

MAPLE ROUTINES FOR BOSONS ON CURVED MANIFOLDS*

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Exactly solvable models for fields coupled to gravity within a general-relativistic analytical approach require a huge volume of computations. Therefore, the need of iterations on nonlinear equations and the necessity of permanent checking the results justifies the idea of building a complete algorithm for this problem. After presenting a MAPLE set of procedures and routines for the study of coupled Klein–Gordon–Maxwell–Einstein system of equations on a curved manifold, we focus on two particular metrics and derive the explicit form of field equations describing the analyzed configurations.

Key word: Klein-Gordon equation, Maxwell equation, Einstein equation.

1. INTRODUCTION

In the last three decades, field theories on curved manifolds with significant applications to Cosmology have been intensively investigated leading to various exciting results that shed quite a new light on our understanding of the Universe [1]. Besides the historic Einstein’s Universe, which continues to play an important role for academic exercises and tests on various semi-classical or quantum field dynamics, [2–4], exactly solvable models for fields coupled to gravity have been a main target of investigations.

After the mid 1980’s, a major interest has been focused on macroscopic stable boson stars, since they have been considered to provide a considerable fraction of the non-baryonic part of dark matter, [5].

The problem has been successfully worked out in low dimensional gravity [4], while in four dimensions, fields interacting via gravity has been investigated mainly by numerical calculation [6].

Nevertheless, using the Newtonian approximation, the whole analysis gets greatly simplified allowing interesting and inspiring investigations, as for example

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the process of gravitational-radiation emission from an excited boson star [5]. Since the general-relativistic analytical study of the coupled field equations is of a real interest for a better understanding of different stellar configurations as well as for a numerical-functional combined iterative treatment which describes the dynamics of charged boson nebulae. In a series of recent articles, [7], we got the first-order approximating solutions to the system of Klein–Gordon–Maxwell–Einstein equations. Moreover, we have studied the feedback of gravity and electric field on the charged scalar source, using a perturbative approach. The huge volume of computations, the need of iterations on the nonlinear equations, the necessity of permanent checking the results are fully justifying the idea of building a complete algorithm for this problem.

2. FIELDS EQUATIONS ON CURVED SPACE-TIME

In the present paper, we are going to derive the Klein–Gordon–Maxwell–Einstein system of equations, for a complex scalar field minimally coupled to a spherically symmetric space-time. Employing a pseudo-orthonormal tetradic frame, $\{e_a\}_{a=1,4}$, in order to have a Minkowskian metric tensor

$$\eta_{ab} = \text{diag}[1, 1, 1, -1] \quad (1)$$

a charged massive boson, coupled to the electromagnetic field, is described by the $SO(3) \times U(1)$ -gauge invariant Lagrangian density of the form

$$L = \eta^{ab} \bar{\Phi}_{,a} \Phi_{,b} + m_0^2 \bar{\Phi} \Phi + \frac{1}{4} F^{ab} F_{ab} \quad (2)$$

The gauge covariant – read

$$\Phi_{,a} = \Phi_{|a} - ieA_a \Phi$$

and respectively

$$\bar{\Phi}_{,a} = \bar{\Phi}_{|a} + ieA_a \bar{\Phi} \quad (3)$$

The Maxwell tensor

$$F_{ab} = A_{b;a} - A_{a;b} \quad (4)$$

is expressed in the terms of the Levi-Civita covariant derivative of the four-potential $\{A_a\}_{a=1,4}$, *i.e.*

$$A_{a;b} = A_{alb} - A_c \Gamma^c_{ab} \quad (5)$$

By varying with respect to different fields, we come to the Klein–Gordon–Maxwell (KGM) system of equations [8]:

$$\square \Phi - m_0^2 \Phi = 2ieA^c \Phi_{|c} + e^2 A^c A_c \Phi \quad \text{and its h.c.} \quad (6)$$

and

$$F^{ab}{}_{;c} = -ie\eta^{ab} \left[\bar{\Phi} (\Phi_{|b} - ieA_b \Phi) - (\bar{\Phi}_{|b} - ieA_b \bar{\Phi}) \Phi \right] \quad (7)$$

Building up the energy-momentum tensor [7, 9]

$$T_{ab} = \bar{\Phi}_{;a} \Phi_{;b} + \bar{\Phi}_{;b} \Phi_{;a} + F_{ac} F_b{}^c - \eta_{ab} L. \quad (8)$$

it can be derived the Einstein equation

$$G_{ab} = kT_{ab}, \quad (9)$$

where the tensor G_{ab} have the explicit form as.

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \quad (10)$$

3. REVIEW OF MAPLE ROUTINES

In order to build an efficient tool for the analysis of the system (6–8), using a set of procedures and routines, let us describe the structure and the main features of the procedures of the formalism.

Actually, two major parts of the programs can be detected: the first one is regarding the tensorial structure of the objects needed in the algorithm, while the second one contains line-commands specific to each version.

The first part of the program starts after initializing the main used package, with a set of definitions for the entire set of necessary objects. For example, using the MAPLE platform, one can write (for the particular case of using polar coordinates)

```
> coord := [r, theta, phi, t];
> Mg := array(1..4, 1..4, symmetric, [(1,1)=g11(coord[1], ..., coord[4]),
(1,2)=g12(coord[1], coord[2], coord[3], coord[4]), ...,
... ..
(4,4)=g44(coord[1], coord[2], coord[3], coord[4])])
> g := create([-1, -1], op(Mg));
```

Now, using the MAPLE platform, one is able to compute the first and second order derivatives of the metric tensor g_{ab} , the Christoffel symbols, $\Gamma^a{}_{bc}$, the Riemann and Ricci tensors and finally, the Einstein tensor G_{ab} components, as it follows:

```
> D1g := d1metric(g, coord);
> D2g := d2metric(D1g, coord);
```

```
> Cf1:=Christoffel(D1g);
... ..
> Einstein1:=lin_com(1,Rc,1/2*Rsc,g);
```

In order to get a simpler set of differential equations, it is more convenient to introduce a pseudo-orthonormal tetradic frame, $\{e_a\}_{a=1,4}$ with the metric tensor of Minkowskian spacetime, (1). The transformation tensors h_1 and respectively h_{1inv} , defined in the following command lines, are needed to turn to the tetradic frame.

```
> Mgc:=array(1..4,1..4,symmetric,[(1,1)=1,(1,2)=0..... (4,4)= -1]);
> g_c:=create([-1,-1],op(Mgc));
> frame(g_c, h1inv, const_g, coord);
> eval(h1_inv);
... ..
> change_basis(g_c,h1,h1inv);
```

The definition of the gauge-covariant derivatives of the complex scalar field, (3), is an important step in building the system lagrangian and in writing down the KGM coupled equations. In order to get a coherent structure, the terms are defined as tensors. The main advantage of this form is the simplicity on the transformation between the two considered frames. The gauge fields $\{A_a\}_{a=1,4}$ can be defined in a direct manner and, using the tensor structure of definition, it can be easily used in building the gauge derivatives of the scalar field.

```
> P:=create([],Phi(coord[1],coord[2],coord[3],coord[4]));
> P_conjugate:=create([],conjugate(Phi(coord[1],.....,coord[4])));
> P_bar_a:=partial_diff(P,coord);
... ..
> A_gauge:=create([-1],array(1..4, [(1)=A[1](coord[1],.....,coord[4]),
(2)=A[2](coord[1],coord[2],coord[3],coord[4]),
(3)=A[3](coord[1],coord[2],coord[3],coord[4]),.....
> P_bar_a_2:=prod(A_gauge,P);
> P_coma_a:=lin_com(1,P_bar_a,-I*E,P_bar_a_2);
> P_bar_a_conjugate:=prod(P_conjugate,A_gauge);
.....
```

Further, using a tensor for the interaction term in the Klein–Gordon equations, the evolution equation for the complex scalar field Φ reads:

```
> prod(A_gauge,P_bar_a_conjugate);
> J_Klein_Gordon1:=prod(A_gauge_up,P_bar_a,[1,1]);
> J_Klein_Gordon_temp:=prod(A_gauge,P);
> J_Klein_Gordon2:=prod(A_gauge_up,J_Klein_Gordon_temp,[1,1]);
```

```

> .....
> Klein_Gordon1:=prod(g_inv,P_bar_a,[2,1]);
> Klein_Gordon2:=partial_diff(Klein_Gordon1,coord);
> Klein_Gordon3:=contract(Klein_Gordon2,[1,2]);
> Klein_Gordon_kinetic:=lin_com(1/det_g,Klein_Gordon3);
> Klein_Gordon5:=prod(P_conjugate,P);
> .....
> Eq_Klein_Gordon:=Klein_GordonM=J_Klein_GordonM;

```

The next step is devoted to the Maxwell tensor F_{ab} , defined in (4), and to the corresponding Maxwell equations (7). The currents and the interaction term are put in the same tensorial structure.

```

> Fab1:=cov_diff(A_gauge, coord,Cf2);
> Fba2:=permute_indices(Fab1,[2,1]);
> Fab:=lin_com(1,Fab1,-1,Fba2);
> .....
> JMaxwell2:=prod(P,P_coma_a_conjugate);
> JMaxwell3:=lin_com(1,JMaxwell1,-1,JMaxwell2);
> JMaxwell4:=prod(g_inv,JMaxwell3,[2,1]);
> JMaxwell:=lin_com(-I*E,JMaxwell4);

```

One may notice that these steps are sufficient to build the Maxwell equations and, by using the tensors: h_1 whose character is $[+1, -1]$ and his invert $-h_{1inv}$, one can turn to the initial frame system.

```

> Fab_b:=cov_diff(Fab,coord,Cf2);
> Fab_b1:=partial_diff(Fab_up,coord);
> Fab_b2:=prod(Fab_up,Cf2,[1,2]);
> .....;
> JMaxwellM:=JMaxwell[compts];
> for i from 1 to 4 do eqMaxwell[i]:=MaxwellM[i]=JMaxwellM[i] end do;

```

At this point, an important achievement is the building of necessary Lorentz condition. In order to introduce the whole lagrangian of the system, given by (2), and to build the energy–momentum tensor involved in the Einstein equations one needs the following procedure part. This can generate a rank zero tensor-type object which can be employed in writing down the energy-momentum tensor, T_{ab} , given by (8):

```

> L1:=prod(P_coma_a_conjugate,P_coma_a);
> Lkinetic:=prod(g_inv,L1,[1,1],[2,2]);
> L2:=prod(P_conjugate,P);
> Lmass:=lin_com(m0^2,L2);
> .....;

```

```
> Tab:=lin_com(1,Tab1,1,Tba1,1,Tab2,-1,Tab3);
> J_Einstein:=lin_com(Kappa,Tab);
```

Putting together the objects defined in previous steps of the algorithm, the Einstein equations (9) explicitly are:

```
> EinsteinM:=Einstein1[compts];
> J_EinsteinM:=J_Einstein[compts];
> for i from 1 to 4 do : for j from 1 to 4 do
  EqEinstein[i,j]:=EinsteinM[i,j]=J_EinsteinM[i,j] end do : end do;
> .....
```

Finally, one has to impose specific *ansatz* conditions, in order to obtain a set of simpler equations, easier to manipulate and of course, much interesting for didactic reasons.

In the last part of this short presentation, will be briefly discussed only procedures used in order to obtain a first order perturbative solution for the coupled field equations' system. This approach starts with the physically reasonable assumption that the charged scalar field is the main source of both the electromagnetic and gravitational fields. Considering it, in the first instance, could be neglected the feedbacks of gravity and electromagnetism on the charged scalar source [2, 7, 9].

To succeed in this purpose, it has to build field equations in an Euclidean approximation, using null Christoffel symbol values for the field covariant derivative and null metric tensor functions. In a coherent approaching, it should be computed all the necessary elements for the Klein - Gordon - Maxwell system equations.

```
> Fab_euclidean1:=partial_diff(A_gauge,coord);
> Fab_euclidean2:=permute_indices(Fab_euclidean1,[2,1]);
> Fab_euclidean:=lin_com(1,Fab_euclidean1,-1,Fab_euclidean2);
> Fab_euclidean_temp1:=prod(g_rigid_inv,Fab_euclidean,[2,1]);
> Fab_euclidean_up:=prod(g_rigid_inv,Fab_euclidean_temp1,[2,2]);
>....
> Maxwell_Eq_euclidean:=contract(Fab_euclidean_up_coma_b,[2,3]);
> Maxwell_EqM_euclidean:=Maxwell_Eq_euclidean[compts];
>....
```

At this point, it should be underlined the necessary elimination of the mixed partial differential terms from the Maxwell's equations. Using the Lorentz condition and deriving it in respect with the corresponding coordinate, can be obtained a coupled electromagnetic field equations system without mixed partial differential terms.

In the next step, it has to use the partial differential equation package functions (*PDEtools*) to build the first order perturbative solutions. In this order,

for the Klein -Gordon scalar field equation it should be considered the free field case.

```
> with(PDEtools);
> .....
> Eq_Klein_Gordon_1order:=Klein_GordonM_Euclidean=0;
```

Using this scalar field solution, can be further focussed on the Maxwell system equations. Eliminating the phase factors, could be built approximate analytical solution for the large distance domain or, could be identified particular solutions using a specific procedure designed to investigate this possibility [7].

However, in the most studied cases, for an analytical approaching, can be obtained only approximate solutions of different rank.

4. SOME SPECIFIC RESULTS

In this section, we are going to consider only one particular metric case and employ the formalism described previously to write down the system of coupled equations governing the corresponding configurations.

The analysis is performed for a curved spacetime, with the line-element

$$ds^2 = e^{2f} (dr)^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - e^{2h} (dt)^2 \quad (11)$$

where f and h are functions of r and t , which has been employed in the study of boson nebulae [7, 9]. In the pseudo-orthonormal tetradic frame, $\{e_a\}_{a=1,4}$ with the corresponding dual orthonormal base

$$\omega^1 = e^f dr \quad \omega^2 = r d\theta \quad \omega^3 = r \sin \theta d\varphi \quad \omega^4 = e^h dt \quad (12)$$

while the non-vanishing connection coefficients are:

$$\begin{aligned} \Gamma_{12} &= -\frac{1}{r} e^{-f} \omega^2 & \Gamma_{13} &= -\frac{1}{r} e^{-f} \omega^3 \\ \Gamma_{23} &= \frac{ctg(\theta)}{r} \omega^3 & \Gamma_{14} &= -f_{,t} e^{-h} \omega^1 + h_{,r} e^{-f} \omega^4 \end{aligned} \quad (13)$$

As in our previous analysis, [7], we are going to use the minimally symmetric *ansatz* $A_1 = A_1(r, t)$, $A_2 = 0$, $A_3 = 0$, $A_4 = A_4(r, t)$, $\Phi = \Phi(r, t)$, for which the the Klein–Gordon–Maxwell system of equation, (6, 7), becomes

$$\begin{aligned} e^{-2f} \left\{ \Phi_{,rr} + \left[h_{,r} - f_{,r} + \frac{2}{r} \right] \Phi_{,r} \right\} - e^{-2h} \left\{ \Phi_{,tt} + [f_{,t} - f_{,r}] \Phi_{,t} \right\} - m_0^2 \Phi = \\ = 2ieA_1 e^{-f} \Phi_{,r} - 2ieA_4 e^{-h} \Phi_{,t} + e^2 \left[(A_1)^2 - (A_4)^2 \right] \end{aligned} \quad (14)$$

respectively

$$\begin{aligned} e^{-h} \left[e^{-f} (A_{4,r} + h_r A_4) - e^{-h} (A_{1,t} + f_t A_1) \right]_{,t} = \\ = ie \left[e^{-f} (\bar{\Phi} \Phi_{,r} - \bar{\Phi}_{,r} \Phi) - 2ie A_1 \bar{\Phi} \Phi \right] \end{aligned} \quad (15)$$

and

$$\begin{aligned} e^{-f} \left[e^{-f} (A_{4,r} + h_r A_4) - e^{-h} (A_{1,t} + f_t A_1) \right]_{,r} + \\ + 2 \frac{e^{-f}}{r} \left[e^{-f} (A_{4,r} + h_r A_4) \right] - 2 \frac{e^{-f}}{r} \left[e^{-h} (A_{1,r} + f_r A_1) \right] = \\ = ie \left[e^{-h} (\bar{\Phi} \Phi_{,t} - \bar{\Phi}_{,t} \Phi) - 2ie A_4 \bar{\Phi} \Phi \right] \end{aligned} \quad (16)$$

In the same assumptions, the Lorentz condition reads

$$e^{-f} \left[A_{1,r} + \left(h_r + \frac{2}{r} \right) A_1 \right] + e^{-h} \left[e^{-f} (A_{4,t} + f_t A_4) \right] = 0 \quad (17)$$

Let us end with the physically reasonable assumption that the charged scalar field is the main source of both the electromagnetic and gravitational fields [7]. Neglecting the feedbacks of gravity and electromagnetism on the charged scalar source, the equation of motion (14) does simply become the one of an $\ell = 0$ state on a Minkowskian background $f = 0 = h$, *i.e.*:

$$\Phi_{,rr} - \frac{2}{r} \Phi_{,t} - \Phi_{,tt} - m_0^2 \Phi = 0 \quad (18)$$

and it's hermitic conjugated, with the positive-frequency mode solutions

$$\Phi = \frac{N}{r} e^{i(\omega t - kr)}, \quad \bar{\Phi} = \frac{N}{r} e^{-i(\omega t - kr)} \quad (19)$$

where $\omega = \sqrt{k^2 + m_0^2}$.

In the same approximation, $f = 0 = h$, the Maxwell equations (15, 16) and the Lorentz condition (17) are given by:

$$A_{1,rr} + \frac{1}{2r} A_{1,r} - \frac{1}{2r^2} A_1 - A_{1,tt} = -2ek \frac{|N|^2}{r^2} \quad (20)$$

and respectively

$$A_{4,rr} + \frac{2}{r} A_{4,r} - A_{4,tt} = 2e\omega \frac{|N|^2}{r^2} \quad (21)$$

and

$$A_{1,r} + \frac{2}{r} e^{-f} A_1 - A_{4,t} = 0 \quad (22)$$

with the solutions

$$A_1 = ek|N|^2 \quad (23)$$

and

$$A_4(r, t) = 2e\omega|N|^2 \log\left(\frac{r}{r_0}\right) + 2ek\frac{|N|^2}{r}t \quad (24)$$

In the particular case $k = 0$ where the star is just above the passage to the stable excited states, its mode pulsation $\omega = m_0$ being located at the accumulation point of the eigenfrequencies of an excited boson star, the correspondingly linearized Einstein field equations become:

$$\frac{2}{r}h_{,r} - \frac{2}{r^2}f = k\left[\frac{|N|^2}{r^4} - 2e^2m_0^2\frac{|N|^4}{r^2}\right] + k\left[4e^2m_0^2\frac{|N|^4}{r^2}\log\left(\frac{r}{r_0}\right)\right] \quad (25)$$

$$\begin{aligned} h_{,rr} + \frac{1}{r}(h_{,r} - f_{,r}) + f_{,tt} = \\ = k\left[-\frac{|N|^2}{r^4} + 2e^2m_0^2\frac{|N|^4}{r^2}\right] + k\left[4e^2m_0^2\frac{|N|^4}{r^2}\log\left(\frac{r}{r_0}\right)\right] \end{aligned} \quad (26)$$

and

$$\begin{aligned} \frac{2}{r}f_{,r} + \frac{2}{r}f = \\ = k\left[2m_0^2\frac{|N|^2}{r^2} + \frac{|N|^2}{r^4} + 2e^2m_0^2\frac{|N|^4}{r^2}\right] + k\left[4e^2m_0^2\frac{|N|^4}{r^2}\log\left(\frac{r}{r_0}\right)\right] \end{aligned} \quad (27)$$

with the solutions of the form

$$f(r) = \frac{C_1}{r} - \frac{2b}{r^2} + 2c\ln\left(\frac{r}{r_0}\right) + (a - c) \quad (28)$$

respectively

$$h(r) = -\frac{C_1}{r} + C_2 + (a - 2c)\ln\left(\frac{r}{r_0}\right) + 2c\left[\ln\left(\frac{r}{r_0}\right)\right]^2 \quad (29)$$

where we used the definitions

$$a = \kappa m_0^2 |N|^2 \quad b = \kappa |N|^2 \quad c = \kappa e^2 m_0^2 |N|^4$$

while C_1 has the significance of the Schwarzschild mass M . Obviously, by imposing to have the ordinary Minkowski metric at asymptotia, we get $C_2 = 0$ and $a = 2c$.

With these metric functions, solutions of Klein–Gordon–Maxwell–Einstein equations (in a first order approximation), one is able to compute the total charge, particle number, radius and mass of the analyzed configuration [7]. Moreover, while dealing with the feedback of gravity and electric field on the charged scalar source, in the case when the radial wave number k is zero, the transition amplitudes and the source-field regeneration rate have been analytically computed, in a first-order perturbative approach, [7]. Going further, these results can be generalized for the non-vanishing momentum case, in order to deal with the quantum transitions, such as the ones related to gravitoelectric particle creation.

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REFERENCES

1. N. D. Birrell, P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge Univ. Press, Cambridge, 1982.
2. R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, Univ. of Chicago Press, Chicago, 1994.
3. R. K. Kaul, *Gravitation, Gauge Theories and the Early Universe*, Kluwer, Amsterdam, 1989.
4. S. Deser, *General Relativity and Gravitation. One Hundred Years After the Birth of Albert Einstein*, ed. A. Held Plenum Press, New York, 1980.
5. M. Carmeli, *Found. Phys.*, 15, 175 (1985).
6. M. Carmeli, S. Malin, *Found. Phys.*, 15, 1019 (1985).
7. M. A. Dariescu, C. Dariescu, *Found. Phys.*, 1323 (1991); 24, 1577 (1994); 28, 1393 (1998).
8. S. M. Christensen, S. A. Fulling, *Phys. Rev. D* 15, 2088 (1977).
9. C. Dariescu, M. Aura Dariescu, I. Gottlieb, *Teoria cuantică a câmpului*, Iași, (2000).