A NEW CLASS OF SEMI-RIEMANNIAN SUBMERSIONS*

G. E. VÎLCU[1,2]

[1] Petroleum-Gas University of Ploiești, Department of Mathematics and Computer Science,
Blv. București, Nr. 39, Ploiești, Romania
[2] University of Bucharest, Faculty of Mathematics and Computer Science
Research Center in Geometry, Topology and Algebra, Str. Academiei, Nr. 14, Sector 1,
București, Romania
E-mail: gvilcu@mail.upg-ploiesti.ro

Received December 15, 2008

In this paper we obtain some properties concerning the geometry of the semi-Riemannian submersions from paraquaternionic CR-submanifolds of paraquaternionic Kähler manifolds.

1. INTRODUCTION

The study of the Riemannian submersions was initiated by O'Neill [21] and Gray [10]. In [26] B. Watson introduced the concept of 3-submersion, as a Riemannian submersion from an almost contact metric manifold onto an almost quaternionic manifold, which commutes with the structure tensors of type (1, 1); in [16] and [17], this concept has been extended in quaternionic setting. Semi-Riemannian submersions were introduced by O'Neill in [22]. It is well-known that semi-Riemannian submersions are of interest in physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories [4, 5, 7, 13, 14, 24, 25].

On another hand, the concept of paraquaternionic structure has been introduced by P. Libermann [20] as a triplet of endomorphisms of the tangent bundle $\{J_1, J_2, J_3\}$, in which J_1 is almost complex and J_2 and J_3 are almost product structures satisfying some relations of anti-commutation. Under certain conditions on the holonomy group of a metric adapted to such a structure, one arrives to the concept of paraquaternionic Kähler manifold. The study of submanifolds of a paraquaternionic Kähler manifold is a very interesting subject and several types of such submanifolds we can find in the recent literature: Kähler and para-Kähler submanifolds [1], normal semi-invariant submanifolds [2], paraquaternionic CR-submanifolds [12], lightlike submanifolds [15]. In this

* Paper presented at the 4th National Conference on Applied Physics, September 25–26, 2008, Galați, Romania.

paper we obtain some properties for semi-Riemannian submersions from paraquaternionic CR-submanifolds of an paraquaternionic Kähler manifold. In particular we find a sufficient condition for these applications to become harmonic maps.

2. PRELIMINARIES

Let $\overline{M}$ be a smooth manifold. We say that a rank-3 subbundle σ of $End(T\overline{M})$ is an almost paraquaternionic structure on $\overline{M}$ if a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ, such that for all $\alpha \in \{1, 2, 3\}$ we have:

$$J_\alpha^2 = -\epsilon_\alpha Id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = \epsilon_{\alpha+2}J_{\alpha+2} \tag{1}$$

where $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_3 = -1$ and the indices are taken from $\{1, 2, 3\}$ modulo 3.

Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and σ an almost paraquaternionic structure on $\overline{M}$. The metric $\overline{g}$ is said to be adapted to the paraquaternionic structure σ if it satisfies:

$$\overline{g}(J_\alpha X, J_\alpha Y) = \epsilon_\alpha \overline{g}(X, Y), \qquad \alpha \in \{1, 2, 3\}, \tag{2}$$

for all vector fields X, Y on $\overline{M}$ and any local basis $\{J_1, J_2, J_3\}$ of σ, where $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_3 = -1$. In this case, $(\overline{M}, \sigma, \overline{g})$ is said to be an almost paraquaternionic hermitian manifold.

It is clear that any almost paraquaternionic hermitian manifold is of dimension $4m$, $m \geq 1$, and any adapted metric is necessarily of neutral signature $(2m, 2m)$.

Moreover, if the Levi-Civita connection of $\overline{g}$ satisfies the following conditions for all $\alpha \in \{1, 2, 3\}$:

$$\overline{\nabla}_X J_\alpha = -\epsilon_\alpha[\omega_{\alpha+2}(X)J_{\alpha+1} - \omega_{\alpha+1}(X)J_{\alpha+2}] \tag{3}$$

where $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_3 = -1$ and the indices are taken from $\{1, 2, 3\}$ modulo 3, for any vector field X on $\overline{M}$, ω_1, ω_2, ω_3 being local 1-forms over the open for which $\{J_1, J_2, J_3\}$ is a local basis of σ, then $(\overline{M}, \sigma, \overline{g})$ is said to be a paraquaternionic Kähler manifold (see [9]).

We remark that any paraquaternionic Kähler manifold is an Einstein manifold, provided that $dim M > 4$ (see [9]).

Let (M, g) be a semi-Riemannian manifold and let N be an immersed submanifold of M. Then N is said to be a non-degenerate submanifold of M if the restriction of the semi-Riemannian metric g to TN is non-degenerate at each point of N. We denote by the same symbol g the semi-Riemannian metric induced by g on N and by $TN^{\perp}$ the normal bundle to N and for the rest of this section we will assume that the induced metric on N is non-degenerate. Then we have the following orthogonal decomposition:

$$TM = TN \oplus TN^{\perp}.$$

Also, we denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection on M and N, respectively. Then the Gauss formula is given by:

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y) \tag{4}$$

for any $X, Y \in \Gamma(TN)$, where $B : \Gamma(TN) \times \Gamma(TN) \to \Gamma(TN^{\perp})$ is the second fundamental form of N in M.

On the other hand, the Weingarten formula is given by:

$$\overline{\nabla}_X \xi = -A_{\xi} X + \nabla^{\perp}_X \xi \tag{5}$$

for any $X \in \Gamma(TN)$ and $\xi \in \Gamma(TN^{\perp})$, where $-A_{\xi}X$ is the tangent part of $\overline{\nabla}_X \xi$ and $\nabla^{\perp}_X \xi$ is the normal part of $\overline{\nabla}_X \xi$; A_{ξ} and $\nabla^{\perp}$ are called the shape operator of N with respect to ξ and the normal connection, respectively. Moreover, B and A_{ξ} are related by:

$$g(B(X, Y), \xi) = g(A_{\xi} X, Y) \tag{6}$$

for any $X, Y \in \Gamma(TN)$ and $\xi \in \Gamma(TN^{\perp})$ (see [22]).

3. PARAQUATERNIONIC CR-SUBMANIFOLDS

Let N be an n-dimensional non-degenerate submanifold of an almost paraquaternionic hermitian manifold (M, σ, g). We say that (N, g) is a paraquaternionic CR-submanifold of M if there exists a non-degenerate distribution $\mathcal{D} : x \to \mathcal{D}_x \subseteq T_x N$ such that on any $U \cap N$ we have (cf. [12]):

i. $\mathcal{D}$ is a paraquaternionic distribution, *i.e.*

$$J_{\alpha} \mathcal{D}_x = \mathcal{D}_x, \quad \alpha \in \{1, 2, 3\} \tag{7}$$

and

ii. $\mathcal{D}^{\perp}$ is a totally real distribution, *i.e.*

$$J_{\alpha} \mathcal{D}^{\perp}_x \subset T^{\perp}_x N, \quad \alpha \in \{1, 2, 3\} \tag{8}$$

for any any local basis $\{J_1, J_2, J_3\}$ of σ on U and $x \in U \cap N$, where $\mathcal{D}^{\perp}$ is the orthogonal complementary distribution to $\mathcal{D}$ in TN.

A non-degenerate submanifold N of an almost paraquaternionic hermitian manifold (M, σ, g) is called a paraquaternionic (respectively, totally real) submanifold if $\mathcal{D}^{\perp} = 0$ (respectively, $\mathcal{D} = 0$). A paraquaternionic CR-submanifold is said to be proper if it is neither paraquaternionic nor totally real. Some examples of paraquaternionic, totally real and proper paraquaternionic CR-submanifolds we can find in [12].

Let N be a paraquaternionic CR-submanifold of an almost paraquaternionic hermitian manifold (M, σ, g). Then we say that:

i. N is $\mathcal{D}$-geodesic if $B(X, Y) = 0, \quad \forall X, Y \in \Gamma(\mathcal{D})$;

ii. N is $\mathcal{D}^{\perp}$-geodesic if $B(X, Y) = 0, \quad \forall X, Y \in \Gamma(\mathcal{D}^{\perp})$;

iii. N is mixed geodesic if $B(X, Y) = 0, \ \forall X \in \Gamma(\mathcal{D}), \ Y \in \Gamma(\mathcal{D}^{\perp})$. Moreover, if $\mathcal{D}$ is integrable, then N is said to be a mixed foliated paraquaternionic CR-submanifold.

We recall now the following result concerning the integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$.

Theorem 3.1. [12] Let N be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M, σ, g). Then:

i. The distribution $\mathcal{D}^{\perp}$ is integrable.

ii. The paraquaternionic distribution $\mathcal{D}$ is integrable if and only if N is $\mathcal{D}$-geodesic.

4. SEMI-RIEMANNIAN SUBMERSIONS FROM PARAQUATERNIONIC CR-SUBMANIFOLDS

Let (M, g) and (M', g') be two connected semi-Riemannian manifold of index s $(0 \le s \le dim M)$ and s' $(0 \le s' \le dim M')$ respectively, with $s' \le s$. Roughly speaking, a semi-Riemannian submersion is a smooth map $\pi : M \to M'$ which is onto and satisfies the following conditions (cf. [22]):

(i) $\pi_* |_p$ is onto for all $p \in M$;

(ii) The fibres $\pi^{-1}(p')$, $p' \in M'$, are semi-Riemannian submanifolds of M;

(iii) π_* preserves scalar products of vectors normal to fibres.

The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by $\mathcal{V}$ the vertical distribution and by $\mathcal{H}$ the horizontal distribution.

A semi-Riemannian submersion $\pi : M \to M'$ determines, as well as in the Riemannian case (see [21]), two $(1, 2)$ tensor field T and A on M, by the formulas:

$$T(E,F)=T_E F=h\nabla_{vE}vF+v\nabla_{vE}hF \tag{9}$$

and respectively:

$$A(E,F)=A_E F=v\nabla_{hE}hF+h\nabla_{hE}vF \tag{10}$$

for any $E,F\in\Gamma(TM)$, where v and h are the vertical and horizontal projection (see [8]). We remark that for $U,V\in\Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X,Y\in\Gamma(\mathcal{H})$, $A_X Y=\frac{1}{2}v[X,Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$.

An horizontal vector field X on M is said to be basic if X is π-related to a vector field X' on M'. It is clearly that every vector field X' on M' has a unique horizontal lift X to M and X is basic.

If $\pi:M\to M'$ is a semi-Riemannian submersion and X, Y are basic vector fields on M, π-related to X' and Y' on M', then we have the next properties (see [3, 8, 21]):

(i) $h[X,Y]$ is a basic vector field and $\pi_* h[X,Y]=[X',Y']\circ\pi$;

(ii) $h(\nabla_X Y)$ is a basic vector field π-related to $\nabla'_{X'}Y'$, where ∇ and ∇' are the Levi-Civita connections on M and M';

(iii) $[E,U]\in\Gamma(\mathcal{V})$, $\forall U\in\Gamma(\mathcal{V})$ and $\forall E\in\Gamma(TM)$.

Definition 4.1. Let N be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M,σ,g) and (M',σ',g') be an almost paraquaternionic hermitian manifold. A semi-Riemannian submersion $\pi:N\to M'$ is said to be a paraquaternionic CR-submersion if the following conditions are satisfied:

i. $Ker\,\pi_*=\mathcal{D}^\perp$;

ii. For each $p\in M$, $\pi_*:\mathcal{D}_p\to T_{\pi(p)}M'$ is an isometry with respect to each complex and product structure of $\mathcal{D}_p$ and $T_{\pi(p)}M'$, where $T_{\pi(p)}M'$ denotes the tangent space to M' at $\pi(p)$.

By using the Gauss formula and the above remarks, following the techniques from [17], we obtain the next two results.

Theorem 4.2. Let N be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M,σ,g) and (M',σ',g') be an almost paraquaternionic hermitian manifold. If $\pi:N\to M'$ is a paraquaternionic CR-submersion, then (M',σ',g') is a paraquaternionic Kähler manifold.

Theorem 4.3. Let N be a mixed foliated paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M, σ, g) and (M', σ', g') be an almost paraquaternionic hermitian manifold. If $\pi : N \to M'$ is a paraquaternionic CR-submersion, then the fibres are totally geodesic submanifolds and the distribution $\mathcal{D}^{\perp}$ is parallel.

5. SEMI-RIEMANNIAN SUBMERSIONS AND HARMONIC MAPS

The second fundamental form α_f of a map $f : (M, g) \to (N, g')$ between two semi-Riemannian manifolds is defined by:

$$\alpha_f(X, Y) = \widetilde{\nabla}_X f_* Y - f_* \nabla_X Y, \tag{11}$$

for any vector fields X, Y on M, where ∇ is the Levi-Civita connection of M and $\widetilde{\nabla}$ is the pullback of the connection ∇' of N to the induced vector bundle $f^{-1}(TN)$:

$$\widetilde{\nabla}_X f_* Y = \nabla'_{f_* X} f_* Y. \tag{12}$$

The tension field $\tau(f)$ of f is defined as the trace of α_f, *i.e.*:

$$\tau(f)_x = \sum_{i=1}^{m} \alpha_f(e_i, e_i), \tag{13}$$

where $\{e_1, e_2, ..., e_m\}$ is a local orthonormal frame of $T_x M$, $x \in M$.

We say that f is a harmonic map if and only if $\tau(f)$ vanishes at each point $x \in M$.

It is well known that a semi-Riemannian submersion is a harmonic map if and only if its fibres are minimal submanifolds (see [3, 6, 8]). Now, from Theorem 4.3. we deduce that if N is a mixed foliated paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M, σ, g), (M', σ', g') is an almost paraquaternionic hermitian manifold and $\pi : N \to M'$ is a paraquaternionic CR-submersion, then π is a harmonic map; owing to the great importance of these objects, this result can have important applications in theoretical physics, since their quantization leads to examples of conformal quantum field theories (see [11]).

Acknowledgements. I would like to thank S. Ianuș and M. Vișinescu for motivating discussions and valuable comments. This work was supported by the Romanian Academy under grant GAR 28/2008.

REFERENCES

1. D. Alekseevsky, V. Cortés, *The twistor spaces of a para-quaternionic Kähler manifold*, Osaka J. Math. **45** (1), 215–251 (2008).
2. A. Bejancu, *Normal semi-invariant submanifolds of paraquaternionic Kähler manifolds*, Kuwait J. Sci. Eng. **33** (2), 33–46 (2006).
3. A. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, 1987.
4. J. P. Bourguignon, H. B. Lawson, *A mathematician's visit to Kaluza-Klein theory*, Rend. Semin. Mat. Torino, Fasc. Spec. 143–163 (1989).
5. J. P. Bourguignon, H. B. Lawson, *Stability and isolation phenomena for Yang-Mills fields*, Commun. Math. Phys. **79**, 189–230 (1981).
6. J. Eells, J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86**, 109–160 (1964).
7. M. Falcitelli, S. Ianuș, A. M. Pastore, M. Vișinescu, *Some applications of Riemannian submersions in physics*, Rev. Roum. Phys. **48**, 627–639 (2003).
8. M. Falcitelli, S. Ianuș, A. M. Pastore, *Riemannian submersions and related topics*, River Edge, NJ: World Scientific, 2004.
9. E. Garcia-Rio, Y. Matsushita, R. Vasquez-Lorenzo, *Paraquaternionic Kähler manifolds*, Rocky Mountain J. Math. **31**, 237–260 (2001).
10. A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16**, 715–737 (1967).
11. F. Hélein, *Harmonic maps, conservation laws and moving frames*, Cambridge Tracts in Mathematics 150, Cambridge: Cambridge University Press, 2002.
12. S. Ianuș, S. Marchiafava, G. E. Vîlcu, *Paraquaternionic CR-submanifolds of paraquaternionic Kähler manifolds*, preprint (2008).
13. S. Ianuș, M. Vișinescu, *Kaluza-Klein theory with scalar fields and generalized Hopf manifolds*, Class. Quantum Grav. **4**, 1317–1325 (1987).
14. S. Ianuș, M. Vișinescu, *Space-time compactification and Riemannian submersions*, in The mathematical heritage of C. F. Gauss (ed. G. Rassias), World Sci. Publishing, River Edge NJ, 358–371 (1991).
15. S. Ianuș, R. Mazzocco, G. E. Vîlcu, *Real lightlike hypersurfaces of paraquaternionic Kähler manifolds*, Mediterr. J. Math. **3**, 581–592 (2006).
16. S. Ianuș, R. Mazzocco, G. E. Vîlcu, *Harmonic maps between quaternionic Kähler manifolds*, J. Nonlinear Math. Phys. **15** (1), 1–8 (2008).
17. S. Ianuș, R. Mazzocco, G. E. Vîlcu, *Riemannian submersions from quaternionic manifolds*, Acta Appl. Math. **104** (1), 83–89 (2008).
18. S. Ianuș, G. E. Vîlcu, *Some constructions of almost para-hyperhermitian structures on manifolds and tangent bundles*, Int. J. Geom. Methods Mod. Phys. **5** (6), 893–903 (2008).
19. S. Ivanov, S. Zamkovoy, *Para-hermitian and para-quaternionic manifolds*, Differential Geom. Appl. **23**, 205–234 (2005).
20. P. Libermann, *Sur le problème d'équivalence de certaines structures infinitésimales*, Ann. Mat. Purra Appl. **36**, 27–120 (1954).
21. B. O'Neill, *Fundamental equations of submersions*, Michigan Math. J. **39**, 459–464 (1966).
22. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York-London, 1983.
23. M. H. Shahid, F. Al-Solamy, *Submersion of quaternion CR-submanifolds of a quaternion Kaehler manifold*, Math. J. Toyama Univ. **23**, 93–113 (2000).
24. M. Vișinescu, *Space-time compactification induced by nonlinear sigma models, gauge fields and submersions*, Czech. Journ. of Phys. **B37**, 525–528 (1987).
25. B. Watson, *G, G'-Riemannian submersions and nonlinear gauge field equations of general relativity*, in Global Analysis – Analysis on Manifolds, dedic. M. Morse (ed. T. Rassias), Teubner-Texte Math. **57**, 324–349 (1983).
26. B. Watson, *Almost contact metric 3-submersions*, Int. J. Math. Math. Sci. **7** (4), 667–688 (1984).