FRACTAL SPACE-TIME THEORY
AND TIME DEPENDENT GINZBURG-LANDAU MODEL
IN SUPERCONDUCTIVITY*

M. AGOP[1], M. BUZDUGAN[2,3], V. ENACHE[2]

[1] Department of Physics, Technical "Gh. Asachi" University Iași, Iași 700029, România
[2] Faculty of Physics, "Al. I. Cuza" University Iași, Blvd Carol no. 11, Iași 700506, România
[3] mihaelb@catv.embit.ro

Received September 26, 2006

It is shown that, if one introduces the hydrodynamic formulation of Scale Relativity Theory (SRT) into the Time Dependent Ginzburg-Landau (TDGL) model, the London gauge and the zero momentum of the Copper pairs (*i.e.* the London equations) arise naturally. Then, a particular relation between the fractal coefficient, friction coefficient and the (dimensionless) Ginzburg-Landau parameter which yields a natural gauge for the TDGL equation, is obtained. If the value of the real velocity of the Copper pairs tends to zero, the imaginary velocity of the pairs becomes real. The subquantum potential is proportional to the density of the Copper pairs. Moreover, under special circumstances, the superconductor acts as a subquantum medium energy accumulator.

1. INTRODUCTION

In the Ginzburg-Landau theory of phase transitions [1–3], the state of a superconducting material near the critical temperature is described by a complex-valued order parameter ψ, a real valued vector potential $\mathbf{A}$, and, when the state changes with time, a real-valued scalar potential ϕ. The role of ϕ differs from that of ψ and $\mathbf{A}$: the latter is formed by predictive variables, whose evolution is governed by differential equations; the former resembles a Lagrange multiplier more. After suitable non-dimensioning, the equations and boundary conditions satisfied by ψ and $\mathbf{A}$ are as follows [1–3]

$$\eta\left(\frac{\partial}{\partial t}+i\kappa\phi\right)\psi=-\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+\left(1-|\psi|^2\right)\psi \quad \text{in } \Omega\times(0,\infty), \tag{1}$$

$$\frac{\partial \mathbf{A}}{\partial t}+\nabla\phi=-\nabla\times\nabla\times\mathbf{A}+\mathbf{J}_s+\nabla\times\mathbf{H} \quad \text{in } \Omega\times(0,\infty), \tag{2}$$

* Paper presented at the National Conference on Applied Physics, June 9–10, 2006, Galați, Romania

$$\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\cdot\mathbf{n}=-\frac{i}{\kappa}\gamma\psi \qquad \text{on } \partial\Omega\times(0,\infty), \tag{3}$$

$$(\nabla\times\mathbf{A}-\mathbf{H})\times\mathbf{n}=0 \qquad \text{on } \partial\Omega\times(0,\infty). \tag{4}$$

Here,

$$\mathbf{J}_s=\frac{1}{2i\kappa}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)-|\psi|^2\mathbf{A}=-\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]. \tag{5}$$

The domain Ω corresponds to the region occupied by the superconducting material. We assume that it is bounded, $\Omega\subset\Re^D$ with $D=2$ or 3, and that its boundary $\partial\Omega$ is regular enough; **n** denotes the outer unit normal to $\partial\Omega$. As usual, $\nabla\equiv$ grad, $\nabla\times\equiv$ curl, $\nabla\cdot\equiv$ div, and $\nabla^2=\nabla\cdot\nabla\equiv\Delta$. Furthermore, $\mathbf{i}$ is the imaginary unit, and a superscript * denotes complex conjugation. The parameters of the model are η, a (dimensionless) friction coefficient; k, the (dimensionless) Ginzburg-Landau parameter; and γ, a non-negative parameter, which is zero if the supercondcuting material is surrounded by vacuum. The **H** field vector is a given applied magnetic field; in practice, **H** is either time-independent or time-periodic. The quantity $\mathbf{J}_s$ is the so-called super-current or, more correctly, super-current density. The super-current is a phenomenological quantity, which is considered a flux of moving "super-electrons". The super-electrons (or Cooper pairs), whose density is $n_s=|\psi|^2$, are responsible for the superconducting properties of the material. For example, the super-current prevents a magnetic field from penetrating a superconducting region. Note that $\mathbf{E}=-\partial\mathbf{A}/\partial t-\nabla\phi$ is the electric field and $\mathbf{B}=\nabla\times\mathbf{A}$ the magnetic induction. Therefore, equation (2) may be viewed as Ampere's law, $\nabla\times\mathbf{B}=\mathbf{J}$, where the total current **J** is the sum of a "normal" current $\mathbf{J}_n=\mathbf{E}$, the super-current $\mathbf{J}_s$, and the transport current $\mathbf{J}_t=\nabla\times\mathbf{H}$. The normal current obeys Ohm's law; the "normal conductivity" coefficient is equal to one in the chosen system of units.

The system of equations (1)–(5), with appropriate initial conditions, constitutes the TDGL model of superconductivity. In the present paper some interesting results on the superconducting state are obtaines, if one introduces the SRT in the TDGL model.

2. MATHEMATICAL MODEL

Let us write the time-dependent Ginzburg-Landau equation, considering the nondifferentiability (fractality) of space-time for small scales, *i.e.* by introducing the covariant derivative of the scale relativity theory [4–6]

$$\delta/\delta t = \partial_t + \mathbf{V}\cdot\nabla - i\mathcal{D}\Delta \tag{6}$$

where the mean velocity $\mathbf{V}$ is now complex, and $\mathcal{D}$ is a parameter characterizing the fractal behavior of trajectories (diffusion coefficient). Note that in the Ginzburg-Landau theory of phase transitions [5], the state of a superconducting material near the critical temperature is described by a complex-valued order parameter $\psi = Ae^{iS}$, a real-valued vector potential $\mathbf{A}$, and, when the state changes with time, a real-valued scalar potential ϕ.

The initial system of two non-linear partial derivative equations splits into four equations, two real and two imaginary ones.

The real part of the TDGL system using the zero-electric potential gauge [7] is written as follows:

$$\begin{aligned}&\eta\left[\partial_t \ln A + (\mathbf{v}\cdot\nabla)\ln A + (\mathbf{u}\cdot\nabla)S + \mathcal{D}\Delta S + 2\mathcal{D}A^{-1}(\nabla S\cdot\nabla A)\right] = \\ &= \kappa^{-2}A^{-1}\Delta A - \kappa^{-2}(\nabla S)^2 + \kappa^{-2}\mathbf{A}\cdot\nabla S + 1 - \mathbf{A}^2 - A^2\end{aligned} \tag{7}$$

$$\partial_t\mathbf{A} + (\mathbf{v}\cdot\nabla)\mathbf{A} = \Delta\mathbf{A} - \nabla(\nabla\cdot\mathbf{A}) + \left(\kappa^{-1}\nabla S - \mathbf{A}\right)A^2 + \nabla\times\mathbf{H} \tag{8}$$

and the imaginary part:

$$\begin{aligned}&\eta\left\{\partial_t S + (\mathbf{v}\cdot\nabla)S - (\mathbf{u}\cdot\nabla)\ln A - \mathcal{D}A^{-1}\Delta A - \mathcal{D}(\nabla S)^2\right\} = \\ &= \kappa^{-2}\Delta S + 2\kappa^{-2}A^{-1}(\nabla S\cdot\nabla A) - 2\kappa^{-1}\mathbf{A}\cdot\nabla\ln A - \kappa^{-1}\nabla\cdot\mathbf{A}\end{aligned} \tag{9}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{A} + \mathcal{D}\Delta\mathbf{A} = 0 \tag{10}$$

Here we used the complex velocity [4–6]

$$\begin{aligned}&\mathbf{V} = \mathbf{v} + i\mathbf{u}, \qquad \mathbf{v} = 2\mathcal{D}\nabla S \\ &\mathbf{u} = -i\mathcal{D}\nabla\ln\rho, \quad \rho = A^2\end{aligned} \tag{11a-d}$$

It is difficult to solve the system of non-linear partial derivative equations (7)–(10). In order to find an analytical solution here, we try a simplification of this system, first by reducing the problem to the one-dimensional case. Let us project the system along the Ox axis. One gets the real part:

$$\begin{aligned}&\eta\left[\partial_t \ln A + v_x\partial_x \ln A + u_x\partial_x S + \mathcal{D}\partial_{xx}S + 2\mathcal{D}\partial_x S\partial_x \ln A\right] = \\ &= \kappa^{-2}A^{-1}\partial_{xx}A - \kappa^{-2}(\partial_x S)^2 + 2\kappa^{-1}A_x\partial_x S + 1 - A_x^2 - A^2\end{aligned} \tag{12}$$

$$\partial_t A_x + v_x\partial_x A_x = \left(\kappa^{-1}\partial_x S - A_x\right)A^2 + (\nabla\times\mathbf{H})_x \tag{13}$$

and the imaginary one:

$$\eta\left[\partial_t S+v_x\partial_x S-u_x\partial_x\ln A-\mathcal{D}A^{-1}\partial_{xx}A-\mathcal{D}\left(\partial_x S\right)^2\right]= \\ =\kappa^{-2}\partial_{xx}S+2\kappa^{-2}\partial_x S\partial_x\ln A-2\kappa^{-1}A_x\partial_x\ln A-\kappa^{-1}\partial_x A_x \tag{14}$$

$$u_x\partial_x A+\mathcal{D}\partial_{xx}A_x=0 \tag{15}$$

In the resulting real and imaginary parts of equations (12)–(15), we operate the following substitutions (the hydrodynamic formulation of the SRT [4–6]):

$$\mathcal{D}=\hbar/2m,\quad v_x=\left(\hbar/m\right)\partial_x S=2\mathcal{D}\partial_x S,\quad u_x=-\left(\hbar/m\right)\partial_x\ln A=-2\mathcal{D}\partial_x\ln A \tag{16}$$

and it can be easily noticed, that the following relation will result and will be also used in the forthcoming simplifications:

$$A^{-1}\partial_{xx}A=-\left(2\mathcal{D}\right)^{-1}\partial_x u_x+\left(4\mathcal{D}^2\right)^{-1}u_x^2 \tag{17}$$

Finally, after considering the stationary isolated case (the external magnetic field is null $\left(\nabla\times\mathbf{H}\right)_x=0$), and using (16) and (17) we obtain the real and imaginary part of equations (12)–(15):

$$\eta\left(2^{-1}\partial_x v_x-\left(2\mathcal{D}\right)^{-1}v_x u_x\right)= \\ =-\left(2\kappa^2\mathcal{D}\right)^{-1}\partial_x u_x+\left(4\kappa^2\mathcal{D}^2\right)^{-1}\left(u_x^2-v_x^2\right)+\left(\kappa\mathcal{D}\right)^{-1}v_x A_x+1-A_x^2-A^2 \tag{18}$$

$$v_x\partial_x A_x=\left(\left(2\kappa\mathcal{D}\right)^{-1}v_x-A_x\right)A^2 \tag{19}$$

$$\eta\left[\left(4\mathcal{D}\right)^{-1}\left(v_x^2+u_x^2\right)+2^{-1}\partial_x u_x\right]= \\ =-\left(2\kappa^2\mathcal{D}^2\right)^{-1}v_x u_x+\left(\kappa\mathcal{D}\right)^{-1}u_x A_x-\kappa^{-1}\partial_x A_x+\left(2\kappa^2\mathcal{D}\right)^{-1}\partial_x v_x \tag{20}$$

$$u_x\partial_x A_x+\mathcal{D}\partial_{xx}A_x=0 \tag{21}$$

The imaginary parts of the computed set of equations, (19) and (21), offer an interesting result. First of all, in order to satisfy the equation (19), the following two relations result :

$$v_x=2\kappa\mathcal{D}A_x \qquad \partial_x A_x=0 \tag{22a,b}$$

The first relation is nothing but the condition for a zero momentum of the superconducting pair, and is the London equation (for other details see [7]). As it is known, this equation is not gauge invariant, therefore it can be correct only for a particular choice of the gauge. The choice is the London gauge specified by div $\mathbf{A}=0$. The second relation, is nothing but one-dimensional London gauge.

Consequently, this equation is automatically satisfied; the same happens to equation (21). In other words, equations (19) and (21) are nothing but the London equations, which naturally results from the TDGL system completed with the hydrodynamic formulation of the scale relativity (SR).

Now, the system (18)–(21) reduces to the following second degree equation in u_x (by eliminating the partial derivative in x of u_x) and assuming (22a,b) holds:

$$\left(2\kappa^2\mathcal{D}^2\right)^{-1}u_x^2+\eta\kappa A_x u_x+1-A^2+A_x^2=0 \tag{23}$$

which has the straightforward solution:

$$u_{x_{1,2}}=-\eta\kappa^3\mathcal{D}^2A_x\pm\kappa^2\mathcal{D}^2\sqrt{\left(\eta^2\kappa^2-\kappa^{-2}\mathcal{D}^{-2}\right)A_x^2-2\kappa^{-2}\mathcal{D}^{-2}\left(1-A^2\right)} \tag{24}$$

with η a (dimensionless) friction coefficient, κ the (dimensionless) Ginzburg-Landau parameter, $\mathcal{D}$ the diffusion coefficient, A_x the vector potential and A^2 the concentration of the Copper pairs.

Let us analyze the solution (24) which contains some interesting results, as can be notices in the following. If, for the friction coefficient value, one chooses the value $\eta=\pm\sqrt{2}/\kappa^2\mathcal{D}$ that annuls the paranthesis under the square root, then (24) takes a simpler form $u_{x_{1,2}}=\mp\sqrt{2}\kappa\mathcal{D}\left(A_x-i\sqrt{1-A^2}\right)$. First, we can focus on the particular relation $\eta=\pm\sqrt{2}/\kappa^2\mathcal{D}$, among the diffusion coefficient, the (dimensionless) friction coefficient and the (dimensionless) Ginzburg-Landau parameter which gives a new natural gauge for the TDGL equation. Here $\omega\equiv\mathcal{D}=\pm\sqrt{2}/\kappa^2\eta$ resulted in a natural manner, from the TDGL system completed with the hydrodynamic formulation of scale relativity. Secondly, we can get an interesting case if the real velocity of the pair is equated to zero, *i.e.* $v_x=0$ (that is $A_x=0$ – the Copper pairs are "freezing") $u_{x_{1,2}}=\pm i\sqrt{2}\kappa\mathcal{D}\sqrt{1-A^2}=$ $=\pm i\left(\kappa\eta\right)^{-1}\sqrt{1-A^2}$. Here, if we put the real velocity $\mathbf{v}=0$, we still obtain a non-zero velocity $\mathsf{V}=\pm\sqrt{2}\kappa\mathcal{D}\sqrt{1-A^2}=\pm2\left(\kappa\eta\right)^{-1}\sqrt{1-A^2}$ coming from the complex part of Nottale's velocity, in other words the complex velocity $\mathbf{u}$ becomes real.

At the same time, if the complex velocity u is substituted in the subquantum potential, and the expression $\eta=\pm\sqrt{2}/\kappa^2\mathcal{D}$ of the diffusion coefficient, as a result of using the more general $\phi+\omega\,\mathrm{div}A=0$ gauge [7], we obtain $Q=-m\mathcal{D}\nabla\cdot\mathbf{u}-2^{-1}m\mathbf{u}^2=\mp\left(\sqrt{2}m\mathcal{D}/\eta\right)\left(1-A^2\right)$. The subquantum potential takes now a very simple expression which is directly proportional to the density of states of Copper pairs. When the density of states of Copper pairs becomes zero (*i.e.* the material is normal) the subquantum potential has a finite value, $\mp\sqrt{2}m\mathcal{D}/\eta$ and when it becomes 1 (*i.e.* the entire material becomes superconducting), the subquantum potential turns to zero – the entire quantity of energy from the subquantum medium transfers to the superconducting pairs. Consequently, one can assume that the energy from the background subquantum medium can be stocked by transforming all the particles from the environment into Copper pairs and then "freezing" them. The superconductor acts like a subquantum medium energy accumulator.

3. CONCLUSIONS

The main conclusions of the present paper are the followings:

i) the London equations come naturally from the imaginary equations;

ii) the particular relation among the diffusion coefficient, the (dimensionless) friction coefficient and the (dimensionless) Ginzburg-Landau parameter yields a new natural gauge for the TDGL equation;

iii) if one equates to zero the real velocity **v** of the Copper pairs, *i.e.* for a coherence quantum fluid, the imaginary velocity **u** of the Nottale's complex speed turns real;

iv) the subquantum potential takes a very simple expression which is directly proportional to the density of states of the Copper pairs;

v) the energy from the subquantum fluid can be stocked by transforming all the particles from the environment into Copper pairs and then "freezing" them, *i.e.* the superconducting fluid can act like a subquantum medium energy accumulator.

REFERENCES

1. V. L. Ginzburg, L. D. Landau, On the Theory of Superconductivity (1998); Zh. Eksperim. I Teor. Fiz. (USSR) **20**, 1064–1082 (1950); Engl. transl. in D. TER HAAR: L. D. Landau; Men of Physics, Vol. I, Pergamon Press, 138–167 (1965).
2. L. P. Gor'kov, G.M. Eliashberg, Generalizations of the Ginzburg-Landau equations for non-stationary problems in the case of alloys with paramagnetic impurities, Zh. Eksp. Teor. Fiz. **54**, 612–626 (1968); Soviet Phys., JETP 27, 328–334 (1968).
3. N. Galbreath *et al.*, Parallel Solution of the Three-Dimensional Time-Dependent Ginzburg-Landau Equation, Proc. SIAM Conf. On Parallel Processing for Scientific Computing, SIAM, Philapelphia, (1993).
4. L. Nottale, Fractal Space-Time and Microphysics: Towards a Theory of Scale Relativity (World Scientific, Singapore) (1993).

5. M. S. El Naschie, O. E. Rösler, I. Prigogine, Quantum Mechanics, Diffusion and Chaotic Fractals (Elsevier, Oxford), (1995).
6. M. Agop, P. D. Ioannou, C. Gh. Buzea, P. Nica, Hydrodynamic formulation of scale relativity theory and unified superconductivity by means of a fractal string, Physica C **390**, 37–55 (2003).
7. C. P. Poole, H. A. Farach, R. J. Geswich, *Superconductivity*, Academic Press, San Diego, New York, Boston, London, Sydney, Tokyo, Toronto, 1995.