FRACTAL SPACE-TIME THEORY AND SUPERCONDUCTIVITY★

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The hydrodynamic model of the scale relativity theory is established. Then ψ simultaneously becomes wave-function and speed potential and some properties of the superconducting state are given: quantification law of the magnetic flux, Cooper pair through the kink solution and the "accumulator" effect.

1. INTRODUCTION

The Scale Relativity Theory (SRT) extends Einstein's principle of relativity to scale transformation of resolution [1, 2]. It is based on the giving up of the axiom of differentiability of the space-time continuum. Three consequences arise from this with drawl: i) The geodesics of a non-differentiable space-time are fractal and in infinite number: this leads one to use a fluid-like description, $\boldsymbol{v} = \boldsymbol{v}[x(t), t]$; ii) The geometry of space-time becomes fractal, *i.e.*, explicitly resolution dependent: this allows one to describe a non-differentiable physics in terms of differential equations acting in the scale spaces. In such a context, each elementary displacement is then described in terms of the sum, $d\boldsymbol{X} = d\boldsymbol{x} + d\boldsymbol{\xi}$ of a mean classical displacement $d\boldsymbol{x} = \boldsymbol{v}dt$ and of a fractal fluctuation $d\boldsymbol{\xi}$ whose behavior satisfies the principle of SRT. It is such that $\langle d\boldsymbol{\xi}\rangle = 0$ and $\langle d\xi^2\rangle = 2\mathcal{D}dt$; iii) Time reversibility is broken at the infinitesimal level: this can be described in terms of a two-valuedness of the velocity vector, for which we use a complex representation, $\mathbf{V} = \left[\left(\boldsymbol{v}_+ + \boldsymbol{v}_-\right)/2\right] - i\left[\left(\boldsymbol{v}_+ - \boldsymbol{v}_-\right)/2\right]$. These three effects can be combined to construct a complex time-derivative operator

$$\delta/\delta t = \partial_t + \mathbf{V}\cdot\nabla - i\mathcal{D}\Delta \tag{1}$$

where the mean velocity $\mathbf{V}$ is now complex, and $\mathcal{D}$ is a parameter characterizing the fractal behavior of trajectories.

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Since the mean velocity is complex, the same is true of the Lagrange function, then of the generalized action $\mathbf{S}$ as well. Setting $\psi = \exp\left(i\mathbf{S}/2m_0\mathcal{D}\right) = Ae^{iS}$ with A the amplitude and S the phase the wave function, velocity $\mathbf{V}$ has the expression

$$\mathbf{V} = -2i\mathcal{D}\nabla\left(\ln\psi\right) \tag{2}$$

and the Newton's equation of dynamics, $m\,\delta\mathbf{V}/\delta t = -\nabla U$, can be integrated in terms of a generalized Schrödinger equation:

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial_t\psi = \left(U/2m\right)\psi \tag{3}$$

In the present paper using the hydrodynamic model of the SRT some properties of the superconducting state are given.

2. HYDRODYNAMIC MODEL OF SCALE RELATIVITY THEORY. SUPERCONDUCTIVITY

The hydrodynamic model in the non-differentiable space-time is built, replacing the complex velocity $\mathbf{V}$ in the form

$$\begin{gathered}\mathbf{V} = \mathbf{v} + i\mathbf{u}, \qquad \mathbf{v} = 2\mathcal{D}\nabla S\\ \mathbf{u} = -i\mathcal{D}\nabla\ln\rho, \quad \rho = A^2\end{gathered} \tag{4 a-d}$$

in the Newton's Eq. of dynamics. It firstly follows that

$$\begin{gathered}-m^{-1}\nabla\cdot U = \left\{\partial_t\left(\mathbf{v} - i\mathcal{D}\nabla\ln\rho\right) + \left[\left(\mathbf{v} - i\mathcal{D}\nabla\ln\rho\right)\cdot\nabla\right]\times\right.\\ \left.\times\left(\mathbf{v} - i\mathcal{D}\nabla\ln\rho\right) - i\mathcal{D}\Delta\left(\mathbf{v} - i\mathcal{D}\nabla\ln\rho\right)\right\}\end{gathered} \tag{5}$$

Using the identities

$$\begin{gathered}\Delta\nabla = \nabla\Delta, \qquad \left(\nabla f\cdot\nabla\right)\left(\nabla f\right) = 2^{-1}\nabla\left(\nabla f\right)^2\\ f^{-1}\Delta f = \Delta\ln f + \left(\nabla\ln f\right)^2\end{gathered}, \tag{6 a-c}$$

and separating in (5) the real and the imaginary parts (up to an arbitrary phase factor $\beta(t)$ which may be set to zero by a suitable choice of the phase of ψ) we obtain:

$$\begin{gathered}m(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\nabla(U - Q)\\ \partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0\end{gathered} \tag{7a, b}$$

where Q is the quantum potential and has the expression

$$Q = -2m\mathcal{D}^2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} = -m\mathcal{D}\nabla \cdot \mathbf{u} - \frac{1}{2} m\mathbf{u}^2 \tag{8}$$

The quantum potential depends only on the imaginary part of the complex velocity. Since **u** arises from non-differentiability according to the non-differentiable space model of quantum mechanics, it might be stressed out that the quantum potential comes from the non-differentiability of the quantum space-time (sub-quantum medium).

The wave function of $\psi(\mathbf{r}, t)$ is invariant when its phase changes by an integer multiple of 2π. Indeed, equation (4b) gives:

$$\oint m\mathbf{v}d\mathrm{r} = 2m\mathcal{D}\oint dS = 4\pi nm\mathcal{D}, \quad n = 0, \ \pm 1, \ \pm 2, \ ... \tag{9}$$

a condition of compatibility between the SR hydrodynamic model and the wave mechanics.

For $\mathcal{D} = \hbar/2m^*$ with $m^* = 2m_e$ the mass of the Cooper pair, the relation (9) becomes $\oint \mathbf{p} \cdot d\mathbf{r} = nh$. This result can be identified with the quantification law of the magnetic flux $\phi = n\phi_0$ with $\phi_0 = h/2e$ the fluxon and $n \in Z$. Indeed, the generalized momentum of the Cooper pair in the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ with **A** the potential vector of the magnetic field, $\mathbf{P} = 2m_e\mathbf{v} + 2e\mathbf{A} = \hbar\nabla S + 2e\mathbf{A}$ is null, *i.e.* $\mathbf{P} \equiv 0$. From here, by means of integration, it follows that

$$\hbar\oint \nabla S = \pm 2\pi n\hbar = 2e\oint \mathbf{A}d\mathbf{r} = 2e\iint_{\partial\Sigma} \mathbf{B} \cdot d\boldsymbol{\Sigma} = 2e\phi_e$$

The set of Eqs. (7a, b) represents a complete system of differential Eqs. for the fields $\rho(\mathbf{r}, t)$ and $\mathbf{v}(\mathrm{r}, t)$; relation (9) relates each solution $(\rho, \mathbf{v})_n$ with the wave solution Ψ in a unique way.

The field $\rho(\mathbf{r}, t)$ is a probability distribution, namely the probability of finding the particle in the vicinity $d\mathbf{r}$ of the point $\boldsymbol{r}$ at time t, $dP = \rho d\mathbf{r}$, $\iiint \rho dr = 1$, the space integral being extended over the entire area of the system. Any time variation of the probability density $\rho(\mathbf{r}, t)$ is accompanied by a probability current $\rho\mathbf{v}$ pointing towards or outwards, the corresponding field point $\boldsymbol{r}$ (Eq. (7b)).

The position probability of the real velocity field $\mathbf{v}(\mathrm{r}, t)$ (Eq. (7a)), varies with space and time similar to a hydrodynamic fluid placed in the force-field of an external potential $U(\boldsymbol{r}, t)$ and a quantum potential (8). The fluid (in the sense of a statistical particles ensemble) exhibits, however, an essential difference compared to an ordinary fluid: in a rotation motion $\mathbf{v}(\mathrm{r}, t)$ increases (decreases) with the decreasing (increasing) distance $\boldsymbol{r}$ from the center (Eq. (9)).

The expectation values for the real velocity field and the velocity operator $\hat{\mathbf{v}} = -2i\mathcal{D}\nabla$ of wave mechanics are equal $\langle \mathbf{v} \rangle = \iiint \rho \mathbf{v} d\mathrm{r} = \iiint \Psi^* \hat{\mathbf{v}} \Psi d\mathrm{r} = \langle \hat{\mathbf{v}} \rangle_{WM}$ but in the higher-order, $|n| > 2$, similar identities are invalid, namely $\langle \mathbf{v}^n \rangle \neq \langle \mathbf{v}^n \rangle_{WM}$. The expectation for the 'quantum force' vanishes at all times (theorem of Ehrenfest), *i.e.*, $\langle -\nabla Q \rangle = \iiint \rho(-\nabla Q) d\boldsymbol{r} = 0$ or explicitly $2m\mathcal{D}^2 \iiint \rho \nabla\left(\nabla^2 \sqrt{\rho} / \sqrt{\rho}\right) d\mathbf{r} = m\mathcal{D}^2 \oint (\rho \nabla\nabla \ln \rho) \cdot d\boldsymbol{\sigma} = 0.$

Two types of fractal stationary states are to be distinguished:

i) Dynamic states. For $\partial/\partial t = 0$ and $\mathbf{v} \neq 0$, Eqs. (7a,b) give

$$\nabla\left(\frac{1}{2} m\mathbf{v}^2 + U - 2m\mathcal{D}^2 \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) = 0, \qquad \nabla(\rho\mathbf{v}) = 0 \tag{10a,b}$$

namely

$$\frac{1}{2} m\mathbf{v}^2 + U - 2m\mathcal{D}^2 \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = E, \qquad \rho\mathbf{v} = \nabla \times \mathrm{F} \tag{11a,b}$$

Consequently, inertia $m\mathbf{v}\cdot\nabla\mathbf{v}$, exterior forces ($-\nabla U$), and quantum forces ($-\nabla Q$) are in balance at every field point (Eq. (10 a)). The sum of the kinetic energy $m\mathbf{v}^2/2$, external (U) and quantum potential energy (Q) is invariant, *i.e.*, equal to the integration constant $E \neq E(\boldsymbol{r})$ (Eq. (11a)). $E \equiv \langle E \rangle$ represents the total energy of the dynamic system. The probability flow density $\rho\mathbf{v}$ has no sources (Eq. (10b)), *i.e.* its streamlines are closed (Eq. (11b)).

ii) Static states. For $\partial/\partial t = 0$ and $\mathbf{v} = 0$, Eqs. (7a, b) give

$$\nabla\left(U - 2m\mathcal{D}^2 \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \tag{12}$$

i.e.

$$U - 2m\mathcal{D}^2 \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = E \tag{13}$$

The exterior force ($-\nabla U$) is balanced by the quantum force ($-\nabla Q$) at any field point (Eq. (12)). The sum of the exterior (U) and interior (Q) potential energy is invariant, *i.e.*, equal to the integration constant $E \neq E(\boldsymbol{r})$ (Eq. (13)). $E \equiv \langle E \rangle$ represents the total energy of the fractal static system.

In such a context, let us consider $S = const.$ *i.e.* $\mathbf{v} = 0$, *i.e.* the phase coherence of the quantum fluid. Then the equation (13) with $U = E\rho$ becomes

$$\frac{2m\mathcal{D}^2}{E}\Delta f = f^3 - f \tag{14}$$

where $\rho^{1/2} = f$. In the one-dimensional case $\xi = x\left(E/2m\mathcal{D}^2\right)^{1/2}$, Eq. (14) can be solved in terms of the Jacobian elliptic functions $sn(\xi; k)$ of argument ξ and modulus k [3]. It results

$$f = \pm\sqrt{\frac{2k^2}{1+k^2}}\,sn\left(\frac{\xi - \xi_0}{\sqrt{1+k^2}}; k\right) \tag{15}$$

From here, for $k = 1$ restriction, we obtain the kink solution

$$f = \pm\tanh\left(\frac{\xi - \xi_0}{\sqrt{2}}\right) \tag{16}$$

Eq. (16) comes from a field theory with spontaneous symmetry breaking [4, 5]: the "*f*" field spontaneously breaks the vacuum state symmetry, generating Cooper-type pair. Through the phase coherence of the quantum fluid particles, the quantum fluid becomes superconducting: the kink solution (16) corresponds to the Cooper type pair (for other details see [6, 7]).

By means of relations $U = E\rho$, the energy conservation law takes the form:

$$E\rho - \mathcal{D}\nabla\cdot\mathbf{u} - \left(m_0\mathbf{u}^2/2\right) = E = const. \tag{17}$$

In addition, if the quantum fluid is also incompressible *i.e.* $\nabla\cdot\mathbf{u} = \Delta\ln\rho = 0$, the one-dimensional Eq. (17) admits the speed field

$$u_x = i\sqrt{\frac{2E}{m_0}(1-\rho)} \tag{18}$$

The quantum potential takes now a very simple expression which is directly proportional to the density of states of the Cooper type pairs, *i.e.*

$$Q = -\left(m_0u_x^2/2\right) = E(1-\rho) \tag{19}$$

When the density of states of the Cooper type pairs becomes zero (*i.e.* the quantum fluid is normal), the quantum potential takes a finite value, *E*, and when it becomes *1* (*i.e.* the quantum fluid becomes superconducting), the quantum potential turns to zero – the entire quantity of energy from the subquantic medium transfers to the superconducting pairs. Consequently, one can assume that the energy from the background subquantic medium can be stocked by transforming all the particles from the environment into Cooper type pairs and then 'freezing' them. The quantum fluid acts like a subquantic medium energy accumulator.

3. CONCLUSIONS

The main conclusions of the present paper are the followings:

i) The hydrodynamic formulation of scale relativity theory is established;

ii) ψ simultaneously becomes wave-function and speed potential;

iii) Some properties of the superconducting state are given: quantification law of the magnetic flux, kink solution associated with the Cooper pear and the "accumulator" effect of the quantum fluid.

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