SOLVING OF THE FRACTIONAL NON-LINEAR AND LINEAR SCHRÖDINGER EQUATIONS BY HOMOTOPY PERTURBATION METHOD

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In this paper, the homotopy perturbation method is applied to obtain approximate analytical solutions of the fractional non-linear Schrödinger equations. The solutions are obtained in the form of rapidly convergent infinite series with easily computable terms. We illustrated the ability of the method for solving fractional non linear equation by some examples.

1. INTRODUCTION

The theory of fractional calculus goes back to Leibniz, Liouville, Riemann, Grunwald and Letnikov has been found many application in science and engineering [1–7]. Finding accurate and efficient methods for solving fractional non-linear differential equations (FNLDEs) has been an active research undertaking. Exact solutions of most of the FNLDEs can not be found easily, thus analytical and numerical methods must be used. The *Adomian Decomposition Method* (ADM) was shown to be applicable to linear and nonlinear fractional differential equations [8]. However, another analytic technique for nonlinear problems is called the homotopy perturbation method, first proposed by He [9–14]. In refs [15, 16] homotopy perturbation method is used to solve linear and nonlinear fractional differential equations. The solution of the Schrödinger equation over an infinite integration interval by perturbation methods was given in [17]. Recently, in [18] an analytical approximation to the solution of Schrödinger equations has been obtained. Very recently, analytical and approximate solutions for different nonlinear Schrödinger equation of fractional (NLSF) order involving Caputo derivatives has been obtained by Adomian Decomposition Method [19]. In the present, we obtain the analytical and approximate solutions for fractional nonlinear Schrödinger equation using homotopy perturbation method. The plan of the paper is as follows:

Section 2 is dedicated to the notion of fractional integral and fractional derivatives. Section 3 contains a brief summary of homotopy perturbation method. Section 4 deals with using homotopy perturbation method for solving fractional nonlinear Schrödinger equation. Finally, Section 5 provides some examples for illustrating of the ability of method.

2. FRACTIONAL CALCULUS

In this section we give the definition of the Riemann-Liouville, Caputo derivatives and fractional integral with properties.

Definition 2.1. Let $f(x)$ and $\alpha > 0$ then [3, 4]

$$I_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0, \tag{1}$$

is called as the left sided Riemann-Liouville fractional integral of order α.
The properties of the operator I_x^α are as following [3]:

(1) $$I_x^\alpha I_x^\beta f(x) = I_x^{\alpha+\beta} f(x),$$

(2) $$I_x^\alpha I_x^\beta f(x) = I_x^\beta I_x^\alpha f(x),$$

(3) $$I_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Definition 2.2. Let $u(x,t)$ and $n-1 \le \alpha < n$, then partial Caputo fractional derivatives is defined as:

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n}{\partial \tau^n} u(x,\tau) d\tau, \tag{2}$$

Note that

$$I_t^\alpha D_t^\alpha u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t^k}{k!}. \tag{3}$$

3. HOMOTOPY PERTURBATION METHOD

The homotopy perturbation method (HPM) introduced by He [9–14]. The combination of the perturbation method and the homotopy methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. In this section, basic ideas of this method has been explained.

Let us consider the following general non-linear differential equation

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{4}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{5}$$

where A is a general differential operator, B a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω.
The operator A can be generally divided into linear (L) and non linear (N) parts. Therefore Eq. (4) can be written as follows:

$$L(u) + N(u) - f(r) = 0. \tag{6}$$

Using the homotopy technique, we construct a homotopy $U(r, p) : \Omega \times [0,1] \to \to \mathbb{R}$, which satisfies:

$$H(U, p) = (1-p)[L(U) - L(u_0)] + p[A(U) - f(r)] = 0, \quad p \in [0,1], \ r \in \Omega \tag{7}$$

or

$$H(U, p) = L(U) - L(u_0) + pL(u_0) + p[N(U) - f(r)] = 0, \tag{8}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation for the solution of Eq. (4), which satisfies the boundary conditions. Clearly, from Eqs. (7) and (8) we have

$$H(U, 0) = L(U) - L(u_0) = 0, \tag{9}$$

$$H(U, 1) = A(U) - f(r) = 0. \tag{10}$$

While p changes form zero to unity, $U(r, p)$ varies from $u_0(r)$ to $u(r)$. In topology, this is called homotopy. On account at HPM, first we can use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (7) and (8) can be written as a power series in p:

$$U = U_0 + pU_1 + p^2U_2 + ... \tag{11}$$

Letting $p = 1$, results in the approximate solution of Eq. (4)

$$u = \lim_{p \to 1} U = U_0 + U_1 + U_2 + ... \tag{12}$$

The series (12) is convergent for the most cases [10].

4. FRACTIONAL NON-LINEAR SCHRÖDINGER

The nonlinear Schrödinger equation is a model of the evolution of a one dimensional packet of surface waves on sufficiently deep water. NLS equation describing the evolution nonlinear wave in nonlinear, strongly dispersive, and hyperbolic systems [20]. The propagation of a guided mode in a perfect nonlinear mono-mode fiber is modeled by nonlinear Schrödinger equation [21–23]. In this section we have solved the fractional non linear Schrödinger equation using HPM.

Let us consider the following Schrödinger equation with the following initial condition [24]:

$$i\,D_t^{\alpha}\psi(X,t) = -\frac{1}{2}\nabla^2\psi + V_d(X)\psi + \alpha_d\psi^2 + \beta_d\,|\,\psi\,|^2\,\psi, \tag{13}$$

$$\psi(X,0) = \psi_0, \qquad X \in \mathbb{R}^d$$

where $V_d(X)$ is the trapping potential and β_d is a real constant.

To solve Eq. (13) by homotopy perturbation method, we construct the following homotopy:

$$H(\Psi,p) = \\ = (1-p)(i\,D_t^{\alpha}\Psi) + p\Big(i\,D_t^{\alpha}\Psi + \frac{1}{2}\nabla^2\Psi - V_d(X)\Psi - \alpha_d\Psi^2 - \beta_d\,|\,\Psi\,|^2\,\Psi\Big) = 0 \tag{14}$$

Suppose the solution of Eq. (14) to be as following form

$$\Psi = \Psi_0 + p\Psi_1 + p^2\Psi_2 + \ldots. \tag{15}$$

Substituting (15) into (14), and equating the coefficients of the terms with identical powers of p,

$$p^0 : D_t^{\alpha}\Psi_0 = 0 \tag{16}$$

$$p^1 : D_t^{\alpha}\Psi_1 - i\Big(\frac{1}{2}\nabla^2\Psi_0 - V_d(X)\Psi_0 - \alpha_d\Psi_0^2 - \beta_d\,|\,\Psi_0\,|^2\,\Psi_0\Big) = 0, \\ \Psi_1(X,0) = 0, \tag{17}$$

$$p^2 : D_t^{\alpha}\Psi_2 - i\left(\frac{1}{2}\nabla^2\Psi_1 - V_d(X)\Psi_1 - \alpha_d\sum_{i=0}^{1}\Psi_i\Psi_{1-i} - \right. \\ \left. -\beta_d\sum_{i=0}^{1}\sum_{k=0}^{1-i}\Psi_i\Psi_k\bar{\Psi}_{1-k-i}\right) = 0, \qquad \Psi_2(X,0) = 0, \tag{18}$$

$$p^3 : D_t^{\alpha}\Psi_3 - i\left(\frac{1}{2}\nabla^2\Psi_2 - V_d(X)\Psi_2 - \alpha_d\sum_{i=0}^{2}\Psi_i\Psi_{2-i} - \right.$$
$$\left. -\beta_d\sum_{i=0}^{2}\sum_{k=0}^{2-i}\Psi_i\Psi_k\bar{\Psi}_{2-k-i}\right) = 0, \quad \Psi_3(X,0) = 0, \tag{19}$$

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$$p^j : D_t^{\alpha}\Psi_j - i\left(\frac{1}{2}\nabla^2\Psi_{j-1} - V_d(X)\Psi_{j-1} - \alpha_d\sum_{i=0}^{j-1}\Psi_i\Psi_{j-i-i} - \right.$$
$$\left. -\beta_d\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-i}\Psi_i\Psi_k\bar{\Psi}_{j-k-i}\right) = 0, \quad \Psi_j(X,0) = 0, \tag{20}$$

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where in p^j, there are the multiplication of two series $|\Psi|^2$ and Ψ.
For simplicity we take

$$\Psi_0 = \psi_0 = \psi(X,0). \tag{21}$$

Having this assumption we get the following iterative equation

$$\Psi_j = \frac{i}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\frac{1}{2}\nabla^2\Psi_{j-1} - V_d(X)\Psi_{j-1} - \alpha_d\sum_{i=0}^{j-1}\Psi_i\Psi_{j-i-i} - \right.$$
$$\left. -\beta_d\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\Psi_i\Psi_k\bar{\Psi}_{j-k-i}\right)d\tau, \tag{22}$$

The approximate solution of (13) can be obtained by setting $p = 1$,

$$\psi = \lim_{p\to 1}\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \ldots.$$

5. EXAMPLES

We have used HPM for solving following example for efficiency of the method in fractional nonlinear and linear equation.

5.1. EXAMPLE 1

Consider the following one dimensional Schrödinger equation with the following initial condition [25].

$$i\,D_t^{\alpha}\psi(x,t) = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - |\,\psi\,|^2\,\psi, \qquad t \geq 0, \tag{23}$$

$$\psi(x,0) = e^{ix}.$$

He's homotopy perturbation method consists of the following scheme

$$H(\Psi,p) = (1-p)i\,D_t^{\alpha}\Psi + p\left(i\,D_t^{\alpha}\Psi + \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + |\,\Psi\,|^2\,\Psi\right) = 0. \tag{24}$$

Starting with $\Psi_0 = \psi_0 = \psi(x,0) = e^{ix}$, using (22) we obtain the recurrence relation

$$\Psi_j = \frac{i}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\frac{1}{2}\frac{\partial^2\Psi_{j-1}}{\partial x^2} + \sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\Psi_i\Psi_k\bar{\Psi}_{j-k-i}\right)d\tau,\ \ j = 1,2,3,.... \tag{25}$$

The solution reads

$$\Psi_1(x,t) = \frac{it^{\alpha}}{2\Gamma(\alpha+1)}e^{ix},$$

$$\Psi_2(x,t) = -\frac{t^{2\alpha}}{4\Gamma(2\alpha+1)}e^{ix},$$

$$\Psi_3(x,t) = -\frac{it^{3\alpha}}{8\Gamma(3\alpha+1)}e^{ix},$$

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General form can be written as

$$\Psi_n(x,t) = \frac{(it^{\alpha})^n}{2^n\Gamma(n\alpha+1)}e^{ix}.$$

Finally, solution will be as

$$\psi(x,t) = \lim_{p\to 1}\Psi(x,t) = \sum_{n=0}^{\infty}\frac{(it^{\alpha})^n}{2^n\Gamma(n\alpha+1)}e^{ix}.$$

In Fig. 1, we have presented the graph of solution for the values $\alpha = 0.1$, 0.5 and 0.9.

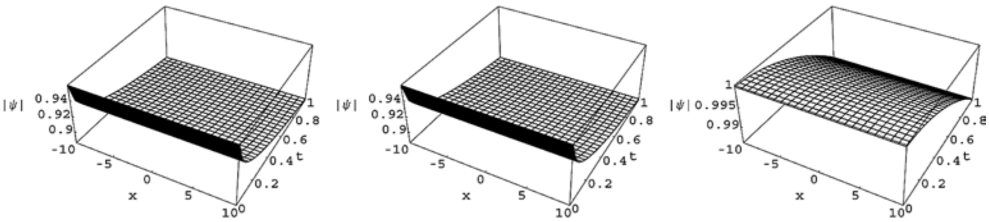

Fig. 1 – Graph of the $|\psi(x,t)|$ corresponding to the values $\alpha = 0.1$, 0.5 and 0.9 from left to right.

5.2. EXAMPLE 2

Consider the following partial differential equation [24]

$$i\,D_t^{\alpha}\psi(x,t) = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \psi\cos^2 x + |\psi|^2\,\psi, \qquad t \geq 0, \tag{26}$$

$$\psi(x,0) = \sin x.$$

We construct a homotopy $\Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$H(\Psi,p) = (1-p)i\,D_t^{\alpha}\Psi + p(i\,D_t^{\alpha}\Psi + \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} - \Psi\cos^2 x - |\Psi|^2\,\Psi = 0. \tag{27}$$

In view of Eq. (22) we have the following scheme

$$\psi_0 = \psi(x,0) = \sin x,$$

$$\Psi_j = \frac{i}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\frac{1}{2}\frac{\partial^2\Psi_{j-1}}{\partial x^2} - \Psi_{j-1}\cos^2 x - \sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\Psi_i\Psi_k\overline{\Psi}_{j-k-i}\right)d\tau, \tag{28}$$

We obtain first few Ψ_j term:

$$\Psi_1(x,t) = -\frac{3it^{\alpha}}{2\Gamma(\alpha+1)}\sin x,$$

$$\Psi_2(x,t) = -\frac{9t^{2\alpha}}{4\Gamma(2\alpha+1)}\sin x,$$

$$\Psi_3(x,t) = \frac{81t^{3\alpha}}{8\Gamma(3\alpha+1)}\sin x,$$

$$\vdots$$

Thus, solution will be as:

$$\psi(x,t)=\lim_{p\to 1}\Psi(x,t)=\sum_{n=0}^{\infty}\frac{(-3it^{\alpha})^{n}}{2^{n}\Gamma(n\alpha+1)}\sin x .$$

5.3. EXAMPLE 3

Consider linear Schrödinger equation as following

$$i\,D_t^{\alpha}\psi(x,t)+i\frac{\partial^2\psi}{\partial x^2}=0, \tag{29}$$

with initial condition

$$\psi(x,0)=1+\sinh(2x).$$

Let construct the following homotopy:

$$H(\Psi,p)=(1-p)i\,D_t^{\alpha}\Psi(x,t)+p\left(i\,D_t^{\alpha}\psi(x,t)+i\frac{\partial^2\Psi}{\partial x^2}\right)=0. \tag{30}$$

Substituting Ψ from Eq. (15) into Eq. (30), rearranging based on powers of p-terms and solving the resulted equations, we have:

$$\Psi_0(x,t)=1+\sinh(2x)$$

$$\Psi_1(x,t)=\frac{-4it^{\alpha}}{\Gamma(\alpha+1)}\sinh(2x)$$

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$$\Psi_n(x,t)=\frac{(-4it^{\alpha})^{n}}{\Gamma(n\alpha+1)}\sinh(2x)$$

The solution of the Eq. (29) when $p\to 1$ will be as follows:

$$\psi(x,t)=\lim_{p\to 1}\Psi(x,t)=1+\sinh(2x)+\sum_{n=1}^{\infty}\frac{(-4it^{\alpha})^{n}}{\Gamma(n\alpha+1)}$$

5.4. EXAMPLE 4

Consider the following three dimensional Schrödinger equation with the following initial condition [24]

$$i\,D_t^{\alpha}\psi(x,t) = -\frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+\psi V(x,y,z)+|\,\psi\,|^2\,\psi,$$
$$t\ge 0,\quad (x,y,z)\in[0,2\pi]\times[0,2\pi]\times[0,2\pi], \tag{31}$$

$$\psi(x,y,z,0)=\sin x\sin y\sin z,$$

where $V(x,y,z)=1-\sin^2 x\sin^2 y\sin^2 z$.
We construct a homotopy $\Omega\times[0,1]\to\mathbb{R}$ which satisfies

$$H(\Psi,p)=(1-p)i\,D_t^{\alpha}\Psi(x,t)+p\left(iD_t^{\alpha}\Psi+\frac{1}{2}\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right)-\right.$$
$$\left.-V(x,y,z)\Psi-|\,\Psi\,|^2\,\Psi\right)=0. \tag{32}$$

In virtue of Eq. (22) we get the recurrence relation

$$\Psi_0=\psi_0=\psi(x,y,z,0)=\sin x\sin y\sin z,$$

$$\Psi_j=\frac{i}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left(\frac{1}{2}\left(\frac{\partial^2\Psi_{j-1}}{\partial x^2}+\frac{\partial^2\Psi_{j-1}}{\partial y^2}+\frac{\partial^2\Psi_{j-1}}{\partial z^2}\right)-\right.$$
$$\left.-V(x,y,z)\Psi_{j-1}-\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\Psi_i\Psi_k\bar{\Psi}_{j-k-i}\right)d\tau. \tag{33}$$

We derive the following results

$$\Psi_1(x,y,z,t)=-\frac{5it^{\alpha}}{2\Gamma(\alpha+1)}\sin x\sin y\sin z,$$

$$\Psi_2(x,y,z,t)=-\frac{25t^{2\alpha}}{4\Gamma(2\alpha+1)}\sin x\sin y\sin z,$$

$$\Psi_3(x,y,z,t)=\frac{25it^{3\alpha}}{8\Gamma(3\alpha+1)}\sin x\sin y\sin z,$$

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$$\Psi_n(x,y,z,t)=\left(\frac{(-5it^{\alpha})^n}{2^n\,\Gamma(n\alpha+1)}\right)\sin x\sin y.$$

Solution of Eq. (31) will be derived by these terms, so

$$\psi(x, y, z, t) = \lim_{p\to 1} \Psi(x, t) = \sum_{n=0}^{\infty} \frac{(-5it^{\alpha})^n}{2^n \Gamma(n\alpha + 1)} \sin x \sin y \sin z.$$

6. CONCLUSION

In the present paper we obtain analytical approximate solution for fractional nonlinear Schrödinger equation by homotopy perturbation method it shows that ability of the homotopy perturbation method in nonlinear fractional equation.

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