EXTRA-DIMENSIONS, CONFORMAL COUPLING AND HIGHER- DERIVATIVE THEORY IN THE BACKGROUND OF RICCI CONSTANT CURVATURE SPACETIME

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Higher derivative gravity theory in $d + 1$ dimensions with conformally non-minimal coupling scalar field and in the background of Ricci constant curvature spacetime is explored. We include the contribution of the ultra-light masses where some important features are revealed and particular case including the Gauss-Bonnet relation is discussed in some details.

Key words: higher-derivative theory, conformally non-minimal coupling, extra-dimensions, ultra-light masses, Gauss-Bonnet relation.

INTRODUCTION

From *high-energy physics points of view*, extended general relativity with higher-curvature effective Lagrangian and the higher dimensional Einstein gravity are considered today as important theories to describe cosmology beyond the standard model. These ideas grew up since the ancient work of Kaluza-Klein on $D = 4 + 1$ compact extra-dimension of the order of Planck lengths, where the isometries of the extra dimensions are gauge symmetries [1–5]. More precisely, we believe today that theories with extra-dimensions represent a *solid bridge between two volcanoes above trouble water*, *i.e.* to solve important problems in theoretical physics and cosmology, in particular the hierarchy problem in standard particle theory and the Big Bang singularity problem in standard cosmology:

HIERARCHY PROBLEM (VOLCANOE 1) → SINGULARTY PROBLEM (VOLCANOE 2)

There exist in literature a lot of extra-dimensional theories following the theoretical progress in mathematical physics:

SUPERGRAVITY → STRING THEORY → M-THEORY

including supersymmetry (SUSY), local SUSY or supergravity (SUGR), supergravities in various spacetime dimensions and their relevance to string dualities, superstrings (string theory with fermions), etc. [6, 7]. The discovery of $D = 11$ supergravity yielding a desirable string theory on the $D = 10$ boundary spacetime produced a revival of the Kaluza-Klein ideas in the early 80's. No dilaton is present but there exist some kind of geometrical moduli, and consequently a Kaluza-Klein dimensional reduction will still give rise to a dilaton-type field describing the size of the compact extra-dimension. This is to say that standard matter fields live on the boundary. Thus we have a new picture of the Universe. The claim of the first string revolution, triggered by the fact that superstrings theory can provide a theory of gravitation instead of a theory of quantum chromodynamics or strong interactions, surprisingly the cancellation of the anomalies and divergences, and the discovery of the heterotic $E_8 \otimes E_8$ or SO_{32} (gauge group) string at strong coupling, showed the interest of the 10 dimensions with six dimensional compactified spaces (Calabi-Yau, orbifolds...). Consequently, new topological objects appear such as non-perturbative strings, T-duality, S-duality, D-branes, Gödel branes (rotating), supermembranes, cosmic strings, colored strings, etc. The Kaluza-Klein scenario became more solid and rigid. All these have inspired new physical and realistic testable scenarios developed to solve the hierarchy problem by putting it in a different setting. The most interesting model concerns the Arkani-Hamed, Dimopoulos and Dvali (ADD) scenario in which our universe is a 3-brane manifold embedded in a higher $D = d + 4$ dimensional space compactified to some small volume [8, 9]. The d dimensions are compact with radius torodial topology that ranges from a fraction of millimeter for $d = 2$ to about $10\,fm$ for $d = 6$ and the standard particle theory is confined on the 3-brane but gravitons can propagate along the whole bulk space. As a result, the gravity is found to be deviated from Newton's standard law only on scales smaller than R which could be as large as a fraction of a millimeter since gravity is tested only down to sizes of about a millimeter. The new fundamental scale of gravity is supposed to be of the order of the electroweak scale in order to solve the famous hierarchy problem. Thus the hierarchy between the Planck and the electroweak (TeV) scale is generated by the large volume of the extra dimensions. The most interesting property of the ADD scenario is that it is compatible with the present experimental data, but at the same time it gives rise to many new phenomena that could be tested in the near future. In addition there are also models where the hierarchy is generated by the curvature of the extra dimensions as it is the case of the Randall-Sundrum (RSII) model with Z_2 symmetry (of the five-dimensional Einstein's field equations) and where the hypersurface geometry of the bulk space-time is AdS_5 (warped bulk geometries with negative cosmological constant and positive tension which corresponds to the vacuum energy density on the brane, but equal to the cosmological constant on the brane) and thus it cannot be factorized [10]. In RSII scenarios, due to the curvature of the bulk space time, Newton's law of gravity can be obtained on the brane of positive tension embedded in an infinite extra-dimensional spacetime manifold and the Einstein gravity is effectively recovered on the brane [11]. Unfortunately, the hierarchy problem persists in this scenario. Consequently, small corrections to Newton's law are generated and the possible scales in the model are constrained to be smaller than a millimeter.

These interested brane world scenarios modify the Einstein gravity drastically at shorter distance than the *AdS* curvature length, while above the *AdS* radius it is effectively approximated by the Einstein's $D = 3 + 1$ gravity. In addition, they give a master role to gravitons mass, whether they are considered massless or massive, as well to ultra-light spin-2 particles [12]. Moreover, the standard $D = 3 + 1$ Friedman-Robertson-Walker (FRW) cosmology is recovered on the brane in the low energy limit. Many different alternatives theoretical models have been developed including the higher derivative gravity theory (HDGT) with an additional quadratic scalar curvature [13–27]. As the standard Einstein-Hilbert action in $d + 1$ dimensions containing a scalar curvature of power one can not stabilize the scalar potential defined in the bulk, deviated theories including small polynomial corrections in Ricci scalar curvature which looks like R^n, R^n, R^2, R^3, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $R\ f(R)$ subject to $\lim f(R)/R = 0$ when $R \to 0$, etc are proposed. When these non-linear terms of scalar curvature (the main ingredient of the higher-derivative gravity, including the Gauss-Bonnet curvature theory) are added to Einstein-Hilbert part of the Lagrangian, it was theoretically found that exact appealing brane cosmological solutions may exist with or without dilatonic potential (acting as the bulk cosmological constant), including the inflationary solution [12, 28]. It is worth mentioning that HDGT with higher-order terms give important information about the physics near the Planck scale (the very early Universe), the physics of the black holes, holographic black holes and probably the leading key of quantum gravity. In all these models, note that the four, five, etc, gravitational constant are closely related and this represent an important characteristic indicating how strong is the Einstein's theory of gravity. Despite the fact that the leading order curvature theories define a certain number of remarkable gravitational properties, what is missing in most of these models is the role of the non-minimal coupling in HDGT near the brane background and the role of the scalar field being an important ingredient of inflationary scenarios.

From *astrophysical points of view*, recent observations of the Type SNIa distant supernovae and of the cosmic microwave background anisotropy have led to the idea that our universe is undergoing a super-expansion or accelerated expansion dominated by a positive cosmological constant (quintessence) such that $\ddot{a} > 0$ with $a(t)$ being the scale factor of the Universe, tending to a flat de-Sitter space-time as predicted by the standard old inflation theory [29–31]. These observations suggest from theoretical point of view the presence of a mysterious dark energy with negative pressure accounting for the missing energy if one really believes inflation theory in all its aspects [32–34]. But in fact, the crucial feature of inflationary and quintessence models are studied when the universe is dominated by a nonminimally coupled scalar field. A series of theoretical arguments in the current literature imply that the investigations of inflationary theory with minimal coupling (NMC) in the framework of general relativity are in fact theoretically inconsistent. Nevertheless, a correct treatment of inflationary cosmology implies highly the presence of non-minimal coupling between the inflaton with scalar field ϕ and the Ricci scalar curvature [35–37].

2. EXTRA-DIMENSIONS, CONFORMAL COUPLING AND HIGHER-DERIVATIVE THEORY IN THE BACKGROUND OF RICCI CONSTANT CURVATURE SPACETIME

Recently, we have investigated a particular cosmological model with complex scalar self-interacting inflation field non-minimally coupled to gravity, based on supergravities argument [38]. It was shown that in the case of non-minimal conformal coupling between the scalar curvature and the density of the scalar field as $L = -(\xi/2)\sqrt{-g}R\phi\phi^*$, $\xi = 1/6$ (ϕ^* is the complex conjugate of ϕ and g is the scalar metric) and for the particular scalar complex potential field (QPF) $\tilde{V}(\phi\phi^*) = -(3/4)m^2 + (3/4)m^2\omega\phi^2\phi^{*2}$, $\omega << 1$, ultra-light masses m ($|m^2| \approx H^2$, H is the Hubble parameter) are implemented naturally in the Einstein field equations (EFE), leading to a cosmological constant Λ in accord with observations. The induced (second) cosmological constant was found to be $\Lambda_m \equiv \Lambda_{induced} \approx -3m^2/4$. These ultra-light masses (ULM) are in fact too low while they may have desirable feature for the description of the accelerated universe and were proved to have important consequences on brane cosmology and higher-derivative gravity theory [39–42]. Moreover, there exists a wide class of cosmological theories for which the conformal coupling have been useful to understand a wide class of astrophysical arguments. The NMC bulk scalar field was found also to have interesting feature in brane world scenario, in particular, the RSII. Recently, it was found that in appropriate regions of ξ, the standard RSII is unstable. In addition, depending on the value of ξ, the model possesses three classes of new static solutions with their corresponding features [43, 44]. So, in this work, we are interested to know some additional features of the NMC+ULM in HDGT, in particular when the action of the theory in $\bar{D} = d + 1$ dimensions looks in the absence of matter like:

$$S = S_{HE} + S(R^2) + S(R_{AB}) + S(R_{ABCD}) + S_{boundary} + S(\varphi) =$$
$$= \int \sqrt{-g}\, d^{\bar{D}}x \left(\left(\frac{R - 2\Lambda_{bulk}}{2\kappa_{\bar{D}}} - \xi_{\bar{D}} \frac{R\varphi^2}{2} \right) + \right.$$
$$\left. + \lambda_{string} \left(aR^2 + bR_{AB}R^{AB} + cR_{ABCD}R^{ABCD} \right) - \frac{1}{2} g^{AB} \nabla_A \varphi \nabla_B \varphi - \tilde{V}(\varphi) \right) + \qquad (1)$$
$$+ S_{boundary}$$

where $S_{HE} = (2\kappa)^{-1} \int d^{\bar{D}}x \sqrt{-g}(R - 2\Lambda)$ is the Einstein-Hilbert gravitational part of the action, $S_{\text{int}} = -(1/2)\xi_{\bar{D}} \int d^{\bar{D}}x \sqrt{-g} R\varphi^2$ is the non-minimal interaction term between the gravitational and the complex scalar fields, $S(\varphi)$ describes the material part of the action associated with the complex scalar field, $\xi_D = (d-1)/4d$ is the conformal non-minimal coupling characterizing the non-minimal coupling to gravity of the scalar field in $D = d+1$ dimensions, Λ_{bulk} is the bulk cosmological constant, λ_{string} is the string coupling set equal to one for simplicity [4, 5],

$$\kappa_{\bar{D}} = \frac{2\left(\bar{D}-1\right)\pi^{\bar{D}/2} G}{\left(\bar{D}-2\right)\left(\frac{\bar{D}}{2}-1\right)!} \qquad (2)$$

is a suitable constant with dimensions L^{d-1} which in four dimensions is equal to $8\pi G$ in natural units, with G being Newton's gravitational constant, a, b and c are constants with dimension L^{3-d} and A, B, C, D... denotes the $d+1$ dimensional space-time indices. For convenience, we decomposed the complex scalar field into its module and phase as $\phi = \sqrt{\gamma}\exp(i\varphi)$. Consequently, $\tilde{V}(\varphi) = V_0 + (3/4)m^2\omega\varphi^4$, $\omega << 1$ and $V_0 = -(3/4)m^2$. The scalar field may be assumed dynamical, *i.e.* tends to a certain minimal value φ_0 when $t \to \infty$. By varying the action with respect to g^{AB}, we obtain the generalized Einstein's field equations:

$$G_{AB} + \kappa_{\bar{D}} \left[-\frac{1}{2} g_{AB} \left(aR^2 + bR_{CD}R^{CD} + cR_{CDEF}R^{CDEF} \right) + \right.$$
$$+ 2\left(aRR_{AB} + bR_{ACBD}R^{CD} + c\left(R_{ACDE}R_B{}^{CDE} - 2R_A{}^C R_{BC} + 2R_{ACBD}R^{CD} \right) \right) -$$
$$\left. - (2a + b + 2c)\left(\nabla_A \nabla_B R - g_{AB} \nabla^2 R \right) + (b + 4c)\nabla^2 G_{AB} \right] + \qquad (3)$$
$$+ k\left(\Lambda_{brane} g_{AB} - \nabla_A \varphi \nabla_B \varphi - \xi_{d+1}\left(g_{AB}\Box - \nabla_A \nabla_B + G_{AB} \right)\varphi^2 + \right.$$
$$\left. + g_{AB}\left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi + V_0 + \frac{3}{4} m^2 \omega \varphi^4 \right) \right) = 0.$$

For a $d+1$ dimensional conformal flat space-time given by [12]

$$ds_{d+1}^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{r} + \left(\frac{l}{2r}\right)^2 dr^2 + \frac{d^{n-1}X^2}{r} \tag{4}$$

where $d^{n-1}X^2$ denotes de $(n-1)$ dimensional flat space metric and l^2 is the square of the curvature radius, equations (2) give:

$$\frac{d(d-1)}{2l^2}\left(\frac{1}{\kappa_{\bar{D}}} - \xi_{d+1}\varphi^2\right) - \frac{d(d-3)}{l^4}\left[d(d+1)a + db + 2c\right] + \\ + \frac{4\Lambda_{brane} + 4V_0 + 3m^2\omega\varphi^4}{4} = 0 \tag{5}$$

For flat Minkowski space-time, $l^2 \to 0$, and consequently the effective brane cosmological constant is:

$$\Lambda_{brane}^{effective} = \frac{-4V_0 - 3m^2\omega\varphi^4}{4} = -V_0 - \frac{3m^2\omega\varphi^4}{4} \tag{6}$$

$$\underset{t\to\infty}{\to} -V_0 - \frac{3m^2\omega\varphi_0^4}{4} \tag{7}$$

$\Lambda_{brane}^{effective} > 0 (< 0)$ according to whether $\varphi_0^4 < (>) -4V_0/3m^2\omega$. Note that for negligible φ_0, $\Lambda_{brane}^{effective} > 0$ (< 0) for $V_0 < (>) 0$. Moreover, one may have $\omega > 0$ ($\tilde{V}(\varphi) = V_0 + (3/4)m^2\omega\varphi^4$) or $\omega < 0$ ($\tilde{V}(\varphi) = V_0 - (3/4)m^2\omega\varphi^4$) and accordingly four types of potential may hold in the theory. As a result, for $V_0 > 0$ and $\omega > 0$, $\Lambda_{brane}^{effective} < 0$ and for $V_0 < 0$ and $\omega < 0$, $\Lambda_{brane}^{effective} > 0$ while for the opposite signs $V_0 > 0$, $\omega < 0$ and $V_0 < 0$, $\omega > 0$, $\Lambda_{brane}^{effective}$ may be set equal to zero if $V_0 = -3m^2\omega\varphi_0^4/4$. If in contrast, the scalar field increases with time, $\left|\Lambda_{brane}^{effective}\right|$ will increases consequently. In order to stop this unwanted behavior, one must assume that the ULM decays in a way they ensure the constancy of $m^2\varphi^4$, *i.e.* $m^2\varphi^4 = cte$. As a result $\left|\Lambda_{brane}^{effective}\right| \to 0$ at late times. From equation (5), the effective Newton's gravitational constant in *D* dimensions is given by:

$$\kappa_{\bar{D}}^{effective} = \frac{\kappa_{\bar{D}}}{1 - \xi_{d+1}\varphi^2\kappa_{\bar{D}}} \underset{\varphi\to\varphi_0}{\to} \frac{\kappa_{\bar{D}}}{1 - \xi_{d+1}\varphi_0^2\kappa_{\bar{D}}} \tag{8.a}$$

$$\underset{\varphi\to\infty}{\to} 0 \tag{8.b}$$

$$\underset{\varphi_0\to 0}{\to} \kappa_{\bar{D}} \tag{8.c}$$

If $\xi_{d+1} \to -\zeta_{d+1}$ (negative non-minimal coupling parameter), the case (a) is of interest if the gravitational constant is dynamical, more precisely, increases with time. In RSII scenario with a NMC scalar field theories and negative non-minimal coupling parameter [47], the scalar field exhibits a unique bound state with tachyon mode implying some kind of instabilities for both the metric and the scalar. This gives a constant $\kappa_{\bar{D}}^{effective}$ as:

$$\kappa_{\bar{D}}^{effective} \approx \frac{\kappa_{\bar{D}}}{1-\xi_{d+1}\varphi_0^2\kappa_{\bar{D}}} \underset{\xi_{d+1}\to-\zeta_{d+1}}{\to} \frac{\kappa_{\bar{D}}}{1+\zeta_{d+1}\varphi_0^2\kappa_{\bar{D}}} \underset{\kappa_{\bar{D}}\propto t^m, m>0}{\to} \frac{1}{\zeta_{d+1}\varphi_0^2} \tag{9}$$

One finds two different values for l^2 assuming $\Lambda_{brane} = 0$ and if at late times $\varphi \to \varphi_0$:

$$\begin{gathered} l^2 = \frac{-d(d-1)\left(\frac{1}{\kappa_{\bar{D}}} - \xi_{d+1}\varphi^2\right)}{\left(4V_0 + 3m^2\omega\varphi^4\right)} \pm \\ \pm \frac{\sqrt{4d^2(d-1)^2\left(\frac{1}{\kappa_{\bar{D}}} - \xi_{d+1}\varphi^2\right)^2 + 4d(d-3)\left(4V_0 + 3m^2\omega\varphi^4\right)\left[d(d+1)a + db + 2c\right]}}{\left(4V_0 + 3m^2\omega\varphi^4\right)} \end{gathered} \tag{10}$$

which are related to the ULM. Assuming that

$$d(d-1)^2\left(\frac{1}{\kappa_{\bar{D}}} - \xi_{d+1}\varphi^2\right)^2 = -(d-3)\left(4V_0 + 3m^2\omega\varphi^4\right)\left[d(d+1)a + db + 2c\right] \tag{11}$$

which gives the effective minimum of the potential of the mass of ultra-light particles on the brane as:

$$V_0^{effective} = -\frac{d(d-1)^2\left(1-\xi_{d+1}\kappa_{\bar{D}}\varphi^2\right)^2}{4(d-3)\left[d(d+1)a + db + 2c\right]} - \frac{3m^2\omega\varphi^4}{4} \tag{12}$$

and one find the non-zero value:

$$\begin{aligned} l^4 &= \frac{d^2(d-1)^2\left(\xi_{d+1}\kappa_{\bar{D}}\varphi^2 - 1\right)^2}{\kappa_{\bar{D}}^2\left(4V_0 + 3m^2\omega\varphi^4\right)^2} = \\ &= \frac{d(3-d)\left(4V_0 + 3m^2\omega\varphi^4\right)\left[d(d+1)a + db + 2c\right]}{\kappa_{\bar{D}}^2\left(4V_0 + 3m^2\omega\varphi^4\right)^2} \end{aligned} \tag{13}$$

and therefore two possible solutions may exist. The first corresponds to de-Sitter case:

$$l_+^2(dS) = \frac{d(d-1)\left(\xi_{d+1}\kappa_{\bar{D}}\varphi^2 - 1\right)}{\kappa_{\bar{D}}\left(4V_0 + 3m^2\omega\varphi^4\right)} = = \frac{\sqrt{d(3-d)\left(4V_0 + 3m^2\omega\varphi^4\right)\left[d(d+1)a + db + 2c\right]}}{\kappa_{\bar{D}}\left(4V_0 + 3m^2\omega\varphi^4\right)} \tag{14.a}$$

$$\underset{\varphi\to\varphi_0}{\to} \frac{d(d-1)\left(\xi_{d+1}\kappa_{\bar{D}}\varphi_0^2 - 1\right)}{\kappa_{\bar{D}}\left(4V_0 + 3m^2\omega\varphi_0^4\right)} = = \frac{\sqrt{d(3-d)\left(4V_0 + 3m^2\omega\varphi_0^4\right)\left[d(d+1)a + db + 2c\right]}}{\kappa_{\bar{D}}\left(4V_0 + 3m^2\omega\varphi_0^4\right)} \tag{14.b}$$

$$\underset{\varphi_0\to 0}{\to} \frac{-d(d-1)}{4V_0\kappa_{\bar{D}}} = \frac{\sqrt{4d(3-d)\left[d(d+1)a + db + 2c\right]V_0}}{4\kappa_{\bar{D}}V_0} \tag{14.c}$$

and tends to zero for increasing scalar field, *i.e.* $l_2^+(dS) \underset{\varphi\to\infty}{\to} 0$.

The second corresponds to Anti-de-Sitter case:

$$l_-^2(AdS) = -\frac{d(d-1)\left(\xi_{d+1}\kappa_{\bar{D}}\varphi^2 - 1\right)}{\kappa_{\bar{D}}\left(4V_0 + 3m^2\omega\varphi^4\right)} = = -\frac{\sqrt{d(3-d)\left(4V_0 + 3m^2\omega\varphi^4\right)\left[d(d+1)a + db + 2c\right]}}{\kappa_{\bar{D}}\left(4V_0 + 3m^2\omega\varphi^4\right)} \tag{15.a}$$

$$\underset{\varphi\to\varphi_0}{\to} -\frac{d(d-1)\left(\xi_{d+1}\kappa_{\bar{D}}\varphi_0^2 - 1\right)}{\kappa_{\bar{D}}\left(4V_0 + 3m^2\omega\varphi_0^4\right)} = = -\frac{\sqrt{d(3-d)\left(4V_0 + 3m^2\omega\varphi_0^4\right)\left[d(d+1)a + db + 2c\right]}}{\kappa_{\bar{D}}\left(4V_0 + 3m^2\omega\varphi_0^4\right)} \tag{15.b}$$

$$\underset{\varphi_0\to 0}{\to} \frac{d(d-1)}{4V_0\kappa_{\bar{D}}} = -\frac{\sqrt{4d(3-d)\left[d(d+1)a + db + 2c\right]V_0}}{4\kappa_{\bar{D}}V_0} \tag{15.c}$$

and tends to zero for increasing scalar field, *i.e.* $l_2^-(AdS) \underset{\varphi\to\infty}{\to} 0$.

Moreover, $l_+^2(dS) \to -d(d-1)/4V_0\kappa_{\bar{D}}$ for too small φ_0. For $V_0 > 0$, $l_+^2(dS) > 0$ if $0 < d < 1$ while for $V_0 < 0$, $l_+^2(dS) > 0$ if $d > 1$. As for the anti-de-

Sitter case, $l_{-}^{2}(AdS)\to d(d-1)/4V_0\kappa_{\bar{D}}$ for too small φ_0. Consequently, for $V_0>0$, $l_{+}^{2}(AdS)<0$ if $0<d<1$ while for $V_0<0$, $l_{+}^{2}(AdS)>0$ if $d>1$. It is worth-mentioning that the minimum of the effective potential tends to

$$V_0^{effective}\underset{\varphi\to 0}{\to}-\frac{d(d-1)^2}{4(d-3)\left[d(d+1)a+db+2c\right]} \tag{16}$$

that is depends on the space dimension and on the parameters a, b, c. Two cases arises:

$$V_0^{effective}=-\frac{3}{4}m^2\to$$
$$\to m^2=\frac{d(d-1)^2}{3(d-3)\left[d(d+1)a+db+2c\right]}\underset{\substack{a=c=-b/4\\ d\neq 1}}{\overset{Gauss-Bonnet}{=}}\frac{d(d-1)}{3a(d-3)(d-2)} \tag{17}$$

$$V_0^{effective}=\frac{3}{4}m^2\to$$
$$\to m^2=\frac{-d(d-1)^2}{3(d-3)\left[d(d+1)a+db+2c\right]}\underset{\substack{a=c=-b/4\\ d\neq 1}}{\overset{Gauss-Bonnet}{=}}\frac{-d(d-1)}{3a(d-3)(d-2)} \tag{18}$$

In order to avoid tachyons, we must ensure to have a positive square mass. For the first general case (negative minimum potential (NMP)), $m^2>0$ if $(d-3)\left[d(d+1)a+db+2c\right]>0$ or in the Gauss-Bonnet special case $d>3$ or $d<2$. For the free parameters a, b, c, the space-time for NMP can be guaranteed as AdS_{d+1} if $d>3$ and $d(d+1)a+db+2c>0$ or $d<3$ and $d(d+1)a+$ $+db+2c<0$. The case $d=0$ and $d=1$ gives a zero ULM. As for the second general case (positive minimum potential (PMP)), $m^2>0$ if $(d-3)\left[d(d+1)a+db+2c\right]<0$ or in the Gauss-Bonnet special case $2<d<3$ or $0<d<1$ (fractal dimensions) [46]. Again, for the free parameters a, b, c the space-time for PMP can be guaranteed as AdS_{d+1} if $d>3$ and $d(d+1)a+db+2c<0$ or $d<3$ and $d(d+1)a+db+2c>0$. The cases $d=2$ and $d=3$ are critical as they yield a divergence of the ULM. Another important characteristic may appear from equation (12) and which corresponds to increasing scalar field with time. In order to avoid the divergence of V_0, we need to assume again that the ULM decays in a way they ensure the constancy of $m^2\varphi^4$, *i.e.* $m^2\varphi^4=cte$. In HDGT, there exist many theoretical evidence that the ULM may decays as $m^2\propto t^{-4}$ while the scalar field φ and the gravitational constant increase with time as $\varphi\propto t$ and $\kappa\propto t^2$ respectively [47], then from equation (12), one may write

$$V_0^{effective} \underset{\varphi \propto t}{\approx} -\frac{3m^2\omega\varphi^4}{4}\left(1+\frac{d(d-1)^2\xi_{d+1}^2\kappa_{\bar{D}}^2}{3m^2\omega(d-3)\left[d(d+1)a+db+2c\right]}\right) \quad (19.a)$$

$$\underset{\substack{a=c=-b/4 \\ d\neq 1}}{\overset{Gauss-Bonnet}{=}} -\frac{3m^2\omega\varphi^4}{4}\left(1+\frac{d(d-1)\xi_{d+1}^2\kappa_{\bar{D}}^2}{3m^2\omega a(d-2)(d-3)}\right) \quad (19.b)$$

In order to insure the stability of V_0, the conformal non-minimal coupling must decays with time in such a way where $\xi_{d+1}^2\kappa_{\bar{D}}^2/m^2$ is constant with time.

3. CONCLUSIONS AND OUTLOOKS

We have explored various solutions of the higher-derivative extra-dimensional gravity theory in the background metric of Ricci constant curvature (BMRCC) by taking into account the conformally coupling, ultra-light masses, scalar field and higher-curvature terms in Einstein field equations. We gave discussed several interesting points. It was found that there exist an effective cosmological constant on the BMRCC which depends on the sign of the minimum effective potential V_0 introduced in the theory and which may vanishes if $V_0 = -3m^2\omega\varphi_0^4/4$. In general, the potential $\tilde{V}$ was found to take an effective value due to the presence of the higher curvature terms as:

$$\tilde{V}(\varphi) = V_0 - V_0^{effective} - \frac{d(d-1)^2\left(1-\xi_{d+1}\kappa_{\bar{D}}\varphi^2\right)^2}{4(d-3)\left[d(d+1)a+db+2c\right]}$$

This relation may yields a relation between the ULM, d and the parameters a, b, c if $|V_0| = 3m^2/4$.

We have found an effective Newton's gravitational constant (ENGC) on the BMRCC which depends on the gravitational coupling constant in higher dimensions (GCCHD). If the GCCHD increases with time while the scalar field decays with time, then for a negative (positive) non-minimal coupling constant, the ENGC is a positive (negative) constant. The square of the curvature radius reveals many interesting properties. It was found that for $V_0 > 0$, $l_+^2(AdS) < 0$ if $0 < d < 1$ while for $V_0 < 0$, $l_+^2(AdS) > 0$ if $d > 1$. The cases $d = 2$ and $d = 3$ are critical as they yield a divergence of the ULM. In order to avoid tachyons, we must ensure to have a positive square mass. For the case of negative minimum potential (NMP), it was found that $m^2 > 0$ if $(d-3)\left[d(d+1)a+db+2c\right] > 0$ or in the Gauss-Bonnet special case $d > 3$ or $d < 2$. For the free parameters a, b, c, the space-time for NMP can be guaranteed as AdS_{d+1} if $d>3$ and $d(d+1)a+db+2c>0$ or $d<3$ and $d(d+1)a+db+2c<0$. As for case of positive minimum potential (PMP), we have $m^2>0$ for $(d-3)\left[d(d+1)a+ +db+2c\right]<0$ or in the Gauss-Bonnet special case $2<d<3$ or $0<d<1$ (fractal dimensions which is widely discussed in literature and have attracted considerable interest recently in string theory and quantum gravity [48–64]). Again, for the free parameters a, b, c, the space-time for PMP can be guaranteed as AdS_{d+1} if $d>3$ and $d(d+1)a+db+2c<0$ or $d<3$ and $d(d+1)a+ +db+2c>0$. Increasing and decreasing power-law type of the scalar field at late times and gravitational constant and decreasing conformal coupling constant, were also discussed where many additional important features are under progress. One may extend the discussed scenarios to black holes physics, black holes holography problems and fractal higher-dimensional space-time with many choice of the coupling to the Ricci scalar [65].

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