HAMILTON-JACOBI TREATMENT OF LAGRANGIAN WITH FERMIONIC AND SCALAR FIELD

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The Hamilton-Jacobi formalism is applied to singular Lagrangian containing variables which are elements of a fermionic and a scalar field. The equations of motion are obtained as total differential equations in many variables. The integrability conditions are examined. Path integral quantization based on Hamilton-Jacobi approach is obtained for the system.

1. INTRODUCTION

In this work we intend to study singular systems with Lagrangians containing elements of a fermionic and a sclar field from the point of view of the Hamilton-Jacobi formalism developed by Güler [1, 2]. The study of such systems through Dirac's generalized Hamiltonian formalism has already been extensively developed in literature [3–5] and will be used for comparative purposes.

Despite the success of Dirac's approach in studying singular systems, which is demonstrated by the wide number of physical systems to which this formalism has been applied, it is always instructive to study singular systems through other formalisms, since different procedures will provide different views for the same problems, even for nonsingular systems. The Hamilton-Jacobi formalism that we study in this work, is applied to a few of physical examples [6-9]. But a better understanding of this approach utility in the studying singular systems is still lacking, and such understanding can only be achieved through its applications to other interesting physical systems. Our aim in this work is to apply the Hamilton-Jacobi approach for singular systems to the case of Lagrangian containing a fermionic and a scalar field, and to compare the results to those obtained through Dirac's method.

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2. HAMILTON-JACOBI APPROACH

In this section, we shall briefly review the Hamilton-Jacobi formulation of constrained systems [1, 2]. The starting point of this method is to consider the Lagrangian $L \equiv L(q_i, \dot{q}_i, \tau)$, $i = 1, 2, \ldots, n$, with the Hess matrix

$$A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, \tau)}{\partial \dot{q}_i \partial \dot{q}_j}, \qquad i, j = 1, 2, \ldots, n, \tag{1}$$

of rank $(n - r)$, $r < n$. Then the r momenta are dependent. The generalized momenta P_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, 2, \ldots, n - r, \tag{2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = n - r + 1, \ldots, n, \tag{3}$$

The singularity of the system enables us to solve eq. (2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, p_a; \tau) \equiv \omega_b. \tag{4}$$

Substituting eq. (4), into eq. (3), we obtain the constraints as

$$H'_\mu = p_\mu + H_\mu(\tau, q_i, p_a) = 0, \tag{5}$$

where

$$H_\mu = -\frac{\partial L}{\partial \dot{q}_\mu}|_{\dot{q}_a \equiv \omega_a}. \tag{6}$$

In this formulation the usual Hamiltonian H_0 is defined as

$$H_0 = -L + p_a \dot{\omega}_a - \dot{q}_\mu H_\mu. \tag{7}$$

Like functions H_μ, the function H_0 is not an explicit function of the velocities $\dot{q}_\nu$. Therefore, the Hamilton-Jacobi function $S(\tau, q_i)$ should satisfy the following set of *Hamilton-Jacobi partial differential equations* (HJPDE) simultaneously for an extremum of the function:

$$H'_\alpha\left(t_\beta, q_\alpha, P_i = \frac{\partial S}{\partial q_i}, P_0 = \frac{\partial S}{\partial t_0}\right) = 0, \tag{8}$$

where

$$\alpha, \beta = 0, n - r + 1, \ldots, n; \qquad a = 1, 2, \ldots, n - r,$$

and

$$H'_\alpha = p_\alpha + H_\alpha. \tag{9}$$

The canonical equations of motion are given as total differential equations in variables t_β,

$$dq_p = \frac{\partial H'_\alpha}{\partial p_p} dt_\alpha, \quad p = 0, 1, \ldots, n; \quad \alpha = 0, n-r+1, \ldots, n, \tag{10}$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \qquad a = 1, \ldots, n-r, \tag{11}$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \qquad \alpha = 0, n-r+1, \ldots, n, \tag{12}$$

$$dZ = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_\alpha} dt_\alpha \right), \tag{13}$$

where

$$Z \equiv S(t_\alpha, q_a), \tag{14}$$

being the action. Thus, the analysis of a constrained system is reduced to solve equations (10–12) with constraints

$$H'_\alpha(t_\beta, q_a, P_i) = 0, \qquad \alpha, \beta = 0, n-r+1, \ldots, n. \tag{15}$$

Since the equations above are total differential equations, integrability conditions should be checked. These equations of motion are integrable if and only if the variations of H'_α vanish identically, that is

$$dH'_\alpha = 0. \tag{16}$$

If they do not vanish identically, then we consider them as new constraints. This procedure is repeated until a complete system is obtained.

In this paper we will study a system of Lagrangian containing elements of a fermionic and a scalar field.

3. DIRAC'S METHOD

Consider a Lagrangian containing elements of a fermionic and a scalar field σ given by

$$L = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) + \frac{1}{2}\partial_\mu \sigma(x) \partial^\mu \sigma(x) - \frac{1}{2} M^2 \sigma^2(x), \quad \mu = 0, 1, 2, 3. \tag{17}$$

We are adopting the Minkowski metric $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$.

The Lagrangian (17) is singular, since the rank of the Hess matrix (1) is one. The generalized momenta (2) and (3) can be written as

$$p=\frac{\partial L}{\partial\dot{\sigma}}=\partial^0\sigma, \tag{18}$$

$$p_\psi=\frac{\partial L}{\partial\dot{\psi}}=i\bar{\psi}\gamma^0=-H_\psi, \tag{19}$$

$$p_{\bar{\psi}}=\frac{\partial L}{\partial\dot{\bar{\psi}}}=0=-H_{\bar{\psi}}. \tag{20}$$

Where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual ψ as a column vector and $\bar{\psi}$ as a row vector implies that p_ψ will be a row vector while $p_{\bar{\psi}}$ will be a column vector.

The usual Hamiltonian H_0 is given as

$$H_0=-L+\omega p+\partial_0\psi\, p_\psi\,|_{p_\psi=-H_\psi}+\partial_0\bar{\psi}\, p_{\bar{\psi}}\,|_{p_{\bar{\psi}}=-H_{\bar{\psi}}}, \tag{21}$$

or

$$H_0=\frac{1}{2}p^2-\bar{\psi}(i\gamma^a\partial_a\psi-m\psi)-\frac{1}{2}(\partial_a\sigma\partial^a\sigma-M^2\sigma^2),\qquad a=1,2,3. \tag{22}$$

Eqs. (19) and (20) lead to the primary constraints

$$H'_\psi=p_\psi+H_\psi=p_\psi-i\bar{\psi}\gamma^0=0, \tag{23}$$

and

$$H'_{\bar{\psi}}=p_{\bar{\psi}}+H_{\bar{\psi}}=p_{\bar{\psi}}=0. \tag{24}$$

respectively. These constraints lead to the total Hamiltonian

$$H_T=H_0+\lambda_\psi H'_\psi+\lambda_{\bar{\psi}}H'_{\bar{\psi}}, \tag{25}$$

or

$$\begin{aligned}H_T=&\frac{1}{2}p^2-\bar{\psi}(i\gamma^a\partial_a\psi-m\psi)-\\&-\frac{1}{2}(\partial_a\sigma\partial^a\sigma-M^2\sigma^2)+\lambda_\psi(p_\psi-i\,\bar{\psi}\,\gamma^0)+\lambda_{\bar{\psi}}p_{\bar{\psi}}.\end{aligned} \tag{26}$$

According to Dirac's method, the time derivative of the primary constraints should be zero, that is

$$\dot{H}'_\psi=\{H'_\psi,H_T\}=-(i\partial_a\bar{\psi}\gamma^a+m\bar{\psi})-i\lambda_{\bar{\psi}}\gamma^0\approx 0, \tag{27}$$

$$\dot{H}'_{\bar{\psi}}=\{H'_{\bar{\psi}},H_T\}=(i\gamma^a\partial_a-m)\psi+i\gamma^0\lambda_\psi\approx 0. \tag{28}$$

Eqs. (27) and (28) fix the multipliers $\lambda_{\bar{\psi}}$ and λ_ψ, respectively as

$$i\lambda_{\bar{\psi}}\gamma^0=-(i\partial_a\gamma^a+m)\bar{\psi}, \tag{29}$$

$$i\gamma^0\,\lambda_\psi=-(i\gamma^a\partial_a-m)\psi. \tag{30}$$

Multiplying eq. (29) from the right and eq. (30) from the left by $-i\gamma^0$, we obtain

$$\lambda_{\overline{\psi}} = -\partial_a \overline{\psi}\gamma^a \gamma^0 + im\overline{\psi}_{(2)}\gamma^0, \tag{31}$$

$$\lambda_{\psi} = -\gamma^0(\gamma^a\partial_a + im)\psi. \tag{32}$$

There are no secondary constraints. Taking suitable linear combinations of constraints, one has to find all numbers of second-class ones, there are

$$\Phi_1 = H'_{\psi} = p_{\psi} - i\overline{\psi}\gamma^0, \tag{33}$$

and

$$\Phi_2 = H'_{\overline{\psi}} = p_{\overline{\psi}}. \tag{34}$$

The total Hamiltonian is vanishing weakly. It can completely be written in terms of second-class constraints as

$$H_T = \frac{1}{2}p^2 - \overline{\psi}(i\gamma^a\partial_a\psi - m\psi) - \frac{1}{2}(\partial_a\sigma\partial^a\sigma - M^2\sigma^2) + \lambda_{\psi}\Phi_1 + \lambda_{\overline{\psi}}\Phi_2. \tag{35}$$

The equations of motion are read as

$$\dot{\sigma} = \{\sigma, H_T\} = p, \tag{36}$$

$$\dot{\psi} = \{\psi, H_T\} = \lambda_{\psi}, \tag{37}$$

$$\overline{\psi} = \{\overline{\psi}, H_T\} = \lambda_{\overline{\psi}}, \tag{38}$$

$$\dot{p} = \{p, H_T\} = -M^2\sigma, \tag{39}$$

$$\dot{p}_{\psi} = \{p_{\psi}, H_T\} = -\overline{\psi}(i\overline{\partial_a}\gamma^a + m), \tag{40}$$

$$\dot{p}_{\overline{\psi}} = \{p_{\overline{\psi}}, H_T\} = (i\gamma^a\partial_a - m)\psi + i\gamma^0\lambda_{\psi}. \tag{41}$$

Differentiating eq. (36) with respect to time, and using eq. (39), we have

$$\ddot{\sigma} + M^2\sigma = 0. \tag{42}$$

Substituting from eqs. (31) and (32) into eqs. (38) and (39) respectively, we get

$$\overline{\psi}(i\overleftarrow{\partial}_{\mu}\gamma^{\mu} + m) = 0, \tag{43}$$

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0. \tag{44}$$

Substituting from eq. (30) into eq. (41), one obtains

$$\dot{p}_{\overline{\psi}} = 0. \tag{45}$$

In the following section the same system will be discussed using Hamilton-Jacobi approach.

4. HAMILTON-JACOBI METHOD

The set of Hamilton-Jacobi partial differential equation (HJPDE) (8) read as

$$H_0' = p_0 + H_0 = p_0 + \frac{1}{2}p^2 - \overline{\psi}(i\gamma^a \partial_a \psi - m\psi) - \frac{1}{2}(\partial_a \sigma \partial^a \sigma - M^2 \sigma^2), \tag{46}$$

$$H_\psi' = p_\psi + H_\psi = p_\psi - i\overline{\psi}\gamma^0 = 0, \tag{47}$$

$$H_{\overline{\psi}}' = p_{\overline{\psi}} + H_{\overline{\psi}} = p_{\overline{\psi}} = 0. \tag{48}$$

Therefore, the total differential equations for the characteristic (10), (11) and (12) are:

$$d\sigma = p\, d\tau, \tag{49}$$

$$dp = -M^2 \sigma\, d\tau, \tag{50}$$

$$dp_\psi = -\overline{\psi}(i\overleftarrow{\partial}_a \gamma^a + m)d\tau, \tag{51}$$

$$dp_{\overline{\psi}} = (i\gamma^a \partial_a - m)\psi d\tau + i\gamma^0 d\psi \tag{52}$$

The integrability conditions $(dH_\alpha' = 0)$ imply that the variation of the constraints H_ψ' and $H_{\overline{\psi}}'$ should be identically zero, that is

$$dH_\psi' = dp_\psi - i\, d\overline{\psi}\gamma^0 = 0, \tag{53}$$

$$dH_{\overline{\psi}}' = dp_{\overline{\psi}} = 0. \tag{54}$$

Substituting from eqs. (51) and (52) into eqs. (53) and (54), respectively we get the following equations of motion:

$$\dot{\sigma} = p. \tag{55}$$

$$\overline{\psi}(i\overleftarrow{\partial}_\mu \gamma^\mu + m) = 0, \tag{56}$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \tag{57}$$

From eqs. (50–52), we get the equations

$$\dot{p} = -M^2 \sigma, \tag{58}$$

$$\dot{p}_\psi = -\overline{\psi}(i\overleftarrow{\partial}_a \gamma^a + m), \tag{59}$$

$$\dot{p}_{\bar{\psi}} = 0. \tag{60}$$

Differentiate eq. (55) with respect to time and using eq. (58), we obtain

$$\ddot{\sigma} + M^2\sigma = 0. \tag{61}$$

5. CONCLUSION

In this paper we have investigated constrained system of Lagrangian containing a fermionic and a scalar field using Dirac's Hamiltonian formulism and Hamilton-Jacobi formulism.

In Dirac method the total Hamiltonian composed by adding the constraints multiplied by Lagrange multipliers to the canonical Hamiltonian. In order to drive the equations of motion, one needs to redefine these unknown multipliers in an arbitrary way.

However, in the Hamilton-Jacobi formulism, there is no need to introduce Lagrange multipliers to the canonical Hamiltonian. Both the consistency conditions and the integrability conditions lead to the same constraints. In Hamilton-Jacobi formulation, the equations of motion are obtained directly by using HJPDES as total differential equations. Also, it is not necessary to distinguish between first and second class constraints.

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