M. Eisenberg, W. Greiner, *Nuclear Theory*, Volume 1: Nuclear Models, 3rd ed. (North Holland Amsterdam 1987).
3. W. Greiner, J. Maruhn, *Nuclear Models*, Springer-Verlag, Heidelberg, 1986.
4. P. Ring, P. Schuck, *The Nuclear Many-Body Problem*, Springer-Verlag, Berlin, Heidelberg, 2000.
5. A. A. Raduta, V. Ceausescu, A. Gheorghe, M. S. Popa, Nucl. Phys. **A 427** (1984) 1.
6. A. A. Raduta, L. Pacearescu, V. Baran, Phys. Rev. **C 67** (2003) 014301.
7. A. A. Raduta, F. D. Aaron, I. I. Ursu, Nucl. Phys. **A 772** (2006) 20.
8. D. Bucurescu *et al.*, Phys. Rev. **C73** (2006) 064309.
9. A. A. Raduta, F. D. Aaron, J. Phys. G: Nucl. Part. Phys. **34** (2007) 2053.
10. D. A. Meyer *et al.*, Phys. Rev. **C 74** 044309 (2006).
11. S. R. Lesher *et al.*, Phys. Rev. **C 66** 051305(R) (2002).

12. D. Bucurescu, private communication.
13. A. Gheorghe, A. A. Raduta, V. Ceausescu, Nucl. Phys. **296** (1978) 228.
14. A. A. Raduta, V. Ceausescu, A. Gheorghe, Nucl. Phys. **A 311** (1978) 118.
15. A. Gheorghe, A. A. Raduta, J. Phys. A: Math Gen. **37** (2004) 10951.
16. M. E. Rose, *Elementary Theory of Angular Momentum,* John Wiley & Sons, New York, 1957.
17. A. Molinari, T. Regge, Phys. Lett. **41 B** (1972) 93.
18. A. Arima, F. Iachello, Ann. Phys. **115** (1978) 325.
19. A. A. Raduta, R. Dreizler, Nucl. Phys. **A258** (1976) 109.
20. T. K. Das, R. M. Dreizler, A. Klein, Phys. Rev. **C2** (1970) 632.