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DIRAC FERMIONS IN DE SITTER AND ANTI-DE SITTER BACKGROUNDS

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Starting with a new theory of symmetries generated by isometries in field theories with spin, one finds the generators of the spinor representation in backgrounds with a given symmetry. In this manner one obtains a collection of conserved operators from which one can chose the complete sets of commuting operators defining quantum modes. In this framework, the quantum modes of the free Dirac field on de Sitter or anti-de Sitter spacetimes can be completely derived in static or moving charts. The discrete quantum modes are presented in the central static charts of the anti-de Sitter spacetime, whose eigenspinors can be normalized. The consequence is that the second quantization can be done in this case in canonical manner. For the free Dirac on de Sitter manifolds this can not be done in static charts being forced to consider the moving ones. The quantum modes of the free Dirac field in these charts are used for writing down the quantum Dirac field and the its one-particle operators.

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1. INTRODUCTION

In general relativity [1–3] the development of the quantum field theory in curved spacetimes [4] give rise to many difficult problems related to the physical interpretation of the one-particle quantum modes that may indicate how to quantize the fields. This is because the form and the properties of the particular solutions of the free field equations [5–9] are strongly dependent on the procedure of separation of variables and, implicitly, on the choice of the local chart (natural frame). Moreover, when the fields have spin the situation is more complicated since then the field equations and, therefore, the form of their particular solutions depend, in addition, on the tetrad gauge in which one works [10, 1]. In these conditions it would be helpful to use the traditional method of the quantum theory in flat spacetime based on the complete sets of commuting operators that determine the quantum modes as common eigenstates and give physical meaning to the constants of the separation of variables which are just the eigenvalues of these operators.

A good step in this direction could be to proceed like in special relativity looking for the generators of the geometric symmetries similar to the familiar momentum, angular momentum and spin operators of the Poincaré covariant field theories [11]. However, the relativistic covariance in the sense of general relativity is too general to play the same role as the Lorentz or Poincaré covariance in special relativity. In its turn the tetrad gauge invariance of the theories with spin represents another kind of general symmetry that is not able to produce itself conserved quantities [1]. Therefore, one must focus only on the isometry transformations that point out the specific spacetime symmetry related to the presence of Killing vectors [1, 3, 12].

Another important problem is how to define the generators of these representations for any spin when some spin parts could appear. In the case of the Dirac field these spin parts were calculated not only in some particular cases [13] but even in the general case of any generator corresponding to a Killing vector in any chart and arbitrary tetrad gauge fixing [14]. Starting with this important result, we have generalized this theory for any spin, formulating the theory of *external symmetry* for any curved manifold [15].

Our approach is a general theory of tetrad gauge invariant fields defined on curved spacetimes with given external symmetries. This predicts how must transform these fields under isometries in order to leave invariant the form of the field equations and to obtain the general form of the generators of these transformations. The basic idea is that the isometries transformations must preserve the position of the local frames with respect to the natural one. Such transformations can be constructed as isometries *combined* with suitable tetrad gauge transformations necessary for keeping unchanged the tetrad field components. In this way we obtain the external symmetry group showing that it is locally isomorphic with the isometry group.

Moreover, we define the operator-valued representations of the external symmetry group carried by spaces of fields with spin. We point out that these are induced by the linear finite-dimensional representations of the $SL(2,\mathbb{C})$ group. This is the motive why the symmetry transformations which leave invariant the field equations have generators with a composite structure. These have the usual orbital terms of the scalar representation and, in addition, specific spin terms which depend on the choice of the tetrad gauge even in the case of the fields with integer spin. In general, the spin and orbital terms do not commute to each other apart from some special gauge fixings where the fields transform manifestly covariant under external symmetry transformations.

As examples we study the central symmetry and the maximal symmetry of the de Sitter (dS) and anti-de Sitter (AdS) spacetimes. In the central charts we define a suitable version of Cartesian tetrad gauge which allowed us recently to find new analytical solutions of the Dirac equation [16, 17]. We show that in this gauge fixing the central symmetry becomes global and, consequently, the spin parts of its generators are the same as those of special relativity [18, 19]. For the dS and AdS spacetimes we calculate the generators of any representation of the external symmetry group in central charts with our Cartesian gauge.

Furthermore, we show how can be used these results for finding the quantum modes of the Dirac field on AdS and dS spacetimes. First we consider (static) central charts where the Dirac equation can be analytically solved. On AdS spacetime we obtain only quantum discrete modes of given energy described by fundamental solutions that can be correctly normalized. Unfortunately, in the case of the dS spacetime we find fundamental solutions corresponding to a continuous energy spectrum that can not be normalized in the generalized sense. In these circumstances we are able to quantize the Dirac field in central charts only in the AdS background. However, quantization of the Dirac field on dS spacetime can be solved in moving charts where we can identify the components of the momentum operator and normalize the fundamental solutions using the momentum representation [20]. Obviously, to this end our theory of external symmetry is crucial.

We start presenting in the second section the basic ideas of the relativistic and gauge covariance which will be embedded in our theory of external symmetry in the next section. The fourth section is devoted to the charts with central symmetry while the dS and AdS symmetries are discussed in section 5. The main features of the Dirac theory in curved manifolds are reviewed in the next section. Section 7 is devoted to the quantum modes of the Dirac field in central charts of AdS and dS spacetime. We show that the second quantization of this field can be performed in canonical manner only in AdS case. For quantizing the Dirac field in dS spacetimes we must choose moving charts where we have a suitable momentum representation. The theory of the Dirac field in these charts and the quantization procedure are given in section 8. Useful formulas are given in Appendix.

We work in natural units with $\hbar = c = 1$.

2. RELATIVISTIC COVARIANCE

In the Lagrangian field theory in curved spacetimes the relativistic covariant equations of scalar, vector or tensor fields arise from actions that are invariant under general coordinate transformations. Moreover, when the fields have spin in the sense of the $SL(2,\mathbb{C})$ symmetry then the action must be invariant under tetrad gauge transformations [10]. The first step to our approach is to embed both these kind of transformations into new ones, called combined transformations, that will help us to understand the relativistic covariance in its most general terms.

2.1. GAUGE TRANSFORMATIONS

Let us consider the curved spacetime M and a local chart (natural frame) of coordinates x^μ, $\mu = 0, 1, 2, 3$. Given a gauge, we denote by $e_{\hat\mu}(x)$ the tetrad fields that define the local (unholonomic) frames, in each point x, and by $\hat{e}^{\hat\mu}(x)$ those defining the corresponding coframes. These fields have the usual orthonormalization properties

$$\hat{e}^{\hat\mu}_{\alpha}\, e^{\alpha}_{\hat\nu} = \delta^{\hat\mu}_{\hat\nu}\,, \quad \hat{e}^{\hat\mu}_{\alpha}\, e^{\beta}_{\hat\mu} = \delta^{\beta}_{\alpha}\,, \quad e_{\hat\mu}\cdot e_{\hat\nu} = \eta_{\hat\mu\hat\nu}\,, \quad \hat{e}^{\hat\mu}\cdot \hat{e}^{\hat\nu} = \eta^{\hat\mu\hat\nu}\,, \tag{1}$$

where $\eta = \mathrm{diag}(1, -1, -1, -1,)$ is the Minkowski metric. From the line element

$$ds^2 = \eta_{\hat\mu\hat\nu} d\hat{x}^{\hat\mu} d\hat{x}^{\hat\nu} = g_{\mu\nu}(x) dx^{\mu} dx^{\nu}\,, \tag{2}$$

expressed in terms of 1-forms, $d\hat{x}^{\hat\mu} = \hat{e}^{\hat\mu}_{\nu} dx^{\nu}$, we get the components of the metric tensor of the natural frame,

$$g_{\mu\nu} = \eta_{\hat\alpha\hat\beta}\hat{e}^{\hat\alpha}_{\mu}\hat{e}^{\hat\beta}_{\nu}, \quad g^{\mu\nu} = \eta^{\hat\alpha\hat\beta} e^{\mu}_{\hat\alpha} e^{\nu}_{\hat\beta}\,. \tag{3}$$

These raise or lower the *natural* vector indices, *i.e.*, the Greek ones ranging from 0 to 3, while for the *local* vector indices, denoted by hat Greeks and having the same range, we must use the Minkowski metric. The local derivatives $\hat\partial_{\hat\nu} = e^{\mu}_{\hat\nu}\partial_{\mu}$ satisfy the commutation rules

$$[\hat\partial_{\hat\mu}, \hat\partial_{\hat\nu}] = e^{\alpha}_{\hat\mu} e^{\beta}_{\hat\nu}(\hat{e}^{\hat\sigma}_{\alpha,\beta} - \hat{e}^{\hat\sigma}_{\beta,\alpha})\hat\partial_{\hat\sigma} = C_{\hat\mu\hat\nu\cdot}^{\cdot\cdot\hat\sigma}\hat\partial_{\hat\sigma} \tag{4}$$

defining the Cartan coefficients which help us to write the *connection* components in local frames as

$$\hat\Gamma^{\hat\sigma}_{\hat\mu\hat\nu} = e^{\alpha}_{\hat\mu} e^{\beta}_{\hat\nu}(\hat{e}^{\hat\sigma}_{\gamma}\Gamma^{\gamma}_{\alpha\beta} - \hat{e}^{\hat\sigma}_{\beta,\alpha}) = \frac{1}{2}\eta^{\hat\sigma\hat\lambda}(C_{\hat\mu\hat\nu\hat\lambda} + C_{\hat\lambda\hat\mu\hat\nu} + C_{\hat\lambda\hat\nu\hat\mu})\,. \tag{5}$$

We specify that this connection is often called *spin* connection (and denoted by $\Omega^{\hat\sigma}_{\hat\mu\hat\nu}$) in order to do not be confused with the Christoffel symbols, $\Gamma^{\gamma}_{\alpha\beta}$, involved in the well-known formulas of the usual covariant derivatives $\nabla_{\mu} = {}_{;\mu}$.

The Minkowski metric $\eta_{\hat\mu\hat\nu}$ remains invariant under the transformations of the *gauge* group of this metric, $G(\eta) = O(3, 1)$. This has as subgroup the Lorentz group, $L^{\uparrow}_{+}$, of the transformations $\Lambda[A(\omega)]$ corresponding to the transformations $A(\omega) \in SL(2,\mathbb{C})$ through the canonical homomorphism [11]. In the standard *covariant* parametrization, with the real parameters $\omega^{\hat\alpha\hat\beta} = -\omega^{\hat\beta\hat\alpha}$, we have

$$A(\omega) = e^{-\frac{i}{2}\omega^{\hat\alpha\hat\beta} S_{\hat\alpha\hat\beta}}\,, \tag{6}$$

where $S_{\hat\alpha\hat\beta}$ are the covariant basis-generators of the $SL(2,\mathbb{C})$ Lie algebra which satisfy

$$\left[S_{\hat{\mu}\hat{\nu}}, S_{\hat{\sigma}\hat{\tau}}\right] = i(\eta_{\hat{\mu}\hat{\tau}}\, S_{\hat{\nu}\hat{\sigma}} - \eta_{\hat{\mu}\hat{\sigma}}\, S_{\hat{\nu}\hat{\tau}} + \eta_{\hat{\nu}\hat{\sigma}}\, S_{\hat{\mu}\hat{\tau}} - \eta_{\hat{\nu}\hat{\tau}}\, S_{\hat{\mu}\hat{\sigma}}). \tag{7}$$

For small values of $\omega^{\hat{\alpha}\hat{\beta}}$ the matrix elements of the transformations Λ can be written as

$$\Lambda[A(\omega)]^{\hat{\mu}\cdot}_{\cdot\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}} + \omega^{\hat{\mu}\cdot}_{\cdot\hat{\nu}} + \dots. \tag{8}$$

Now we assume that M is orientable and time-orientable such that $L^{\uparrow}_{+}$ can be considered as the gauge group of the Minkowski metric [3]. Then the fields with spin can be defined as in the case of the flat spacetime, with the help of the finite-dimensional *linear* representations, ρ, of the $SL(2,\mathbb{C})$ group [11]. In general, the fields $\psi_\rho : M \to V_\rho$ are defined over M with values in the vector spaces V_ρ of the representations ρ. In the following we systematically use the bases of V_ρ labeled only by spinor or vector *local* indices defined with respect to the axes of the local frames given by the tetrad fields. These will not be written explicitly except the cases when this is requested by the concrete calculation needs.

The relativistic covariant field equations are derived from actions [10, 1],

$$\mathcal{S}[\psi_\rho, e] = \int d^4x\sqrt{g}\,\mathcal{L}(\psi_\rho, D^\rho_{\hat{\mu}}\psi_\rho), \quad g = |\det(g_{\mu\nu})|, \tag{9}$$

depending on the matter fields, ψ_ρ, and the components of the tetrad fields, e, which represent the gravitational degrees of freedom. Recently it was shown that this action can be completed adding a term with the integration measure $d^4x\Phi$ (instead of $d^4x\sqrt{g}$) where Φ can be expressed in terms of scalar fields independent on e [21]. This new term allows one to define a new global scale symmetry which, in our opinion, is compatible with the geometric symmetries we study here. Therefore, without to lose generality, we can restrict ourselves to actions of the traditional form (9) in which the canonical variables are the components of the fields ψ_ρ and e.

The covariant derivatives,

$$D^\rho_{\hat{\alpha}} = \hat{e}^\mu_{\hat{\alpha}} D^\rho_\mu = \hat{\partial}_{\hat{\alpha}} + \frac{i}{2}\rho(S^{\hat{\beta}\cdot}_{\cdot\hat{\gamma}})\hat{\Gamma}^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}}, \tag{10}$$

assure the invariance of the whole theory under the *tetrad gauge* transformations,

$$\begin{aligned} &\hat{e}^{\hat{\alpha}}_\mu(x) \to \hat{e}'^{\hat{\alpha}}_\mu(x) = \Lambda[A(x)]^{\hat{\alpha}\cdot}_{\cdot\hat{\beta}}\, \hat{e}^{\hat{\beta}}_\mu(x), \\ &e^\mu_{\hat{\alpha}}(x) \to e'^\mu_{\hat{\alpha}}(x) = \Lambda[A(x)]^{\cdot\hat{\beta}}_{\hat{\alpha}\cdot}\, e^\mu_{\hat{\beta}}(x), \end{aligned} \tag{11}$$

$$\psi_\rho(x) \to \psi'_\rho(x) = \rho[A(x)]\,\psi_\rho(x),$$

determined by the mappings $A : M \to SL(2,\mathbb{C})$ the values of which are the local $SL(2,\mathbb{C})$ transformations $A(x) \equiv A[\omega(x)]$. These mappings can be organized as a group, $\mathcal{G}$, with respect to the multiplication $\times$ defined as $(A' \times A)(x) = A'(x)A(x)$. The notation Id stands for the mapping identity, $Id(x) = 1 \in SL(2,\mathbb{C})$, while A^{-1} is the inverse of A, $(A^{-1})(x) = [A(x)]^{-1}$.

2.2. COMBINED TRANSFORMATIONS

The general coordinate transformations are automorphisms of M which, in the passive mode, can be seen as changes of the local charts corresponding to the same domain of M [3, 12]. If x and x' are the coordinates of a point in two different charts then there is a mapping ϕ between these charts giving the coordinate transformation, $x \to x' = \phi(x)$. These transformations form a group with respect to the composition of mappings, $\circ$, defined as usual, *i.e.* $(\phi' \circ \phi)(x) = \phi'[\phi(x)]$. We denote this group by $\mathcal{A}$, its identity map by *id* and the inverse mapping of ϕ by ϕ^{-1}.

The automorphisms change all the components carrying natural indices including those of the tetrad fields [1] changing thus the positions of the local frames with respect to the natural ones. If we assume that the physical experiment makes reference to the axes of the local frame then it could appear situations when several correction of the positions of these frames should be needed before (or after) a general coordinate transformation. Obviously, these have to be done with the help of suitable gauge transformation associated to the automorphisms. Thus we arrive to the necessity of introducing the *combined* transformations denoted by (A, ϕ) and defined as gauge transformations, given by $A \in \mathcal{G}$, followed by automorphisms, $\phi \in \mathcal{A}$. In this new notation the pure gauge transformations will appear as (A, id) while the automorphisms will be denoted from now by (Id, ϕ).

The effect of a combined transformation (A, ϕ) upon our basic fields, ψ_ρ, e and $\hat{e}$ is $x \to x' = \phi(x)$, $e(x) \to e'(x')$, $\hat{e}(x) \to \hat{e}'(x')$ and $\psi_\rho(x) \to \psi'_\rho(x') = \rho[A(x)]\psi_\rho(x)$ where e' are the transformed tetrads of the components

$$e'^{\mu}_{\hat{\alpha}}[\phi(x)] = \Lambda[A(x)]^{\cdot\,\hat{\beta}}_{\hat{\alpha}\,\cdot}\, e^{\nu}_{\hat{\beta}}(x) \frac{\partial \phi^{\mu}(x)}{\partial x^{\nu}}, \tag{12}$$

while the components of $\hat{e}'$ have to be calculated according to Eqs. (1). Thus we have written down the most general transformation laws that leave the action invariant in the sense that $\mathcal{S}[\psi'_\rho, e'] = \mathcal{S}[\psi_\rho, e]$. The field equations derived from $\mathcal{S}$, written in local frames as $(E_\rho \psi_\rho)(x) = 0$, *covariantly* transform according to the rule

$$(E_\rho \psi_\rho)(x) \to (E'_\rho \psi'_\rho)(x') = \rho[A(x)](E_\rho \psi_\rho)(x), \tag{13}$$

since the operators E_ρ involve covariant derivatives [1].

The association among the transformations of the groups $\mathcal{G}$ and $\mathcal{A}$ must lead to a new group with a specific multiplication. In order to find how looks this new operation it is convenient to use the composition among the mappings A and ϕ (taken only in this order) giving new mappings, $A \circ \phi \in \mathcal{G}$, defined as $(A \circ \phi)(x) = = A[\phi(x)]$. The calculation rules $Id \circ \phi = Id$, $A \circ id = A$ and $(A' \times A) \circ \phi = = (A' \circ \phi) \times (A \circ \phi)$ are obvious. With these ingredients we define the new multiplication

$$(A', \phi') * (A, \phi) = \big((A' \circ \phi) \times A, \phi' \circ \phi\big). \tag{14}$$

It is clear that now the identity is (Id, id) while the inverse of a pair (A, ϕ) reads

$$(A, \phi)^{-1} = (A^{-1} \circ \phi^{-1}, \phi^{-1}). \tag{15}$$

First of all we observe that the operation $*$ is well-defined and represents the composition among the combined transformations since these can be expressed, according to their definition, as $(A, \phi) = (Id, \phi) * (A, id)$. Furthermore, we can convince ourselves that if we perform successively two arbitrary combined transformations, (A, ϕ) and (A', ϕ'), then the resulting transformation is just $(A', \phi') * (A, \phi)$ as given by Eq. (14). This means that the combined transformations form a group with respect to the multiplication $*$. It is not difficult to verify that this group, denoted by $\tilde{\mathcal{G}}$, is the semidirect product $\tilde{\mathcal{G}} = \mathcal{G} \circledS \mathcal{A}$ where $\mathcal{G}$ is the *invariant* subgroup while $\mathcal{A}$ is an usual one.

In the theories involving only vector and tensor fields we do not need to use the combined transformations defined above since the theory is independent on the positions of the local frames. This can be easily shown even in our approach where we use field components with local indices. Indeed, if we perform a combined transformation (A, ϕ) then any tensor field of rank (p, q),

$$\psi^{\hat\alpha_1, \hat\alpha_2, \dots, \hat\alpha_p}_{\hat\beta_1, \hat\beta_2, \dots, \hat\beta_q} = \hat e^{\hat\alpha_1}_{\mu_1} \cdots \hat e^{\hat\alpha_p}_{\mu_p}\, e^{\nu_1}_{\hat\beta_1} \cdots e^{\nu_q}_{\hat\beta_q}\, \psi^{\mu_1, \mu_2, \dots, \mu_p}_{\nu_1, \nu_2, \dots, \nu_q}, \tag{16}$$

transforms according to the representation

$$\rho^{\hat\beta_1,\hat\beta_2,\dots,\hat\beta_q;\hat\alpha'_1,\hat\alpha'_2,\dots,\hat\alpha'_p}_{\hat\alpha_1,\hat\alpha_2,\dots,\hat\alpha_p,;\hat\beta'_1,\hat\beta'_2,\dots,\hat\beta'_q}(A)=\Lambda^{\hat\beta_1\cdot}_{\cdot\hat\beta'_1}(A)\cdots\Lambda^{\cdot\hat\alpha'_1}_{\hat\alpha_1\cdot}(A)\cdots, \tag{17}$$

such that the resulting transformation law of the components carrying natural indices,

$$\psi'^{\mu_1,\dots}_{\nu_1,\dots}(x')=\frac{\partial x'^{\mu_1}}{\partial x^{\sigma_1}}\cdots\frac{\partial x^{\tau_1}}{\partial x'^{\nu_1}}\cdots\psi^{\sigma_1,\dots}_{\tau_1,\dots}(x), \tag{18}$$

is just the familiar one [1]. In other words, in this case the effect of the combined transformations reduces to that of their automorphisms. However, when the half integer spin fields are involved this is no more true and we must use the combined transformations of $\tilde{\mathcal{G}}$ if we want to keep under control the positions of the local frames.

3. EXTERNAL SYMMETRY

In general, the symmetry of any manifold M is given by its isometry group whose transformations leave invariant the metric tensor in any chart. The scalar field transforms under isometries according to the standard scalar representation generated by the orbital generators related to the Killing vectors of M [1, 3, 12]. In the following we present the generalization of this theory of symmetry to fields with spin, for which we have defined the *external* symmetry group and its representations [15].

3.1. ISOMETRIES

There are conjectures when several coordinate transformations, $x\to x'=\phi_\xi(x)$, depend on N independent real parameters, ξ^a (a, b, $c\ldots$ = = 1, 2, …, N), such that $\xi = 0$ corresponds to the identity map, $\phi_{\xi=0}=id$. The set of these mappings is a Lie group [22], $G\in\mathcal{G}$, if they accomplish the composition rule

$$\phi_{\xi'}\circ\phi_\xi=\phi_{f(\xi',\xi)}, \tag{19}$$

where the functions $f:G\times G\to G$ define the group multiplication. These must satisfy $f^a(0,\xi)=f^a(\xi,0)=\xi^a$ and $f^a(\xi^{-1},\xi)=f^a(\xi,\xi^{-1})=0$ where ξ^{-1} are the parameters of the inverse mapping of ϕ_ξ, $\phi_{\xi^{-1}}=\phi_\xi^{-1}$. Moreover, the structure constants of G can be calculated as [23]

$$c_{abc} = \left(\frac{\partial f^c(\xi, \xi')}{\partial \xi^a \partial \xi'^b} - \frac{\partial f^c(\xi, \xi')}{\partial \xi^b \partial \xi'^a} \right)_{|\xi=\xi'=0} . \tag{20}$$

For small values of the group parameters the infinitesimal transformations, $x^\mu \to x'^\mu = x^\mu + \xi^a k_a^\mu(x) + \cdots$, are given by the vectors k_a whose components,

$$k_a^\mu = \frac{\partial \phi_\xi^\mu}{\partial \xi^a}_{|\xi=0} , \tag{21}$$

satisfy the identities

$$k_a^\mu k_{b,\mu}^\nu - k_b^\mu k_{a,\mu}^\nu + c_{abc} k_c^\nu = 0, \tag{22}$$

resulting from Eqs. (19) and (20).

In the following we restrict ourselves to consider only the *isometry* transformations, $x' = \phi_\xi(x)$, which leave invariant the components of the metric tensor [1, 12], *i.e.*

$$g_{\alpha\beta}(x') \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} = g_{\mu\nu}(x). \tag{23}$$

These form the isometry group $G \equiv I(M)$ which is the Lie group giving the symmetry of the spacetime *M*. We consider that this has *N* independent parameters and, therefore, k_a, $a = 1, 2, \dots N$, are independent Killing vectors (which satisfy $k_{a\,\mu;\nu} + k_{a\,\nu;\mu} = 0$). Then their corresponding Lie derivatives form a basis of the Lie algebra *i*(*M*) of the group *I*(*M*) [12].

However, in practice we are interested to find the operators of the relativistic quantum theory related to these geometric objects which describe the symmetry of the background. For this reason we focus upon the operator-valued representations [24] of the group *I*(*M*) and its algebra. The *scalar* field $\psi : M \to \mathbb{C}$ transforms under isometries as $\psi(x) \to \psi'[\phi_\xi(x)] = \psi(x)$. This rule defines the representation $\phi_\xi \to T_\xi$ of the group *I*(*M*) whose operators have the action $\psi' = T_\xi \psi = \psi \circ \phi_\xi^{-1}$. Hereby it results that the operators of infinitesimal transformations, $T_\xi = 1 - i\xi^a L_a + \cdots$, depend on the basis-generators,

$$L_a = -i k_a^\mu \partial_\mu , \quad a = 1, 2, \dots, N, \tag{24}$$

which are completely determined by the Killing vectors. From Eq. (22) we see that they obey the commutation rules

$$[L_a, L_b] = i c_{abc} L_c , \tag{25}$$

given by the structure constants of $I(M)$. In other words they form a basis of the operator-valued representation of the Lie algebra $i(M)$ in a carrier space of scalar fields. Notice that in the usual quantum mechanics the operators similar to the generators L_a are called often *orbital* generators.

3.2. THE GROUP OF EXTERNAL SYMMETRY

Now the problem is how may transform under isometries the whole geometric framework of the theories with spin where we explicitly use the local frames. Since the isometry is a general coordinate transformation it changes the relative positions of the local and natural frames. This fact may be an impediment when one intends to study the symmetries of the theories with spin induced by those of the background. For this reason it is natural to suppose that the good symmetry transformations we need are combined transformations in which the isometries are preceded by appropriate gauge transformations such that not only the form of the metric tensor should be conserved but the form of the tetrad field components too.

Thus we arrive at the main point of our theory. We introduce the *external symmetry* transformations, (A_ξ, ϕ_ξ), as combined transformations involving isometries and corresponding gauge transformations necessary to *preserve the gauge*. We assume that in a fixed gauge, given by the tetrad fields e and $\hat{e}$, A_ξ is defined by

$$\Lambda[A_\xi(x)]^{\hat{\alpha}\cdot}_{\cdot\hat{\beta}} = \hat{e}^{\hat{\alpha}}_{\mu}[\phi_\xi(x)]\frac{\partial\phi^{\mu}_{\xi}(x)}{\partial x^{\nu}}e^{\nu}_{\hat{\beta}}(x), \tag{26}$$

with the supplementary condition $A_{\xi=0}(x) = 1 \in SL(2,\mathbb{C})$. Since ϕ_ξ is an isometry Eq. (23) guarantees that $\Lambda[A_\xi(x)] \in L^{\uparrow}_{+}$ and, implicitly, $A_\xi(x) \in SL(2,\mathbb{C})$. Then the transformation laws of our fields are

$$(A_\xi, \phi_\xi): \quad \begin{array}{rcrcl} x & \to & x' & = & \phi_\xi(x), \\ e(x) & \to & e'(x') & = & e[\phi_\xi(x)], \\ \hat{e}(x) & \to & \hat{e}'(x') & = & \hat{e}[\phi_\xi(x)], \\ \psi_\rho(x) & \to & \psi_\rho(x') & = & \rho[A_\xi(x)]\psi_\rho(x). \end{array} \tag{27}$$

The mean virtue of these transformations is that they leave *invariant* the form of the operators of the field equations, E_ρ, in local frames. This is because the components of the tetrad fields and, consequently, the covariant derivatives in local frames, $D^{\rho}_{\hat{\mu}}$, do not change their form.

For small ξ^a the covariant $SL(2,\mathbb{C})$ parameters of $A_\xi(x) \equiv A[\omega_\xi(x)]$ can be written as $\omega_\xi^{\hat\alpha\hat\beta}(x) = \xi^a \Omega_a^{\hat\alpha\hat\beta}(x) + \cdots$ where, according to Eqs. (6), (8) and (26), we have

$$\Omega_a^{\hat\alpha\hat\beta} \equiv \frac{\partial \omega_\xi^{\hat\alpha\hat\beta}}{\partial \xi^a}_{|\xi=0} = \left(\hat e^{\hat\alpha}_\mu\, k^\mu_{a,\nu} + \hat e^{\hat\alpha}_{\nu,\mu} k^\mu_a\right) e^\nu_{\hat\lambda} \eta^{\hat\lambda\hat\beta} . \tag{28}$$

We must specify that these functions are antisymmetric if and only if k_a are Killing vectors. This indicates that the association among isometries and the gauge transformations defined by Eq. (26) is correct.

It remains to show that the transformations (A_ξ, ϕ_ξ) form a Lie group related to $I(M)$. Starting with Eq. (26) we find that

$$(A_{\xi'} \circ \phi_\xi) \times A_\xi = A_{f(\xi',\xi)} , \tag{29}$$

and, according to Eqs. (14) and (19), we obtain

$$(A_{\xi'}, \phi_{\xi'}) * (A_\xi, \phi_\xi) = (A_{f(\xi',\xi)}, \phi_{f(\xi',\xi)}), \tag{30}$$

and $(A_{\xi=0}, \phi_{\xi=0}) = (Id, id)$. Thus we have shown that the pairs (A_ξ, ϕ_ξ) form a Lie group with respect to the operation $*$. We say that this is the external symmetry group of M and we denote it by $S(M) \subset \tilde{\mathcal{G}}$. From Eq. (30) we understand that $S(M)$ is *locally isomorphic* with $I(M)$ and, therefore, the Lie algebra of $S(M)$, denoted by $s(M)$, is isomorphic with $i(M)$ having the same structure constants. In our opinion, $S(M)$ must be isomorphic with the universal covering group of $I(M)$ since it has anyway the topology induced by $SL(2,\mathbb{C})$ which is simply connected. In general, the number of group parameters of $I(M)$ or $S(M)$ (which is equal to the number of the independent Killing vectors of M) can be $0 \le N \le 10$.

The form of the external symmetry transformations is strongly dependent on the choice of the local chart as well as that of the tetrad gauge. If we change simultaneously the gauge and the coordinates with the help of a combined transformation (A, ϕ) then each $(A_\xi, \phi_\xi) \in S(M)$ transforms as

$$(A_\xi, \phi_\xi) \to (A'_\xi, \phi'_\xi) = (A, \phi) * (A_\xi, \phi_\xi) * (A, \phi)^{-1} \tag{31}$$

which means that

$$A'_\xi = \left\{\left[\left(A \circ \phi_\xi\right) \times A_\xi\right] \times A^{-1}\right\} \circ \phi^{-1} , \tag{32}$$

$$\phi'_\xi = \left(\phi \circ \phi_\xi\right) \circ \phi^{-1}. \tag{33}$$

Obviously, these transformations define automorphisms of $S(M)$.

3.3. REPRESENTATIONS

The last of Eqs. (27) which gives the transformation law of the field ψ_ρ defines the operator-valued representation $(A_\xi, \phi_\xi) \to T^\rho_\xi$ of the group $S(M)$,

$$(T^\rho_\xi \psi_\rho)[\phi_\xi(x)] = \rho[A_\xi(x)]\psi_\rho(x). \tag{34}$$

The mentioned invariance under these transformations of the operators of the field equations in local frames reads

$$T^\rho_\xi E_\rho (T^\rho_\xi)^{-1} = E_\rho . \tag{35}$$

Since $A_\xi(x) \in SL(2,\mathbb{C})$ we say that this representation is *induced* by the representation ρ of $SL(2,\mathbb{C})$ [24, 25]. As we have shown in Sec. 2.2, if ρ is a vector or tensor representation (having only integer spin components) then the effect of the transformation (34) upon the components carrying natural indices is due only to ϕ_ξ. However, for the representations with half integer spin the presence of A_ξ is crucial since there are no natural indices. In addition, this allows us to define the generators of the representations (34) for any spin.

The basis-generators of the representations of the Lie algebra $s(M)$ are the operators

$$X^\rho_a = i\frac{\partial T^\rho_\xi}{\partial \xi^a}_{|\xi=0} = L_a + S^\rho_a , \tag{36}$$

which appear as sums among the orbital generators defined by Eq. (24) and the *spin terms* which have the action

$$(S^\rho_a \psi_\rho)(x) = \rho[S_a(x)]\psi_\rho(x). \tag{37}$$

This is determined by the form of the *local* $SL(2,\mathbb{C})$ generators,

$$S_a(x) = i\frac{\partial A_\xi(x)}{\partial \xi^a}_{|\xi=0} = \frac{1}{2}\Omega^{\hat\alpha\hat\beta}_a(x) S_{\hat\alpha\hat\beta} , \tag{38}$$

that depend on the functions (28). Furthermore, if we derive Eq. (29) with respect to ξ and ξ' then from Eqs. (8), (20) and (28), after a few manipulations, we obtain the identities

$$\eta_{\hat\alpha\hat\beta}\left(\Omega_a^{\hat\alpha\hat\mu}\Omega_b^{\hat\beta\hat\nu}-\Omega_b^{\hat\alpha\hat\mu}\Omega_a^{\hat\beta\hat\nu}\right)+k_a^\mu\Omega_{b,\mu}^{\hat\mu\hat\nu}-k_b^\mu\Omega_{a,\mu}^{\hat\mu\hat\nu}+c_{abc}\Omega_c^{\hat\mu\hat\nu}=0. \tag{39}$$

Hereby it results that

$$[S_a^\rho, S_b^\rho]+[L_a, S_b^\rho]-[L_b, S_a^\rho]=ic_{abc}S_c^\rho\,, \tag{40}$$

and, according to Eq. (25), we find the expected commutation rules

$$[X_a^\rho, X_b^\rho]=ic_{abc}X_c^\rho\,. \tag{41}$$

Thus we have derived the basis-generators of the operator-valued representation of $s(M)$ induced by the linear representation ρ of $SL(2,\mathbb{C})$. All the operators of this representation commute with the operator E_ρ since, according to Eqs. (35) and (36), we have

$$[E_\rho, X_a^\rho]=0, \quad a=1,2,...,N\,. \tag{42}$$

Therefore, for defining quantum modes we can use the set of commuting operators containing the Casimir operators of $s(M)$, the operators of its Cartan subalgebra and E_ρ.

Finally, we must specify that the basis-generators (36) of the representations of the $s(M)$ algebra can be written in covariant form as

$$X_a^\rho=-ik_a^\mu D_\mu^\rho+\frac{1}{2}k_{a\mu;\nu}\,e_{\hat\alpha}^\mu\,e_{\hat\beta}^\nu\,\rho(S^{\hat\alpha\hat\beta}), \tag{43}$$

generalizing thus the important result obtained in Ref. [14] for the Dirac field.

3.4. MANIFEST COVARIANCE

The action of the operators X_a^ρ depends on the choice of many elements: the natural coordinates, the tetrad gauge, the group parametrization and the representation ρ. What is important here is that they are strongly dependent on the tetrad gauge fixing even in the case of the representations with integer spin. This is because the covariant parametrization of the $SL(2,\mathbb{C})$ algebra is defined with respect to the axes of the local frames. In general, if we consider the representation $(A_\xi, \phi_\xi)\to T_\xi^\rho$ and we perform the transformation (31) then it results the *equivalent* representation, $(A'_\xi, \phi'_\xi)\to T'^\rho_\xi$. Its generators calculated from Eqs. (32) indicate that in this case the equivalence relations are much more complicated than those of the usual theory of linear representations. Without to enter in other technical details we specify that if we change only the gauge with the help of the transformation (A, id) then the local $SL(2,\mathbb{C})$ generators (38) transform as

$$\begin{aligned} S_a(x) \to S'_a(x) = A(x)S_a(x)A(x)^{-1} + \\ + k_a^{\sigma}(x)\Lambda[A(x)]_{\hat{\alpha}\hat{\mu},\sigma}\Lambda[A(x)]^{\cdot\hat{\mu}}_{\hat{\beta}\cdot} S^{\hat{\alpha}\hat{\beta}}, \end{aligned} \tag{44}$$

while the orbital parts do not change their form. This means that the gauge transformations change, in addition, the commutation relations among the spin and orbital parts of the generators X_a^{ρ}.

The consequence is that we can find gauge fixings where the local $SL(2,\mathbb{C})$ generators $S_a(x)$, $a = 1, 2, \ldots, n$ $(n \le N)$, corresponding to a subgroup $H \subset S(M)$, are independent on x and, therefore, $[S_a^{\rho}, L_b] = 0$ for all $a = 1, 2, \ldots,$ n and $b = 1, 2, \ldots, N$. Then the operators S_a^{ρ}, $a = 1, 2, \ldots, n$ are just the basis-generators of an usual linear representation of H and the field ψ_{ρ} behaves *manifestly covariant* under the external symmetry transformations of this subgroup. Of course, when $H = S(M)$ we say simply that the field ψ_{ρ} is manifest covariant.

The simplest examples are the manifest covariant fields of special relativity. Since here the spacetime M is flat, the metric in Cartesian coordinates is $g_{\mu\nu} = \eta_{\mu\nu}$ and one can use the *inertial* (local) frames with $e^{\mu}_{\nu} = \hat{e}^{\mu}_{\nu} = \delta^{\mu}_{\nu}$. Then the isometries are just the transformations $x' = \Lambda[A(\omega)]x - a$ of the Poincaré group, $\mathcal{P}_+^{\uparrow} = T(4)Ⓢ L_+^{\uparrow}$ [11]. If we denote by $\xi^{(\mu\nu)} = \omega^{\mu\nu}$ the $SL(2,\mathbb{C})$ parameters and by $\xi^{(\mu)} = a^{\mu}$ those of the translation group $T(4)$, then it is a simple exercise to calculate the basis-generators

$$X^{\rho}_{(\mu)} = i\partial_{\mu}, \tag{45}$$

$$X^{\rho}_{(\mu\nu)} = i(\eta_{\mu\alpha}x^{\alpha}\partial_{\nu} - \eta_{\nu\alpha}x^{\alpha}\partial_{\mu}) + \rho(S_{\mu\nu}), \tag{46}$$

which show us that ψ_{ρ} transforms manifestly covariant. On the other hand, it is clear that the group $S(M) \equiv \tilde{\mathcal{P}}_+^{\uparrow} = T(4)Ⓢ SL(2,\mathbb{C})$ is just the universal covering group of $I(M) \equiv \mathcal{P}_+^{\uparrow}$.

In general, there are many cases of curved spacetimes for which one can choose suitable local frames allowing one to introduce manifest covariant fields with respect to a subgroup $H \subset S(M)$ or even the whole group $S(M)$. In our opinion, this is possible only when H or $S(M)$ are at most subgroups of $\tilde{\mathcal{P}}_+^{\uparrow}$.

4. THE CENTRAL SYMMETRY

Let us take as first example the spacetimes M which have spherically symmetric static chart that will be called here *central* charts (or central frames). These manifolds have the isometry group $I(M)=T(1)\otimes SO(3)$ of time translations and space rotations.

4.1. CENTRAL CHARTS

In a central chart $\{t,\vec{x}\}$, with Cartesian coordinates $x^0=t$ and x^i $(i, j, k \ldots = 1, 2, 3)$, the metric tensor is time-independent and transforms manifestly covariant under the rotations $R\in SO(3)$ of the space coordinates,

$$t'=t,\quad x'^i=R^{i\cdot}_{\cdot j}(\omega)x^j=x^i+\omega^{i\cdot}_{\cdot j}x^j+\cdots,\tag{47}$$

denoted simply by $x\to x'=Rx$. Here the most general form of the line element,

$$ds^2=g_{\mu\nu}(x)dx^\mu dx^\nu=A(r)dt^2-[B(r)\delta_{ij}+C(r)x^ix^j]dx^idx^j,\tag{48}$$

may involve three functions, A, B and C, depending only on the Euclidean norm of $\vec{x}$, $r=|\vec{x}|$. In applications it is convenient to replace these functions by new ones, u, v and w, defined as

$$A=w^2,\quad B=\frac{w^2}{v^2},\quad C=\frac{1}{r^2}\left(\frac{w^2}{u^2}-\frac{w^2}{v^2}\right).\tag{49}$$

Other useful central charts are those with spherical coordinates, $\{t, r, \theta, \phi\}$, commonly associated with the Cartesian space ones, x^i. Here the line elements are

$$ds^2=w^2dt^2-\frac{w^2}{u^2}dr^2-\frac{w^2}{v^2}r^2(d\theta^2+\sin^2\theta d\phi^2),\tag{50}$$

and, consequently, we have

$$\sqrt{g}=B[A(B+r^2C)]^{1/2}=\frac{w^4}{uv^2}.\tag{51}$$

The new functions we introduced here have simple transformation laws under the isotropic dilatations which change only the radial coordinate, $r\to r'(r)$, without to affect the central symmetry of the line element. These transformations,

$$u'(r')=u(r)\left|\frac{dr'(r)}{dr}\right|,\quad v'(r')=v(r)\frac{r'(r)}{r},\quad w'(r')=w(r),\tag{52}$$

allow one to choose desired forms for the functions u, v and w.

4.2. THE CARTESIAN GAUGE

The Cartesian gauge in central charts was mentioned long time ago [26] but it is less used in concrete problems since it leads to complicated calculations in spherical coordinates. However, in Cartesian coordinates this gauge has the advantage of explicitly pointing out the global central symmetry of the manifold. In Refs. [16] we have proposed a version of Cartesian gauge in central charts with Cartesian coordinates that preserve the manifest covariance under rotations (47) in the sense that the 1-forms $d\hat{x}^{\hat{\mu}} = \hat{e}^{\hat{\mu}}_{\alpha}(x)dx^{\alpha}$ transform as

$$d\hat{x}^{\hat{\mu}} \to d\hat{x}'^{\hat{\mu}} = \hat{e}^{\hat{\mu}}_{\alpha}(x')dx'^{\alpha} = (Rd\hat{x})^{\hat{\mu}}. \tag{53}$$

If the line element has the form (48) then a good choice of the tetrad fields with the above property is

$$\hat{e}^0_0 = \hat{a}(r), \quad \hat{e}^0_i = \hat{e}^i_0 = 0, \quad \hat{e}^i_j = \hat{b}(r)\delta_{ij} + \hat{c}(r)x^i x^j \tag{54}$$

$$e^0_0 = a(r), \quad e^0_i = e^i_0 = 0, \quad e^i_j = b(r)\delta_{ij} + c(r)x^i x^j, \tag{55}$$

where, according to Eqs. (3), (48) and (49), we must have

$$\hat{a} = w, \quad \hat{b} = \frac{w}{v}, \quad \hat{c} = \frac{1}{r^2}\left(\frac{w}{u} - \frac{w}{v}\right), \tag{56}$$

$$a = \frac{1}{w}, \quad b = \frac{v}{w}, \quad c = \frac{1}{r^2}\left(\frac{u}{w} - \frac{v}{w}\right). \tag{57}$$

When one defines the metric tensor such that $g_{\mu\nu}{}_{|r=0} = \eta_{\mu\nu}$ then $u(0)^2 = v(0)^2 = w(0)^2 = 1$. Moreover, it is natural to take $\alpha(0) = 0$. In other respects, from Eqs. (56) and (57) we see that the function w must be positively defined in order to keep the same sense for the time axes of the natural and local frames. In addition, it is convenient to consider that the function u is positively defined too. However, the function $v = \eta_P\,|v|$ has the sign given by the relative parity η_P which takes the value $\eta_P = 1$ when the space axes of the local frame at $x = 0$ are parallel with those of the natural frame, and $\eta_P = -1$ if these are anti-parallel.

Now we have all the elements we need to calculate the generators of the representations T^ρ of the group $S(M)$. If we denote by $\xi^{(0)}$ the parameter of the time translations and by $\xi^{(i)} = \varepsilon_{ijk}\omega^{jk}/2$ the parameters of the rotations (47), we find that the local $SL(2,\mathbb{C})$ generators of Eq. (38) are just the $su(2)$ ones, *i.e.* $S_{(i)}(x) = S_i = \varepsilon_{ijk}S_{jk}/2$, such that the basis-generators read

$$X^{\rho}_{(0)} \equiv H = i\partial_t, \quad X^{\rho}_{(i)} \equiv J_{(i)} = L_{(i)} + \rho(S_i) \tag{58}$$

where $L_{(i)}=-i\varepsilon_{ijk}x^j\partial_k$ are the usual components of the orbital angular momentum. Thus we obtain that the group $S(M)=T(1)\otimes SU(2)$ is the universal covering group of $I(M)$. Its transformations are gauge transformations $A_{\xi}\in SU(2)$, independent on x, combined with the isometries of $I(M)$ given by $x\to x'=R(A_{\xi})x$ and $t\to t'=t-\xi^{(0)}$. This means that, in this gauge, the field ψ_ρ transforms manifestly covariant. The generators have the usual physical significance, namely H is the Hamiltonian operator while $J^{\rho}_{(i)}$ are the components of the *total* angular momentum operator. Moreover, the total angular momentum is conserved in the sense that $[H, J_{(i)}]=0$.

Concluding we can say that, in our Cartesian gauge, the local frames play the same role as the usual Cartesian rest frames of the central sources in flat spacetime since their axes are just those of projections of the angular momenta.

4.3. THE DIAGONAL GAUGE

In other gauge fixings the basis-generators are quite different. A tetrad gauge largely used in central charts with spherical coordinates is the *diagonal* gauge defined by the 1-forms [13]

$$d\hat{x}^0_s=w\,dt,\quad d\hat{x}^1_s=\frac{w}{u}dr,\quad d\hat{x}^2_s=r\frac{w}{v}d\theta,\quad d\hat{x}^3_s=r\frac{w}{v}\sin\theta\, d\phi. \tag{59}$$

In this gauge the angular momentum operators of the canonical basis (where $J_{(\pm)}=J_{(1)}\pm iJ_{(2)}$) are [13]

$$J^{\rho}_{(\pm)}=L_{(\pm)}+\frac{e^{\pm i\phi}}{\sin\theta}\rho(S_{23}),\quad J^{\rho}_{(3)}=L_{(3)}. \tag{60}$$

Thus one obtains a representation of $SU(2)$ where the spin terms do not commute with the orbital ones and, therefore, the field ψ_ρ does not transform manifestly covariant under rotations. In this case we can say that the spin part of the central symmetry remains partially hidden because of the diagonal gauge which determines special positions of the local frames with respect to the natural one. However, when this is an impediment one can change at anytime this gauge into the Cartesian one using a simple local rotation. For the flat spacetimes these transformations and their effects upon the Dirac equation are studied in Ref. [27]. We note that the form of the spin generators as well as that of the mentioned rotation depend on the enumeration of the 1-forms (59).

5. THE dS AND AdS SYMMETRIES

The backgrounds with highest external symmetry are the hyperbolic manifolds, namely the dS and the AdS spacetimes. These are exact solutions of the vacuum Einstein equations with cosmological constant Λ_c. We shall briefly discuss simultaneously both these manifolds which will be denoted by M_ϵ where $\epsilon = 1$ for dS case and $\epsilon = -1$ for AdS one. Our goal here is to calculate the generators of the representations of the group $S(M_\epsilon)$ induced by those of $SL(2,\mathbb{C})$.

The dS and AdS spacetimes are hyperboloids in the (4 + 1) or (3 + 2)-dimensional flat spacetimes, M^5_ϵ, of coordinates Z^A, $A, B, \ldots = 0, 1, 2, 3, 5$, and the metric $\eta(\epsilon)^5 = \text{diag}(1,-1,-1,-1,-\epsilon)$. The equation of the hyperboloid of radius $R = 1/\hat{\omega} = \sqrt{3/|\Lambda_c|}$ reads

$$\eta_{AB}(\epsilon)Z^A Z^B = -\epsilon R^2 . \tag{61}$$

From their definitions it results that the dS or AdS spacetimes are homogeneous spaces of the pseudo-orthogonal groups $SO(4, 1)$ or $SO(3, 2)$ which play the role of gauge groups of the metric $\eta(\epsilon)$ (for $\epsilon = 1$ and $\epsilon = -1$ respectively) and represent just the isometry groups of these manifolds, $G[\eta(\epsilon)] = I(M_\epsilon)$. Then it is natural to use the *covariant* real parameters $\omega^{AB} = -\omega^{BA}$ since in this parametrization the orbital basis-generators of the representations of $G[\eta(\epsilon)]$, carried by the spaces of functions over M^5_ϵ, have the usual form

$$L^5_{AB} = i\left[\eta_{AC}(\epsilon)Z^C\partial_B - \eta_{BC}(\epsilon)Z^C\partial_A\right]. \tag{62}$$

They will give us directly the orbital basis-generators of the representations of $S(M_\epsilon)$ in the carrier spaces of the functions defined over dS or AdS spacetimes.

5.1. CENTRAL CHARTS OF AdS AND dS SPACETIME

The hyperboloid equation can be solved in Cartesian dS/AdS coordinates, $x^0 = t$ and x^i $(i = 1, 2, 3)$, which satisfy

$$Z^5 = \hat{\omega}^{-1}\chi_\epsilon(r)\begin{cases}\cosh\hat{\omega}t & \text{if } \epsilon = 1\\ \cos\hat{\omega}t & \text{if } \epsilon = -1\end{cases} \qquad Z^0 = \hat{\omega}^{-1}\chi_\epsilon(r)\begin{cases}\sinh\hat{\omega}t & \text{if } \epsilon = 1\\ \sin\hat{\omega}t & \text{if } \epsilon = -1\end{cases} \tag{63}$$
$$Z^i = x^i ,$$

where we have denoted $\chi_\epsilon(r) = \sqrt{1-\epsilon\hat{\omega}^2 r^2}$. The line elements

$$
\begin{aligned}
ds^2 &= \eta_{AB}(\epsilon) dZ^A dZ^B = \\
&= \chi_\epsilon(r)^2 dt^2 - \frac{dr^2}{\chi_\epsilon(r)^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2),
\end{aligned} \tag{64}
$$

are defined on the radial domains $D_r = [0, 1/\hat{\omega})$ or $D_r = [0, \infty)$ for dS or AdS respectively.

We calculate the Killing vectors and the orbital generators of the external symmetry in the Cartesian coordinates defined by Eq. (63) and the mentioned parametrization of $I(M_\epsilon)$ starting with the identification $\xi^{(AB)} = \omega^{AB}$. Then, from Eqs. (24) and (62), after a little calculation, we obtain the orbital basis-generators

$$L_{(05)} = \frac{i\epsilon}{\hat{\omega}}\partial_t\,, \tag{65}$$

$$L_{(j5)} = \frac{i\epsilon}{\hat{\omega}}\chi_\epsilon(r)\begin{pmatrix}\cosh\hat{\omega}t\\ \cos\hat{\omega}t\end{pmatrix}\partial_j + \frac{ix^j}{\chi_\epsilon(r)}\begin{pmatrix}\sinh\hat{\omega}t\\ \sin\hat{\omega}t\end{pmatrix}\partial_t\,, \tag{66}$$

$$L_{(0j)} = \frac{i}{\hat{\omega}}\chi_\epsilon(r)\begin{pmatrix}\sinh\hat{\omega}t\\ \sin\hat{\omega}t\end{pmatrix}\partial_j + \frac{ix^j}{\chi_\epsilon(r)}\begin{pmatrix}\cosh\hat{\omega}t\\ \cos\hat{\omega}t\end{pmatrix}\partial_t\,, \tag{67}$$

$$L_{(ij)} = -i(x^i\,\partial_j - x^j\,\partial_i). \tag{68}$$

Furthermore, we consider the Cartesian tetrad gauge defined by Eqs. (54)–(57) where, according to Eq. (64), we have

$$u(r) = \chi_\epsilon(r)^2\,, \quad v(r) = w(r) = \chi_\epsilon(r). \tag{69}$$

In addition we take $\alpha = 0$. In this gauge we obtain the following local $SL(2,\mathbb{C})$ generators

$$S_{(05)}(x) = 0\,, \tag{70}$$

$$
\begin{aligned}
S_{(j5)}(x) = S_{0j}\begin{pmatrix}\sinh\hat{\omega}t\\ \sin\hat{\omega}t\end{pmatrix} + \frac{1}{r^2}[\chi_\epsilon(r) - 1]\Bigg[\epsilon\frac{S_{jk}x^k}{\hat{\omega}}\begin{pmatrix}\cosh\hat{\omega}t\\ \cos\hat{\omega}t\end{pmatrix} - \\
- \frac{S_{0k}x^k x^j}{\chi_\epsilon(r)}\begin{pmatrix}\sinh\hat{\omega}t\\ \sin\hat{\omega}t\end{pmatrix}\Bigg],
\end{aligned} \tag{71}
$$

$$
\begin{aligned}
S_{(0j)}(x) = S_{0j}\begin{pmatrix}\cosh\hat{\omega}t\\ \cos\hat{\omega}t\end{pmatrix} + \frac{1}{r^2}[\chi_\epsilon(r) - 1]\Bigg[\frac{S_{jk}x^k}{\hat{\omega}}\begin{pmatrix}\sinh\hat{\omega}t\\ \sin\hat{\omega}t\end{pmatrix} - \\
- \frac{S_{0k}x^k x^j}{\chi_\epsilon(r)}\begin{pmatrix}\cosh\hat{\omega}t\\ \cos\hat{\omega}t\end{pmatrix}\Bigg],
\end{aligned} \tag{72}
$$

$$S_{(ij)}(x) = S_{ij} \,. \tag{73}$$

With their help we can write the action of the spin terms (37) and, implicitly, that of the basis-generators $X^{\rho}_{(AB)} = L_{(AB)} + S^{\rho}_{(AB)}$ of the representations of $S(M_\epsilon)$ induced by the representations ρ of $SL(2,\mathbb{C})$. Hereby it is not difficult to show that $S(M_\epsilon)$ is isomorphic with the universal covering group of $I(M_\epsilon)$ which in both cases $(\epsilon = \pm 1)$ is a subgroup of the $SU(2,2)$ group. As expected, in central charts and Cartesian gauge the fields transform manifestly covariant only under the transformations of the subgroup $SU(2) \subset S(M_\epsilon)$.

6. THE DIRAC FIELD IN CURVED BACKGROUNDS

In what follows we study the Dirac particles *freely* moving on dS or AdS backgrounds without to affect the geometry. The problems of this type are called often Dirac perturbations on a given manifold.

The main purpose is to find the quantum modes of the Dirac field determined by complete sets of commuting operators. The Noether theorem applied to our theory of external symmetry give us classical conserved quantities or conserved operators in quantum theory, corresponding to the generators of the group *S*(*M*). Thus, the quantum theory is equipped with a large algebra of conserved operators among them we can select the systems of commuting operators we need for defining the quantum modes which will help us to perform the second quantization of the Dirac field in canonical manner.

6.1. THE DIRAC FIELD

The form of the Dirac field in curves backgrounds depends on two basic elements: the choice of the tetrad gauge and the representation of the Dirac's γ-matrices. These satisfy

$$\left\{\gamma^{\hat\alpha}, \gamma^{\hat\beta}\right\} = 2\eta^{\hat\alpha\hat\beta} \tag{74}$$

and give the generators of the reducible spinor representation of the $SL(2,C)$ group [19] we denote from now directly by $S^{\hat\alpha\hat\beta}$ instead of $\rho(S^{\hat\alpha\hat\beta})$. These generators,

$$S^{\hat\alpha\hat\beta} = \frac{i}{4}\left[\gamma^{\hat\alpha}, \gamma^{\hat\beta}\right], \tag{75}$$

are self-adjoint, $\overline{S}^{\hat\alpha\hat\beta} = S^{\hat\alpha\hat\beta}$, (with respect to the Dirac adjoint defined as $\overline{X} = \gamma^0 X^+ \gamma^0$) and satisfy

$$\left[S^{\hat\alpha\hat\beta},\gamma^{\hat\sigma}\right]=i(\eta^{\hat\beta\hat\sigma}\gamma^{\hat\alpha}-\eta^{\hat\alpha\hat\sigma}\gamma^{\hat\beta}),\tag{76}$$

$$\left[S_{\hat\alpha\hat\beta},S_{\hat\sigma\hat\tau}\right]=i(\eta_{\hat\alpha\hat\tau}\,S_{\hat\beta\hat\sigma}-\eta_{\hat\alpha\hat\sigma}\,S_{\hat\beta\hat\tau}+\eta_{\hat\beta\hat\sigma}\,S_{\hat\alpha\hat\tau}-\eta_{\hat\beta\hat\tau}\,S_{\hat\alpha\hat\sigma}),\tag{77}$$

In the chiral representation of the spinor field the crucial role is played by the matrix $\gamma^5=-i\gamma^0\gamma^1\gamma^2\gamma^3$ which is diagonal in this representation. We note that the set of matrices $\{S^{\hat\alpha\hat\beta},\gamma^{\hat\mu},\gamma^{\hat\mu}\gamma^5,\gamma^5\}$ form the basis of a fundamental representation of the *su*(2, 2) algebra[1].

The theory of the Dirac field ψ of mass *m*, defined on the space domain *D*, starts with the gauge invariant action [4],

$$\mathcal{S}[\psi]=\int d^4x\sqrt{g}\left\{\frac{i}{2}\left[\overline{\psi}\gamma^{\hat\alpha}D_{\hat\alpha}\psi-(\overline{D_{\hat\alpha}\psi})\gamma^{\hat\alpha}\psi\right]-m\overline{\psi}\psi\right\}\tag{78}$$

where $\overline{\psi}=\psi^+\gamma^0$ is the Dirac adjoint of ψ and

$$D_{\hat\alpha}=\hat\partial_{\hat\alpha}+\frac{i}{2}S^{\hat\beta\cdot}_{\cdot\hat\gamma}\hat\Gamma^{\hat\gamma}_{\hat\alpha\hat\beta}\tag{79}$$

are the covariant derivatives of the spinor field given by Eq. (10). From the action (78) it results the Dirac equation

$$E_D\psi-m\psi=0,\quad E_D=i\gamma^{\hat\alpha}D_{\hat\alpha}\,.\tag{80}$$

After a few manipulations the Dirac operator E_D can be put in a more comprehensive form as

$$E_D=i\gamma^{\hat\alpha}e^{\mu}_{\hat\alpha}\partial_\mu+\frac{i}{2}\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}e^{\mu}_{\hat\alpha}\right)\gamma^{\hat\alpha}-\frac{1}{4}\left\{\gamma^{\hat\alpha},S^{\hat\beta\cdot}_{\cdot\hat\gamma}\right\}\hat\Gamma^{\hat\gamma}_{\hat\alpha\hat\beta}\,.\tag{81}$$

On the other hand, starting with the conservation of the electric charge, one can define the time-independent relativistic scalar product of two spinors is [4]. When $e^0_i=0$, $i=1,2,3$, this takes the simple form

$$(\psi,\psi')=\int_D d^3x\,\mu(x)\overline{\psi}(x)\gamma^0\psi'(x),\tag{82}$$

where

$$\mu(x)=\sqrt{g(x)}\,e^0_0(x)\tag{83}$$

is the specific weight function of the Dirac field. In the central chart with the metric (50) this weight function reads

[1] For extended bibliography see Ref. [28].

$$\mu = \frac{1}{b^2(b+r^2c)} = \frac{w^3}{uv^2} \tag{84}$$

since g is given by Eq. (51).

Our theory of external symmetry offers us the framework we need to calculate the conserved quantities predicted by the Noether theorem. Starting with the infinitesimal transformations of the one-parameter subgroup of $S(M)$ generated by X_a corresponding to the Killing vector k_a, we find that there exists the conserved current $\Theta^{\mu}[X_a]$ which satisfies $\Theta^{\mu}[X_a]_{;\mu} = 0$. Using the action (78) we obtain the concrete form of these currents,

$$\Theta^{\mu}[X_a] = -\tilde{T}^{\mu\cdot}_{\cdot\nu} k_a^{\nu} + \frac{1}{4}\overline{\psi}\left\{\gamma^{\hat{\alpha}}, S^{\hat{\beta}\hat{\gamma}}\right\}\psi\, e^{\mu}_{\hat{\alpha}}\, \Omega_{a\hat{\beta}\hat{\gamma}}\,, \tag{85}$$

where the functions $\Omega_{a\hat{\beta}\hat{\gamma}}$ are defined by Eq. (28) while

$$\tilde{T}^{\mu\cdot}_{\cdot\nu} = \frac{i}{2}\left[\overline{\psi}\gamma^{\hat{\alpha}} e^{\mu}_{\hat{\alpha}}\partial_{\nu}\psi - (\overline{\partial_{\nu}\psi})\gamma^{\hat{\alpha}} e^{\mu}_{\hat{\alpha}}\psi\right] \tag{86}$$

is a notation for a part of the stress-energy tensor of the Dirac field [1, 4]. Finally, it is clear that the conserved quantity corresponding to the Killing vector k_a is the real number

$$\int_D d^3x\sqrt{g}\,\Theta^0[X_a] = \frac{1}{2}\left[\langle\psi, X_a\psi\rangle + \langle X_a\psi, \psi\rangle\right]. \tag{87}$$

We note that it is premature to interpret this formula as an expectation value or to speak about Hermitian conjugation of the operators X_a with respect to the scalar product (82), before specifying the boundary conditions on D.

What is important here is that this result is useful in quantization giving directly the one-particle operators of the quantum field theory when ψ becomes a field operator. Indeed, starting with the form (87) of the conserved quantities, we find that for any self-adjoint generator X of the spinor representation of the group $S(M)$ there exists a *conserved* one-particle operator of the quantum field theory which can be calculated simply as

$$\mathbf{X} =: \langle\psi, X\psi\rangle : \tag{88}$$

respecting the normal ordering $(::)$ of the operator products [29]. The quantization rules must be postulated such that the standard algebraic properties

$$[\mathbf{X}, \psi(x)] = -X\psi(x), \quad [\mathbf{X}, \mathbf{X}'] =: \langle\psi, [X, X']\psi\rangle : \tag{89}$$

should be accomplished.

6.2. THE REDUCED DIRAC EQUATION IN CENTRAL CHARTS

In what follows we consider manifolds M with central symmetry having either charts with Cartesian coordinates $\{t, \vec{x}\}$ and the line element (48) or charts with spherical coordinates $\{t, r, \theta, \phi\}$ where we prefer to work with metrics of the form (50). In these charts we chose the Cartesian gauge defined by Eqs. (54), (55), (56) and (57) which bring the rotations in manifest covariant form.

The Dirac equation with central symmetry can be written replacing the concrete form of the tetrad components in Eq. (81). First, we observe that the last term of this equation does not contribute when the metric is spherically symmetric. The argument is that $\{\gamma^{\hat{\alpha}}, S^{\hat{\beta}\hat{\gamma}}\} = \varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\cdot}_{\cdot\cdot\cdot\hat{\lambda}}\gamma^5\gamma^{\hat{\lambda}}$ (with $\varepsilon^{0123} = 1$) is completely antisymmetric, while the Cartan coefficients resulted from (54) and (55) have no such type of components. Furthermore, we bring the remaining equation in a simpler form defining the *reduced* Dirac field, $\tilde{\psi}$, as

$$\psi(x) = \chi(r)\tilde{\psi}(x). \tag{90}$$

where

$$\chi = \left[\sqrt{g}(b + r^2 c)\right]^{-1/2} = b\sqrt{a} = vw^{-3/2}. \tag{91}$$

In this way we obtain the *reduced* Dirac equation in the central chart $(t, \vec{x})$ with Cartesian coordinates and Cartesian tetrad gauge,

$$i\left\{a(r)\gamma^0\partial_t + b(r)(\vec{\gamma}\cdot\vec{\partial}) + c(r)(\vec{\gamma}\cdot\vec{x})\left[1 + (\vec{x}\cdot\vec{\partial})\right]\right\}\tilde{\psi}(x) = m\tilde{\psi}(x). \tag{92}$$

We observe that this equation is invariant under time translations which means that the energy is conserved. For this reason it is convenient to bring it in Hamiltonian form,

$$\tilde{H}_D\tilde{\psi} = i\partial_t\tilde{\psi}, \tag{93}$$

using the operators

$$\tilde{H}_D = -i\frac{u(r)}{r^2}(\gamma^0\gamma^i x^i)\left(1 + x^i\partial_i\right) - i\frac{v(r)}{r^2}(\gamma^i x^i)K + w(r)\gamma^0 m, \tag{94}$$

$$K = \gamma^0\left(2\vec{S}\cdot\vec{L} + 1\right). \tag{95}$$

The operator K which concentrates all the angular terms of $\tilde{H}_D$ as well as the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ defined by Eq. (58) commute with $\tilde{H}_D$ and satisfy $K^2 = \vec{J}^2 + \frac{1}{4}$. Consequently, all the properties related to the conservation of the angular momentum, including the separation of variables in spherical coordinates, will be similar to those of the usual Dirac theory in Minkowski spacetimes. Thus in Eq. (93) one can separate the spherical variables with the help of the four-components angular spinors $\Phi^{\pm}_{m_j,\kappa_j}$ used in the central problems of special relativity [19]. These are eigenspinors of the complete set $\{\vec{J}^2, J_3, K\}$ corresponding to the eigenvalues $\{j(j+1), m_j, -\kappa_j\}$ where κ_j can take only the values $\pm(j+1/2)$. We note that for each set of quantum numbers (j, κ_j, m_j) there are two orthogonal spinors, denoted by the superscripts ±, as defined in Ref. [19].

We must specify that in curved manifolds this conjecture can be obtained only in our Cartesian tetrad gauge where the representations of the group $SU(2)$ are manifest covariant since $[L_i, S_j]=0$. In other gauge fixings where L_i and $S_i(x)$ do not commute among themselves the form of Eqs. (94) and (95) may be different [13, 7] so that the spinors $\Phi^{\pm}_{m_j,\kappa_j}$ become useless. For example, in the diagonal gauge defined by Eq. (59) one must consider another type of angular spinors, constructed with the help of the so called spin-weighted harmonics [30].

In other respects, we can verify that in our gauge the discrete transformations, P, C and T, have in this gauge the same significance and action as those of special relativity [18, 19]. This is because the form of the Hamiltonian operator (94) is close to that meet in the flat case. We shall see later that the charge conjugation transforms each particular solution of positive frequency of Eq. (92) into the corresponding one of negative frequency.

6.3. THE RADIAL PROBLEM

In the central chart $\{t, r, \theta, \phi\}$ where the line element can be put in the form (50) we consider the particular solutions of the original Dirac equation (80) having positive frequency and the energy E,

$$U_{E,j,\kappa_j,m_j}(x) = U_{E,j,\kappa_j,m_j}(t,r,\theta,\phi) = \\ = \frac{v(r)}{rw(r)^{3/2}}\left[f^{(+)}_{E,\kappa_j}(r)\Phi^{+}_{m_j,\kappa_j}(\theta,\phi) + f^{(-)}_{E,\kappa_j}(r)\Phi^{-}_{m_j,\kappa_j}(\theta,\phi)\right] e^{-iEt}\,. \tag{96}$$

The reduced part of these solutions, $\tilde{U} = U/\chi$, satisfy Eq. (93) from which, after the separation of angular variables, we obtain the radial equations

$$\left[u(r)\frac{d}{dr} + v(r)\frac{\kappa_j}{r}\right] f^{(+)}(r) = [E + w(r)m] f^{(-)}(r), \tag{97}$$

$$\left[-u(r)\frac{d}{dr}+v(r)\frac{\kappa_j}{r}\right]f^{(-)}(r)=[E-w(r)m]f^{(+)}(r), \tag{98}$$

where we omitted the indices E and κ_j of the radial functions. In practice, these radial equations can be written directly starting with the line element put in the form Eq. (50) from which we can identify the functions u, v, and w we need.

The angular spinors are normalized such that the angular integral of the scalar product (82) does not contribute and, consequently, this reduces to the radial integral. By using Eqs. (90) and (96) we find that this is

$$\left(\tilde{\psi}_1,\tilde{\psi}_2\right)=\int_{D_r}\frac{dr}{u(r)}\left\{\left[f_1^{(+)}(r)\right]^* f_2^{(+)}(r)+\left[f_1^{(-)}(r)\right]^* f_2^{(-)}(r)\right\} \tag{99}$$

where D_r is the radial domain corresponding to D. What is remarkable here is that the radial weight function $\mu\chi^2=1/u$, resulted from Eqs. (128) and (91), is just that we need in order to have $(u\partial_r)^+=-u\partial_r$. This means that the operators of the left-hand side of the radial equations are related between themselves through the Hermitian conjugation with respect to the scalar product (99).

Hereby it results that the operator

$$H_r=\left|\begin{array}{cc} mw & -u\dfrac{d}{dr}+\kappa_j\dfrac{v}{r}\\ u\dfrac{d}{dr}+\kappa_j\dfrac{v}{r} & -mw\end{array}\right| \tag{100}$$

is self-adjoint. This is just the *radial* Hamiltonian, which allows one to write the Eqs. (97) and (98) as the eigenvalue problem

$$H_r\mathcal{F}=E\mathcal{F}, \tag{101}$$

where the two-dimensional eigenvectors $\mathcal{F}=(f^{(+)},f^{(-)})^T$ have their own scalar product,

$$(\mathcal{F}_1,\mathcal{F}_2)=\int_{D_r}\frac{dr}{u(r)}\mathcal{F}_1^+\mathcal{F}_2\,, \tag{102}$$

as it results from (99). Thus we have obtained an *independent* radial problem which has to be solved in each particular case separately using appropriate methods.

First of all, we look for possible transformations which should simplify the radial equations. It is known that the transformations of the space coordinates of a natural frame with static metric do not change the quantum modes. In our case we can change only the radial coordinate without to affect the central symmetry. A good choice is the chart where the radial coordinate is defined by

$$r_s(r) = \int \frac{dr}{u(r)} + const. \tag{103}$$

so that $r_s(0) = 0$. This chart will be called *special* central chart (frame). In the following we shall use only this frame starting with metrics with $u = 1$ while the subscript s will be omitted.

In radial problems of the Dirac theory in flat spacetime [19] the manifest supersymmetry play an important role in solving the radial equations. For this reason we look for transformations of the form $\mathcal{F} \to \hat{\mathcal{F}} = U\mathcal{F}$ and $H_r \to \hat{H}_r = UH_rU^{-1}$ which could simplify the radial problem pointing out a *hidden* supersymmetry. This means that the transformed Hamiltonian has the supersymmetric form

$$\hat{H}_r = \begin{vmatrix} \nu & -\frac{d}{dr} + W \\ \frac{d}{dr} + W & -\nu \end{vmatrix} \tag{104}$$

where ν must be a constant and W is the resulting superpotential [16]. If the radial problem has this property, then the second order equations for the components $\hat{f}^{(+)}$ and $\hat{f}^{(-)}$ of $\hat{\mathcal{F}}$ can be obtained from $\hat{H}_r{}^2\hat{\mathcal{F}} = E^2\hat{\mathcal{F}}$. These equations,

$$\left(-\frac{d^2}{dr^2} + W(r)^2 \mp \frac{dW(r)}{dr} + \nu^2\right)\hat{f}^{(\pm)}(r) = E^2\hat{f}^{(\pm)}(r), \tag{105}$$

represent the starting point for finding analytical solutions.

The simplest radial problems are those with manifest supersymmetry, for which the original radial Hamiltonian H_r has the form (138). These are generated by the metrics of the central manifolds $\mathbb{R} \times M_3$ which have $w = 1$. In the special frames (where $u = 1$) these are determined only by the arbitrary function v which gives the superpotential $W = \kappa_j v/r$. The most popular example is the three-dimensional sphere with the time trivially added, $\mathbb{R} \times S^3$. A more complicated situation is when we need to use a suitable transformation U in order to point out the supersymmetry. These are problems with hidden supersymmetry which are similar with that of the Dirac particle in external Coulomb field, known from special relativity. However, examples of radial problems in which the supersymmetry is much more hidden will be discussed in the next section.

7. DIRAC FIELD IN CENTAL CHARTS OF THE AdS AND dS SPACETIMES

In our approach the Dirac equation in the central charts (frames) of AdS or dS backgrounds can be analytically solved in terms of Gauss hypergeometric functions [16,17]. One obtains thus the fundamental solutions giving the quantum modes. Since the central frames are static, these solutions are energy eigenspinors corresponding to discrete or continuous energy spectra. When these spinors can be normalized (in usual or generalized sense) then the canonical quantization of the Dirac field in these frames may be done.

7.1. DIRAC QUANTUM MODES IN AdS SPACETIME

We consider first the AdS spacetime, M, and we present how can be selected the quantum modes of the Dirac field and to write down the normalized energy eigenspinors of the regular modes. The final objective is to quantize this field [16].

The AdS spacetime M of radius R is defined by Eq. (61) for $\epsilon = -1$. Changing the notation, we denote from now $\omega = 1/R$ instead of $\hat{\omega}$. This manifold has a special central chart $\{t, r, \theta, \phi\}$ with the line element [31]

$$ds^2 = \sec^2 \omega r \left[dt^2 - dr^2 - \frac{1}{\omega^2} \sin^2 \omega r \, (d\theta^2 + \sin^2 \theta \, d\phi^2) \right]. \tag{106}$$

The radial domain of this chart is $D_r = [0, \pi/2\omega)$ because of the event horizon at $r = \pi/2\omega$. We specify that here we take $t \in (-\infty, \infty)$ which defines in fact the universal covering spacetime (CAdS) of AdS [31].

Now, from Eq. (118) we can identify the functions $w(r) = \sec \omega r$ and $v(r) = \omega r \csc \omega r$. With their help and using the notation $k = m/\omega$ we obtain the radial Hamiltonian,

$$H_r = \begin{vmatrix} \omega k \sec \omega r & -\dfrac{d}{dr} + \omega \kappa_j \csc \omega r \\ \dfrac{d}{dr} + \omega \kappa_j \csc \omega r & -\omega k \sec \omega r \end{vmatrix}, \tag{107}$$

which has to give us the radial functions of the particular solutions (96).

Despite of the fact that this is not obvious at all, this Hamiltonian has a hidden supersymmetry that can be pointed out with the help of the local rotation $\mathcal{F} \to \hat{\mathcal{F}} = R\mathcal{F} = \left(\hat{f}^{(+)}, \hat{f}^{(-)} \right)^T$ produced by

$$R(r) = \begin{vmatrix} \cos \dfrac{\omega r}{2} & -\sin \dfrac{\omega r}{2} \\ \sin \dfrac{\omega r}{2} & \cos \dfrac{\omega r}{2} \end{vmatrix}. \tag{108}$$

Indeed, after a few manipulations we find that the transformed (*i.e.* rotated and translated) Hamiltonian,

$$\hat{H}_r = RH_rR^T - \frac{\omega}{2}1_{2\times 2}, \tag{109}$$

has supersymmetry since it has the requested specific form with diagonal constant terms [19] and superpotential [16] of the Pöschl-Teller type [32] .

In these circumstances the new eigenvalue problem

$$\hat{H}_r\hat{\mathcal{F}} = \left(E - \frac{\omega}{2}\right)\hat{\mathcal{F}}, \tag{110}$$

involving the transformed radial wave functions $\hat{f}^{(\pm)}$, leads to a pair of second order equations

$$\left(-\frac{d^2}{dr^2} + \omega^2\frac{k(k\mp 1)}{\cos^2\omega r} + \omega^2\frac{\kappa_j(\kappa_j \pm 1)}{\sin^2\omega r}\right)\hat{f}^{(\pm)}(r) = \omega^2\epsilon^2\hat{f}^{(\pm)}(r), \tag{111}$$

where we denoted $\epsilon = E/\omega - 1/2$. These equations have analytical solutions [16],

$$\begin{aligned}\hat{f}^{(\pm)}(r) = N_\pm \sin^{2s_\pm}\omega r\cos^{2p_\pm}\omega r \times \\ \times F\left(s_\pm + p_\pm - \frac{\epsilon}{2}, s_\pm + p_\pm + \frac{\epsilon}{2}, 2s_\pm + \frac{1}{2}, \sin^2\omega r\right).\end{aligned} \tag{112}$$

where F are the Gauss hypergeometric functions [33]. Their real parameters are defined as

$$2s_\pm(2s_\pm - 1) = \kappa_j(\kappa_j \pm 1), \tag{113}$$

$$2p_\pm(2p_\pm - 1) = k(k\mp 1), \tag{114}$$

while $N_\pm$ are normalization factors. The next step is to select the suitable values of these parameters and to calculate N_+/N_-such that the functions $\hat{f}^{(\pm)}$ should be solutions of the transformed radial problem (137), with a good physical meaning. This can be achieved only when F is a polynomial selected by a suitable quantization condition since otherwise F is strongly divergent for $\sin^2\omega r \to 1$. Then the functions $\hat{f}^{(\pm)}$ will be square integrable with normalization factors calculated according to the condition

$$(\mathcal{F},\mathcal{F}) = (\hat{\mathcal{F}},\hat{\mathcal{F}}) = \int_{D_r} dr\left(\left|\hat{f}^{(+)}(r)\right|^2 + \left|\hat{f}^{(-)}(r)\right|^2\right) = 1, \tag{115}$$

resulted from the fact that the matrix (135) is orthogonal.

The discrete energy spectrum is given by the particle-like CAdS quantization conditions

$$\epsilon = 2(n_\pm + s_\pm + p_\pm), \quad \epsilon > 0, \tag{116}$$

that must be compatible with each other, *i.e.*

$$n_+ + s_+ + p_+ = n_- + s_- + p_- . \tag{117}$$

Hereby we see that there is only one independent *radial* quantum number, $n_r = 0, 1, 2, \ldots$. In addition, we shall use the orbital quantum number l of the spinor $\Phi^+_{m_j,\kappa_j}$ [18], as an auxiliary quantum number. On the other hand, if we express (141) in terms of Jacobi polynomials, we observe that these functions remain square integrable for $2s_\pm > -1/2$ and $2p_\pm > -1/2$. Since $l = 0, 1, 2, \ldots$ we are forced to select only the positive solutions of Eqs. (113). The different solutions of Eqs. (114) defines the boundary conditions of the allowed quantum modes, like in the case of scalar modes [5]. We say that for $k > -1/2$ the values $2p_+ = k$ and $2p_- = k + 1$ define the boundary conditions of *regular* modes. The other possible values, $2p_+ = -k + 1$ and $2p_- = -k$, define the *irregular* modes when $k < 1/2$. Obviously, for $-1/2 < k < 1/2$ both these modes are possible. We note that the AdS quantization conditions require, in addition, k to be a half integer. Then it is clear that the domains of k corresponding to the regular and respectively irregular modes can not overlap with each other. Anyway, in our opinion, the problem of the meaning of the irregular modes as well as that of the relation between these kind of modes is sensitive and may be carefully analyzed. For this reason we restrict ourselves to write down only the energy eigenspinors of the regular modes on CAdS.

Let us take first $\kappa_j = -(j + 1/2) = -l - 1$. Then the positive solutions of (113) are $2s_+ = l + 1$ and $2s_- = l + 2$ while, according to (117), we must have $n_+ = n_r$ and $n_- = n_r - 1$. For these values of parameters, the functions $\hat{f}^{(\pm)}$ given by Eqs. (141) and (116) represent a correct solution of the transformed radial problem (137) only if

$$\frac{N_-}{N_+} = -\frac{2n_r}{2l+3}. \tag{118}$$

Furthermore, it is easy to express (141) in terms of Jacobi polynomials and to calculate the normalization factors according to Eq. (115). Thus we arrive at the result,

$$\begin{aligned}\hat{f}^{(+)}(r)_{|\kappa_j=-(j+1/2)} = N\left[\frac{n_r + k + l + 1}{n_r + l + \frac{1}{2}}\right]^{\frac{1}{2}} \times \\ \times \sin^{l+1}\omega r \cos^k \omega r P_{n_r}^{(l+\frac{1}{2},\, k-\frac{1}{2})}(\cos 2\omega r),\end{aligned} \tag{119}$$

$$\hat{f}^{(-)}(r)_{|\kappa_j=-(j+1/2)} = -N\left[\frac{n_r+k+l+1}{n_r+l+\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$\times \sin^{l+2}\omega r \cos^{k+1}\omega r P_{n_r-1}^{(l+\frac{3}{2},\, k+\frac{1}{2})}(\cos 2\omega r),$$

where

$$N = \eta\sqrt{2\omega}\left[\frac{n_r!\Gamma(n_r+k+l+1)}{\Gamma\left(n_r+l+\frac{1}{2}\right)\Gamma\left(n_r+k+\frac{1}{2}\right)}\right]^{\frac{1}{2}}. \tag{120}$$

is defined up to the phase factor η. Notice that from Eq. (118) we understand that the second equation of Eqs. (119) gives $\hat{f}^{(-)} = 0$ for $n_r = 0$.

For $\kappa_j = j + 1/2 = l$ we use the same procedure finding that $2s_+ = l + 1$, $2s_- = l$, $n_+ = n_- = n_r$ and

$$\frac{N_-}{N_+} = \frac{2l+1}{2n_r+2k+1}. \tag{121}$$

In this case the normalized radial wave functions are

$$\hat{f}^{(+)}(r)_{|\kappa_j=j+1/2} = N\left[\frac{n_r+k+\frac{1}{2}}{n_r+l+\frac{1}{2}}\right]^{\frac{1}{2}}\times$$

$$\times \sin^{l+1}\omega r \cos^{k}\omega r P_{n_r}^{(l+\frac{1}{2},\, k-\frac{1}{2})}(\cos 2\omega r), \tag{122}$$

$$\hat{f}^{(-)}(r)_{|\kappa_j=j+1/2} = N\left[\frac{n_r+l+\frac{1}{2}}{n_r+k+\frac{1}{2}}\right]^{\frac{1}{2}}\times$$

$$\times \sin^{l}\omega r \cos^{k+1}\omega r P_{n_r}^{(l-\frac{1}{2},\, k+\frac{1}{2})}(\cos 2\omega r).$$

The energy levels result from Eq. (116). Bearing in mind that $\omega k = m$ and $\omega\epsilon = E - \omega/2$, and defining the *principal* quantum number $n = 2n_r + l$ we obtain the discrete energy spectrum [16]

$$E_n = m + \omega\left(n + \frac{3}{2}\right), \quad n = 0, 1, 2, ... \tag{123}$$

These levels are degenerated. For a given n our auxiliary quantum number l takes either all the odd values from 1 to n, if n is odd, or the even values from 0 to n, if n is even. In both cases we have $j = l \pm 1/2$ for each l, which means that

$j = 1/2, 3/2, \ldots, n + 1/2$. The selection rule for κ_j is more complicated since it is determined by both the quantum numbers n and j. If n is even then the even κ_j are positive while the odd κ_j are negative. For odd n we are in the opposite situation, with odd positive or even negative values of κ_j. Thus it is clear that for each given pair (n, j) we have only one value of κ_j. With these specifications and by taking into account that for each j we have $2j + 1$ different values of m_j, we can conclude that the degree of degeneracy of the level E_n is $(n + 1)(n + 2)$.

Since the solutions (119) and (122) are completely determined by the values of n and j, we denote by $\hat{f}^{(\pm)}_{n,j}$ the radial wave functions (119) and (122). With their help we can write the functions $f^{(\pm)}_{n,j}$ (*i.e.* the components of $\mathcal{F}$) using the inverse of the transformation (135). Then from Eq. (96) we find the definitive form of the normalized particle-like energy eigenspinors of the regular modes,

$$\begin{aligned}U_{n,j,m_j}(x) = \omega \csc \omega r \cos^{3/2} \omega r \times \\ \times\Bigg[\left(\cos\frac{\omega r}{2}\hat{f}^{(+)}_{n,j}(r) + \sin\frac{\omega r}{2}\hat{f}^{(-)}_{n,j}(r)\right)\Phi^{+}_{m_j,\kappa_j}(\theta,\phi) + \\ +\left(-\sin\frac{\omega r}{2}\hat{f}^{(+)}_{n,j}(r) + \cos\frac{\omega r}{2}\hat{f}^{(-)}_{n,j}(r)\right)\Phi^{-}_{m_j,\kappa_j}(\theta,\phi)\Bigg]e^{-iE_n t}.\end{aligned} \tag{124}$$

The antiparticle-like energy eigenspinors can be derived directly by using the charge conjugation [29]. These are

$$V_{n,j,m_j} = (U_{n,j,m_j})^c \equiv C(\overline{U}_{n,j,m_j})^T, \quad C = i\gamma^2\gamma^0. \tag{125}$$

Furthermore, we can verify that all these normalized energy eigenspinors have good orthogonality properties obeying

$$\begin{aligned}\int_D d^3x\,\mu(\vec{x})\overline{U}_{n,j,m_j}(x)\gamma^0 U_{n',j',m'_j}(x) = \\ = \int_D d^3x\,\mu(\vec{x})\overline{V}_{n,j,m_j}(x)\gamma^0 V_{n',j',m'_j}(x) = \delta_{n,n'}\delta_{j,j'}\delta_{m_j,m'_j},\end{aligned} \tag{126}$$

$$\begin{aligned}\int_D d^3x\,\mu(\vec{x})\overline{U}_{n,j,m_j}(x)\gamma^0 V_{n',j',m'_j}(x) = \\ = \int_D d^3x\,\mu(\vec{x})\overline{V}_{n,j,m_j}(x)\gamma^0 U_{n',j',m'_j}(x) = 0,\end{aligned} \tag{127}$$

where

$$\mu(\vec{x}) = \frac{w(r)^3}{v(r)^2} = \frac{1}{\omega^2 r^2}\sin^2\omega r \sec^3\omega r \tag{128}$$

is the specific relativistic weight function [16]. Of course, the factors ω and $1/\omega^2$ can be removed simultaneously from Eqs. (124) and respectively (128).

The final result is that for $m \geq \omega/2$, when only regular modes are allowed, the quantum Dirac field on CAdS reads

$$\psi(x) = \sum_{n,j,m_j} \left[U_{n,j,m_j}(x) a_{n,j,m_j} + V_{n,j,m_j}(x) b^{\dagger}_{n,j,m_j} \right]. \tag{129}$$

This field can be quantized supposing that the particle $(a,\ a^{\dagger})$ and antiparticle $(b,\ b^{\dagger})$ operators satisfy usual anticommutation relations from which the non-vanishing ones are

$$\left\{ a_{n,j,m_j}, a^{\dagger}_{n',j',m'_j} \right\} = \left\{ b_{n,j,m_j}, b^{\dagger}_{n',j',m'_j} \right\} = \delta_{n,n'} \delta_{j,j'} \delta_{m_j,m'_j} \mathbf{1}, \tag{130}$$

where **1** is the identity operator. Then the one-particle operators obtained via Noether theorem, using Eq. (88) have correct forms and satisfy the conditions (89). After a little calculation we obtain the Hamiltonian operator

$$\mathbf{H} = \sum_{n,j,m_j} E_n \left[a^{\dagger}_{n,j,m_j} a_{n,j,m_j} + b^{\dagger}_{n,j,m_j} b_{n,j,m_j} \right], \tag{131}$$

while the charge operator reads

$$\mathbf{Q} = \sum_{n,j,m_j} \left[a^{\dagger}_{n,j,m_j} a_{n,j,m_j} - b^{\dagger}_{n,j,m_j} b_{n,j,m_j} \right]. \tag{132}$$

The angular operators diagonal in this basis $\vec{\mathbf{J}}^2$, $\mathbf{J}_3$ and **K**, take the same form as **H** but depending on the eigenvalues $j(j+1)$, m_j and κ_j respectively. Thus, we conclude that the complete set of commuting operators that defines the quantum modes is $\{\mathbf{H}, \mathbf{Q}, \vec{\mathbf{J}}^2, \mathbf{K}, \mathbf{J}_3\}$. All these operators have similar structures and properties like those of the usual quantum field theory in flat spacetime.

7.2. DIRAC QUANTUM MODES IN dS SPACETIME

The first solution of the Dirac equation on dS backgrounds was found in Ref. [6] based on a diagonal gauge in central charts. Other interesting solutions in the null tetrad gauge of these charts were derived recently in Ref. [9]. However, here we restrict ourselves to present only the solutions that can be obtained in our Cartesian gauge [17].

Let us consider the problem of the massive Dirac particle on dS background M of radius $R = 1/\omega$ defined by Eq. (61) for $\epsilon = 1$. There exists a special central chart $\{t, r, \theta, \phi\}$ with the line element

$$ds^2 = \frac{1}{\cosh^2 \omega r}\left[dt^2 - dr^2 - \frac{1}{\omega^2}\sinh^2 \omega r\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{133}$$

from which we can identify the functions v and w we need for writing down the radial Hamiltonian,

$$H_r = \begin{vmatrix} \frac{\omega k}{\cosh \omega r} & -\frac{d}{dr} + \frac{\omega \kappa_j}{\sinh \omega r} \\ \frac{d}{dr} + \frac{\omega \kappa_j}{\sinh \omega r} & -\frac{\omega k}{\cosh \omega r} \end{vmatrix}, \tag{134}$$

with the same notation, $k = m/\omega$, as in the previous case. This is the basic piece of the radial problem which must determine the radial functions $f^{(\pm)}$ of the solutions (96).

This radial problem has a hidden supersymmetry like in the AdS case [17]. This can be pointed out with the help of the transformation $\mathcal{F} \to \hat{\mathcal{F}} = U(r)\mathcal{F}$ where $\mathcal{F} = (f^{(+)}, f^{(-)})^T$, $\hat{\mathcal{F}} = (\hat{f}^{(+)}, \hat{f}^{(-)})^T$ and

$$U(r) = \begin{vmatrix} \cosh\frac{\omega r}{2} & -i\sinh\frac{\omega r}{2} \\ i\sinh\frac{\omega r}{2} & \cosh\frac{\omega r}{2} \end{vmatrix}. \tag{135}$$

A little calculation shows us that the transformed Hamiltonian,

$$\hat{H}_r = U(r) H_r U^{-1}(r) - i\frac{\omega}{2} 1_{2\times 2}, \tag{136}$$

which gives the new eigenvalue problem

$$\hat{H}_r \hat{\mathcal{F}} = \left(E - i\frac{\omega}{2}\right)\hat{\mathcal{F}}, \tag{137}$$

has supersymmetry since it has the requested specific form,

$$\hat{H}_r = \begin{vmatrix} \nu & -\frac{d}{dr} + W \\ \frac{d}{dr} + W & -\nu \end{vmatrix}. \tag{138}$$

Here $\nu = \omega(k - i\kappa_j)$ is a constant and

$$W(r) = \omega(ik \tanh \omega r + \kappa_j \coth \omega r) \tag{139}$$

is the superpotential of the radial problem [17]. The transformed radial wave functions $\hat{f}^{(\pm)}$ satisfy the second order equations resulted from the square of Eq. (137). These are

$$\left(-\frac{d^2}{dr^2}-\omega^2\frac{ik(ik\pm 1)}{\cosh^2\omega r}+\omega^2\frac{\kappa_j(\kappa_j\pm 1)}{\sinh^2\omega r}\right)\hat{f}^{(\pm)}(r)=\omega^2\varepsilon^2\hat{f}^{(\pm)}(r), \tag{140}$$

where we have denoted $\epsilon = E/\omega - i/2$.

The solutions of Eqs. (140) are well-known to be expressed in terms of Gauss hypergeometric functions [33], $F_\pm(y)\equiv F(\alpha_\pm,\beta_\pm,\gamma_\pm,y)$, depending on the new variable $y=-\sinh^2\omega r$, as

$$\hat{f}^{(\pm)}(y)=N_\pm(1-y)^{p_\pm}y^{s_\pm}F_\pm(y) \tag{141}$$

where

$$\alpha_\pm=s_\pm+p_\pm+\frac{i\epsilon}{2},\quad \beta_\pm=s_\pm+p_\pm-\frac{i\epsilon}{2},\quad \gamma_\pm=2s_\pm+\frac{1}{2}, \tag{142}$$

$N_\pm$ are normalization factors while the parameters $p_\pm$ and $s_\pm$ are related with k and κ_j through

$$2s_\pm(2s_\pm-1)=\kappa_j(\kappa_j\pm 1), \tag{143}$$

$$2p_\pm(2p_\pm-1)=ik(ik\pm 1). \tag{144}$$

Furthermore, we have to find the suitable values of these parameters such that the functions (141) should be solutions of the transformed radial problem (137). If we replace (141) in (137), after a few manipulation, we obtain

$$\begin{aligned}&y(1-y)\frac{dF_\pm(y)}{dy}-y\left(p_\pm\pm\frac{ik}{2}\right)F_\pm(y)+(1-y)\left(s_\pm\pm\frac{\kappa_j}{2}\right)F_\pm(y)=\\&=\frac{\eta}{2}\frac{N_\mp}{N_\pm}\left(\kappa_j\pm\frac{1}{2}+i\frac{M\pm E}{\omega}\right)y^{s_\mp-s_\pm+1/2}(1-y)^{p_\mp-p_\pm+1/2}F_\mp(y),\end{aligned} \tag{145}$$

where $\eta=\pm 1$. These equations are nothing else than the usual identities of hypergeometric functions if the values of $s_\pm$, $p_\pm$ and N_+/N_- are correctly matched. First we observe that the differences s_+-s_- and p_+-p_- must be half-integer since we work with analytic functions of y. This means that the allowed groups of solutions of (143) are

$$\begin{aligned}&2s_+^1=-\kappa_j, &&2s_+^2=\kappa_j+1,\\&2s_-^1=-\kappa_j+1, &&2s_-^2=\kappa_j,\end{aligned} \tag{146}$$

while Eq. (144) gives us

$$\begin{aligned}&2p_+^1=-ik, &&2p_+^2=ik+1,\\&2p_-^1=-ik+1, &&2p_-^2=ik.\end{aligned} \tag{147}$$

On the other hand, it is known that the hypergeometric functions of (141) are analytical on the domain $D_y = (-\infty, 0]$, corresponding to D_r, only if $\Re(\gamma_\pm) > \Re(\beta_\pm) > 0$ [33]. Moreover, their factors must be regular on this domain including $y = 0$. Obviously, both these conditions are accomplished if we take $s_\pm > 0$. We specify that there are no restrictions upon the values of the parameters $p_\pm$.

We have hence all the possible combinations of parameter values giving the solutions of the second order equations (140) which satisfy the transformed radial problem. These solutions will be denoted by (a, b), $a, b = 1, 2$, understanding that the corresponding parameters are $s_\pm^a$, $p_\pm^b$, as given by (146) and (147), and

$$\alpha_\pm^{(a,b)} = s_\pm^a + p_\pm^b + \frac{i\epsilon}{2}, \quad \beta_\pm^{(a,b)} = s_\pm^a + p_\pm^b - \frac{i\epsilon}{2}, \quad \gamma_\pm^{(a)} = 2s_\pm^a + \frac{1}{2}. \tag{148}$$

The condition $s_\pm > 0$ requires to chose $a = 1$ when $\kappa_j = -j - 1/2$, and $a = 2$ if $\kappa = j + 1/2$. Thus, for each given set (E, j, κ_j) we have a pair of different radial solutions, with $b = 1, 2$. Therefore, it is convenient to denote the transformed radial wave functions (141) by $\hat{f}^{(\pm)}_{E,\kappa_j,a,b}$, bearing in mind that the value of a determines that of κ_j. The last step is to calculate the values of N_+/N_-. From (145) it results

$$\kappa_j = -j - \frac{1}{2}: \qquad \eta\frac{N_-^{(1,1)}}{N_+^{(1,1)}} = -\frac{\alpha_+^{(1,1)}}{\gamma_+^{(1)}}, \qquad \eta\frac{N_-^{(1,2)}}{N_+^{(1,2)}} = \frac{\beta_+^{(1,2)}}{\gamma_+^{(1)}} - 1, \tag{149}$$

$$\kappa_j = j + \frac{1}{2}: \qquad \eta\frac{N_+^{(2,1)}}{N_-^{(2,1)}} = 1 - \frac{\beta_-^{(2,1)}}{\gamma_-^{(2)}}, \qquad \eta\frac{N_+^{(2,2)}}{N_-^{(2,2)}} = \frac{\alpha_-^{(2,2)}}{\gamma_-^{(2)}}. \tag{150}$$

Notice that here $\gamma_+^{(1)} = \gamma_-^{(2)} = j + 1$.

Now we can restore the form of the original radial wave functions of (96) using the inverse of the transformation (135). In our new notation these wave functions are [17]

$$f^{(\pm)}_{E,\kappa_j,a,b}(r) = \cosh\frac{\omega r}{2}\,\hat{f}^{(\pm)}_{E,\kappa_j,a,b}(r) \pm i\sinh\frac{\omega r}{2}\,\hat{f}^{(\mp)}_{E,\kappa_j,a,b}(r). \tag{151}$$

For very large r (when $y \to -\infty$) the hypergeometric functions behave as $F_\pm(y) \sim (-y)^{-\alpha_\pm}$ [33]. Thereby we deduce that $\hat{f}^{(\pm)}_{E,\kappa_j,a,b} \sim \exp(-i\varepsilon\omega r)$ and

$$f^{(\pm)}_{E,\kappa_j,a,b} \sim e^{-iEr}. \tag{152}$$

This means that the functions (151) represent tempered distributions corresponding to a continuous energy spectrum. On the other hand, Eq. (152) indicates that this energy spectrum covers the whole real axis, as it seems to be natural since the metric is not asymptotically flat. Anyway, it is clear that the energy spectrum is continuous, without discrete part, while the energy levels are infinitely degenerated since there are no restrictions upon the values of j, which can be any positive half-integer.

In these conditions, the fundamental solutions can not be normalized in the generalized sense and, therefore, the quantization can not be done in this chart.

8. DIRAC FERMIONS IN MOVING FRAMES OF THE dS SPACETIME

The first solutions of the Dirac equation in moving charts with spherical coordinates of the dS manifolds were derived in Ref. [8] and normalized in Ref. [34]. Other solutions of this equation in moving frames with Cartesian coordinates have been reported [35] but these solutions are not correctly normalized. We have constructed the normalized solutions in this gauge using the helicity basis in momentum representation [20]. Our theory of external symmetry offered us the operators we need for deriving the normalized fundamental solutions and quantizing the Dirac field in usual manner.

8.1. OBSERVABLES IN dS SPACETIME

We remain in the case of the de Sitter spacetime, of radius $R = 1/\omega$, for which we keep the definition (61) and the previous notations. This manifold is the homogeneous space of the pseudo-orthogonal group $SO(4, 1)$. This group is in the same time the gauge group of the metric $\eta^5(1) = (1, -1, -1, -1, -1)$ and the isometry group, $I(M)$, of the dS spacetime. For this reason it is convenient to use the covariant real parameters $\xi^{AB} = -\xi^{BA}$ since in this case the orbital basis-generators of the representation of $SO(4, 1)$, carried by the space of the scalar functions over M^5, have the standard form (62). They will give us directly the orbital basis-generators $L_{(AB)}$ of the scalar representations of $I(M)$. Indeed, starting with the functions $Z^A(x)$ that solve the equation (61) in the chart $\{x\}$, one can write down the operators (62) in the form (24), finding thus the generators $L_{(AB)}$ and implicitly the components $k^{\mu}_{(AB)}(x)$ of the Killing vectors associated to the parameters ξ^{AB} [15]. Furthermore, one has to calculate the spin parts $S_{(AB)}$, according to Eqs. (38) and (28), arriving to the final form of the basis-generators $X_{(AB)} = L_{(AB)} + S_{(AB)}$ of the spinor representation of $S(M)$.

In the dS spacetime there are many static or moving charts of physical interest. First we consider the moving local chart $\{t, r, \theta, \phi\}$ associated to the

Cartesian one $\{t, \vec{x}\}$ with the line element

$$ds^2 = dt^2 - e^{2\omega t} d\vec{x}^2 . \tag{153}$$

The time $t \in (-\infty, \infty)$ of this chart is interpreted as the *proper* time of an observer at $\vec{x} = 0$. Another moving chart which play here a special role is the chart $\{t_c, \vec{x}\}$ with the *conformal* time t_c and Cartesian spaces coordinates x^i defined by

$$\begin{aligned} Z^0 &= -\frac{1}{2\omega^2 t_c}\left[1-\omega^2({t_c}^2 - r^2)\right] \\ Z^5 &= -\frac{1}{2\omega^2 t_c}\left[1+\omega^2({t_c}^2 - r^2)\right] \\ Z^i &= -\frac{1}{\omega t_c} x^i \end{aligned} \tag{154}$$

with $r = |\vec{x}|$. This chart covers only a half of the manifold M, for $t_c \in (-\infty, 0)$ and $\vec{x} \in D \equiv \mathbb{R}^3$. Nevertheless, it has the advantage of a simple conformal flat line element [4],

$$ds^2 = \frac{1}{\omega^2 t_c^2}\left(dt_c^2 - d\vec{x}^2\right). \tag{155}$$

Moreover, the conformal time t_c is related to the proper time t through

$$\omega t_c = -e^{-\omega t} . \tag{156}$$

In what follows we study the Dirac field in the chart $\{t, \vec{x}\}$ using the conformal time as a helpful auxiliary ingredient. The form of the line element (155) allows one to choose the simple Cartesian gauge with the non-vanishing tetrad components [8]

$$e^0_0 = -\omega t_c , \quad e^i_j = -\delta^i_j\, \omega t_c , \quad \hat{e}^0_0 = -\frac{1}{\omega t_c}, \quad \hat{e}^i_j = -\delta^i_j \frac{1}{\omega t_c}. \tag{157}$$

In this gauge the Dirac operator reads

$$\begin{aligned} E_D &= -i\omega t_c\left(\gamma^0 \partial_{t_c} + \gamma^i \partial_i\right) + \frac{3i\omega}{2}\gamma^0 = \\ &= i\gamma^0 \partial_t + i e^{-\omega t}\gamma^i \partial_i + \frac{3i\omega}{2}\gamma^0 \end{aligned} \tag{158}$$

and the weight function of the scalar product (82) is

$$\mu = (-\omega t_c)^{-3} = e^{3\omega t} . \tag{159}$$

The next step is to calculate the basis-generators $X_{(AB)}$ of the spinor representation of $S(M)$ in this gauge since these are the main operators that commute with E_D. The group $SO(4,1)$ includes the subgroup $E(3) = T(3)Ⓢ SO(3)$ which is just the isometry group of the 3-dimensional Euclidean space of our moving charts, $\{t_c, \vec{x}\}$ and $\{t, \vec{x}\}$, formed by $\mathbb{R}^3$ translations, $x^i \to x^i + a^i$, and proper rotations, $x^i \to R^{i\cdot}_{\cdot j} x^j$ with $R \in SO(3)$ [11]. Therefore, the basis-generators of its universal covering group, $\tilde{E}(3) = T(3)Ⓢ SU(2) \subset S(M)$, can be interpreted as the components of the momentum, $\vec{P}$, and total angular momentum, $\vec{J}$, operators. The problem of the Hamiltonian operator seems to be more complicated, but we know that in the mentioned static central charts with the static time t_s this is $H = \omega X_{(05)} = i\partial_{t_s}$ [15]. Thus the Hamiltonian operator and the components of the momentum and total angular momentum operators (P^i and $J^i = \varepsilon_{ijk} J_{jk}/2$ respectively) can be identified as being the following basis-generators of $S(M)$

$$H \equiv \omega X_{(05)} = -i\omega(t_c \partial_{t_c} + x^i \partial_i) \tag{160}$$

$$P^i \equiv \omega\left(X_{(5i)} - X_{(0i)}\right) = -i\partial_i \tag{161}$$

$$J_{ij} \equiv X_{(ij)} = -i(x^i \partial_j - x^j \partial_i) + S_{ij} \tag{162}$$

after which one remains with the three basis-generators

$$N^i \equiv X_{(5i)} + X_{(0i)} = \omega(t_c^2 - r^2)P^i + 2x^i H + 2\omega(S_{i0} t_c + S_{ij} x^j), \tag{163}$$

which do not have an immediate physical significance. The $SO(4,1)$ transformations corresponding to these basis-generators and the associated isometries of the chart $\{t_c, \vec{x}\}$ are briefly presented in Appendix.

In the other local chart, $\{t, \vec{x}\}$, we have the same operators $\vec{P}$ and $\vec{J} = \vec{L} + \vec{S}$ (with $\vec{L} = \vec{x} \times \vec{P}$) whose components are the $\tilde{E}(3)$ generators, while the Hamiltonian operator takes the form

$$H = i\partial_t + \omega \vec{x} \cdot \vec{P}, \tag{164}$$

where the second term, due to the external gravitational field, leads to the commutation rules

$$[H, \ P^i] = i\omega P^i. \tag{165}$$

We observe that in this chart the operators $K^i \equiv X_{(0i)}$ are the analogous of the basis-generators of the Lorentz boosts of $SL(2,\mathbb{C})$ since in the limit of $\omega \to 0$, when (153) equals the Minkowski line element, the operators $H = P^0$, P^i, J^i and K^i become the generators of the spinor representation of the group $T(4)\circledS SL(2,\mathbb{C})$ (*i.e.* the universal covering group of the Poincaré group [11, 19]).

In both the charts we used here the generators (160)–(163) are self-adjoint with respect to the scalar product (82) with the weight function (159) if we consider the usual boundary conditions on $D \equiv \mathbb{R}^3$. Therefore, for any generator X we have $\langle X\psi, \psi'\rangle = \langle \psi, X\psi'\rangle$ if (and only if) ψ and ψ' are solutions of the Dirac equation which behave as tempered distributions or square integrable spinors with respect to the scalar product (82). Moreover, all these generators commute with the Dirac operator E_D.

8.2. POLARIZED PLANE WAVE SOLUTIONS

The plane wave solutions of the Dirac equation with $m \neq 0$ must be eigenspinors of the momentum operators P^i corresponding to the eigenvalues p^i, with the same time modulation as the spherical waves. Therefore, we have to look for particular solutions in the chart $\{t_c, \vec{x}\}$ involving either positive or negative frequency plane waves. Bearing in mind that these must be related among themselves through the charge conjugation, we assume that, in the standard representation of the Dirac matrices (with diagonal γ^0 [19]), they have the form

$$\psi^{(+)}_{\vec{p}} = \begin{pmatrix} f^+(t_c)\alpha(\vec{p}) \\ g^+(t_c)\dfrac{\vec{\sigma}\cdot\vec{p}}{p}\alpha(\vec{p}) \end{pmatrix} e^{i\vec{p}\cdot\vec{x}} \tag{166}$$

$$\psi^{(-)}_{\vec{p}} = \begin{pmatrix} g^-(t_c)\dfrac{\vec{\sigma}\cdot\vec{p}}{p}\beta(\vec{p}) \\ f^-(t_c)\beta(\vec{p}) \end{pmatrix} e^{-i\vec{p}\cdot\vec{x}} \tag{167}$$

where $p = |\vec{p}|$, σ_i denotes the Pauli matrices while α and β are arbitrary Pauli spinors depending on $\vec{p}$. Replacing these spinors in the Dirac equation given by (158) and denoting $k = m/\omega$ and $\nu_\pm = \frac{1}{2} \pm ik$, we find equations of the form (202) whose solutions can be written in terms of Hankel functions as

$$f^+ = (-f^-)^* = C t_c^2 e^{\pi k/2} H^{(1)}_{\nu_-}(-p t_c) \tag{168}$$

$$g^{+} = (-g^{-})^{*} = Ct_c^2 e^{-\pi k/2} H^{(1)}_{\nu_+}(-pt_c). \tag{169}$$

The integration constant C will be calculated from the ortonormalization condition in the momentum scale.

The plane wave solutions are determined up to the significance of the Pauli spinors α and β. For $\vec{p} \neq 0$ these can be treated as in the flat case [36, 19] since, in the tetrad gauge (157), the spaces of these spinors carry unitary linear representations of the $\tilde{E}(3)$ group. Indeed, a transformation (27) produced by $(A, \phi_{A,\vec{a}}) \in \tilde{E}(3) \subset S(M)$ where $A \in SU(2)$ and $\vec{a} \in \mathbb{R}^3$ involves the usual linear isometry of $E(3)$, $x^i \to x'^{\,i} = \phi^i_{A,\vec{a}}(\vec{x}) \equiv \Lambda^{i\cdot}_{\cdot j}(A) x^j + a^i$ with $\Lambda(A) \in SO(3)$, and the global transformation $\psi(t,\vec{x}) \to \psi'(t,\vec{x}') = \rho(A)\psi(t,\vec{x})$. Consequently, the Pauli spinors transform according to the unitary (linear) representation

$$\alpha(\vec{p}) \to e^{-i\vec{a}\cdot\vec{p}}\, A\,\alpha[\Lambda(A)^{-1}\vec{p}\,] \tag{170}$$

(and similarly for β) that preserves orthogonality. This means that any pair of orthogonal spinors $\tilde{\xi}_\sigma(\vec{p})$ with polarizations $\sigma = \pm 1/2$ (obeying $\tilde{\xi}^{+}_{\sigma}\tilde{\xi}_{\sigma'} = \delta_{\sigma\sigma'}$) represents a good basis in the space of Pauli spinors

$$\alpha(\vec{p}) = \sum_{\sigma} \tilde{\xi}_\sigma(\vec{p}) a(\vec{p},\sigma) \tag{171}$$

whose components, $a(\vec{p},\sigma)$, are the particle wave functions in momentum representation. According to the standard interpretation of the negative frequency terms [36, 19], the corresponding basis of the space of β spinors is defined by

$$\beta(\vec{p}) = \sum_{\sigma} \tilde{\eta}_\sigma(\vec{p})[b(\vec{p},\sigma)]^{*}, \quad \tilde{\eta}_\sigma(\vec{p}) = i\sigma_2[\tilde{\xi}_\sigma(\vec{p})]^{*} \tag{172}$$

where $b(\vec{p},\sigma)$ are the antiparticle wave functions. It remains to choose a specific basis, using supplementary physical assumptions. Since it is not sure that the so called spin basis [36] can be correctly defined in the dS geometry, we prefer the *helicity* basis. This is formed by the orthogonal Pauli spinors of helicity $\lambda = \pm 1/2$ which fulfill

$$\vec{\sigma}\cdot\vec{p}\;\tilde{\xi}_\lambda(\vec{p}) = 2p\lambda\,\tilde{\xi}_\lambda(\vec{p}), \quad \vec{\sigma}\cdot\vec{p}\;\tilde{\eta}_\lambda(\vec{p}) = -2p\lambda\,\tilde{\eta}_\lambda(\vec{p}). \tag{173}$$

The desired particular solutions of the Dirac equation with $m \neq 0$ result from our starting formulas (166) and (167) where we insert the functions (168) and (169) and the spinors (171) and (172) written in the helicity basis (173). It remains to calculate the normalization constant C with respect to the scalar product (82) with the weight function (159). After a few manipulations, in the chart $\{t,\vec{x}\}$, it turns out that the final form of the fundamental spinor solutions of positive and negative frequencies with momentum $\vec{p}$ and helicity λ is

$$U_{\vec{p},\lambda}(t,\vec{x}) = iN \begin{pmatrix} \frac{1}{2} e^{\pi k/2} H^{(1)}_{\nu_-}(qe^{-\omega t})\,\tilde{\xi}_\lambda(\vec{p}) \\ \lambda\, e^{-\pi k/2} H^{(1)}_{\nu_+}(qe^{-\omega t})\,\tilde{\xi}_\lambda(\vec{p}) \end{pmatrix} e^{i\vec{p}\cdot\vec{x}-2\omega t} \tag{174}$$

$$V_{\vec{p},\lambda}(t,\vec{x}) = iN \begin{pmatrix} -\lambda\, e^{-\pi k/2} H^{(2)}_{\nu_-}(qe^{-\omega t})\,\tilde{\eta}_\lambda(\vec{p}) \\ \frac{1}{2} e^{\pi k/2} H^{(2)}_{\nu_+}(qe^{-\omega t})\,\tilde{\eta}_\lambda(\vec{p}) \end{pmatrix} e^{-i\vec{p}\cdot\vec{x}-2\omega t}\,, \tag{175}$$

where we introduced the new parameter $q = p/\omega$ and

$$N = \frac{1}{(2\pi)^{3/2}}\sqrt{\pi q}\,. \tag{176}$$

Using Eqs. (201) and (203), it is not hard to verify that these spinors are charge-conjugated to each other,

$$V_{\vec{p},\lambda} = (U_{\vec{p},\lambda})^c = \mathcal{C}(\bar{U}_{\vec{p},\lambda})^T\,, \quad \mathcal{C} = i\gamma^2\gamma^0\,, \tag{177}$$

satisfy the ortonormalization relations

$$\left\langle U_{\vec{p},\lambda}, U_{\vec{p}\,',\lambda'}\right\rangle = \left\langle V_{\vec{p},\lambda}, V_{\vec{p}\,',\lambda'}\right\rangle = \delta_{\lambda\lambda'}\delta^3(\vec{p}-\vec{p}\,'), \tag{178}$$

$$\left\langle U_{\vec{p},\lambda}, V_{\vec{p}\,',\lambda'}\right\rangle = \left\langle V_{\vec{p},\lambda}, U_{\vec{p}\,',\lambda'}\right\rangle = 0, \tag{179}$$

and represent a *complete* system of solutions in the sense that

$$\int d^3p \sum_\lambda \left[U_{\vec{p},\lambda}(t,\vec{x})U^+_{\vec{p},\lambda}(t,\vec{x}\,') + V_{\vec{p},\lambda}(t,\vec{x})V^+_{\vec{p},\lambda}(t,\vec{x}\,')\right] = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,'). \tag{180}$$

Let us observe that the factor $e^{-3\omega t}$ is exactly the quantity necessary to compensate the weight function (159). Other important properties are

$$P^i\, U_{\vec{p},\lambda} = p^i\, U_{\vec{p},\lambda}\,, \quad P^i\, V_{\vec{p},\lambda} = -p^i\, V_{\vec{p},\lambda}\,, \tag{181}$$

$$W\, U_{\vec{p},\lambda} = p\lambda U_{\vec{p},\lambda}\,, \quad W\, V_{\vec{p},\lambda} = -p\lambda V_{\vec{p},\lambda}\,, \tag{182}$$

where

$$W = \vec{J}\cdot\vec{P} = \vec{S}\cdot\vec{P} \tag{183}$$

is the helicity operator which is analogous to the time-like component of the four-component Pauli-Lubanski operator of the Poincaré algebra [11]. Thus, we arrive at the conclusion that the fundamental solutions (174) and (175) form a complete system of common eigenspinors of the operators P^i and W. Since the spin was fixed a priori by choosing the representation ρ, we consider that the complete set of commuting operators which determines separately each of the sets of the particle or antiparticle eigenspinors is $\{E_D, \vec{S}^2, P^i, W\}$. Finally we note that these solutions can be redefined at any time with other momentum dependent phase factors as

$$U_{\vec{p},\lambda} \to e^{i\chi(\vec{p})}U_{\vec{p},\lambda}, \quad V_{\vec{p},\lambda} \to e^{-i\chi(\vec{p})}V_{\vec{p},\lambda}, \quad \chi(\vec{p}) \in \mathbb{R}, \tag{184}$$

without to affect the above properties.

In the case $m = 0$ (when $k = 0$) it is convenient to consider the chiral representation of the Dirac matrices (with diagonal γ^5 [36]) and the chart $\{t_c, \vec{x}\}$. We find that the fundamental solutions in helicity basis of the left-handed massless Dirac field,

$$\begin{aligned} U^0_{\vec{p},\lambda}(t_c, \vec{x}) &= \lim_{k\to 0} \frac{1-\gamma^5}{2} U_{\vec{p},\lambda}(t_c, \vec{x}) = \\ &= \left(\frac{-\omega t_c}{2\pi}\right)^{3/2} \begin{pmatrix} (\frac{1}{2}-\lambda)\tilde{\xi}_\lambda(\vec{p}) \\ 0 \end{pmatrix} e^{-ipt_c + i\vec{p}\cdot\vec{x}} \end{aligned} \tag{185}$$

$$\begin{aligned} V^0_{\vec{p},\lambda}(t_c, \vec{x}) &= \lim_{k\to 0} \frac{1-\gamma^5}{2} V_{\vec{p},\lambda}(t_c, \vec{x}) = \\ &= \left(\frac{-\omega t_c}{2\pi}\right)^{3/2} \begin{pmatrix} (\frac{1}{2}+\lambda)\tilde{\eta}_\lambda(\vec{p}) \\ 0 \end{pmatrix} e^{ipt_c - i\vec{p}\cdot\vec{x}}, \end{aligned} \tag{186}$$

are non-vanishing only for positive frequency and $\lambda = -1/2$ or negative frequency and $\lambda = 1/2$, as in Minkowski spacetime. Obviously, these solutions have similar properties as (177)–(182).

8.3. QUANTIZATION

The quantization can be done considering that the wave functions in momentum representation, $a(\vec{p},\lambda)$ and $b(\vec{p},\lambda)$, become field operators (so that $b^* \to b^\dagger$) [36]. Then the quantum field which satisfies the Dirac equation with $m \neq 0$ in the chart $\{t, \vec{x}\}$ reads

$$\begin{aligned} \psi(t, \vec{x}) &= \psi^{(+)}(t, \vec{x}) + \psi^{(-)}(t, \vec{x}) = \\ &= \int d^3p \sum_\lambda \left[U_{\vec{p},\lambda}(x) a(\vec{p},\lambda) + V_{\vec{p},\lambda}(x) b^\dagger(\vec{p},\lambda) \right]. \end{aligned} \tag{187}$$

We assume that the particle $(a, a^\dagger)$ and antiparticle $(b, b^\dagger)$ operators must fulfill the standard anticommutation relations in the momentum representation, from which the non-vanishing ones are

$$\{a(\vec{p},\lambda), a^\dagger(\vec{p}\,',\lambda')\} = \{b(\vec{p},\lambda), b^\dagger(\vec{p}\,',\lambda')\} = \delta_{\lambda\lambda'}\delta^3(\vec{p}-\vec{p}\,'), \tag{188}$$

since then the equal-time anticommutator takes the *canonical* form

$$\{\psi(t,\vec{x}), \overline{\psi}(t,\vec{x}\,')\} = e^{-3\omega t}\gamma^0\delta^3(\vec{x}-\vec{x}\,'), \tag{189}$$

as it results from (180). In general, the partial anticommutator functions,

$$\tilde{S}^{(\pm)}(t,t',\vec{x}-\vec{x}\,') = i\{\psi^{(\pm)}(t,\vec{x}),\ \overline{\psi}^{(\pm)}(t',\vec{x}\,')\}, \tag{190}$$

and the total one $\tilde{S} = \tilde{S}^{(+)} + \tilde{S}^{(-)}$ are rather complicated since for $t \neq t'$ we have no more identities like (203) which should simplify their time-dependent parts. In any event, these are solutions of the Dirac equation in both their sets of coordinates and help one to write the Green functions in usual manner. For example, from the standard definition of the Feynman propagator [36],

$$\tilde{S}_F(t,t',\vec{x}-\vec{x}\,') = i\langle 0|T[\psi(x)\overline{\psi}(x')]|0\rangle = \tag{191}$$

$$= \theta(t-t')\tilde{S}^{(+)}(t,t',\vec{x}-\vec{x}\,') - \theta(t'-t)\tilde{S}^{(-)}(t,t',\vec{x}-\vec{x}\,'), \tag{192}$$

we find that

$$[E_D(x)-m]\tilde{S}_F(t,t',\vec{x}-\vec{x}\,') = -e^{-3\omega t}\delta^4(x-x'). \tag{193}$$

Another argument for this quantization procedure is that the one-particle operators given by the Noether theorem have similar structures and properties like those of the quantum theory of the free fields in flat spacetime. Indeed, from Eq. (87) we obtain the one-particle operators of the form (88) which satisfy the Eqs. (89). The diagonal one-particle operators result directly using Eqs. (178)–(182). In this way we obtain the momentum components

$$\mathbf{P}^i =: \langle\psi, P^i\psi\rangle := \int d^3p\, p^i \sum_\lambda \left[a^\dagger(\vec{p},\lambda)a(\vec{p},\lambda) + b^\dagger(\vec{p},\lambda)b(\vec{p},\lambda)\right] \tag{194}$$

and the helicity (or Pauli-Lubanski) operator

$$\mathbf{W} =: \langle\psi, W\psi\rangle := \int d^3p \sum_\lambda p\lambda\left[a^\dagger(\vec{p},\lambda)a(\vec{p},\lambda) + b^\dagger(\vec{p},\lambda)b(\vec{p},\lambda)\right]. \tag{195}$$

The definition (88) holds for the generators of internal symmetries too, including the particular case of $X = 1$ when the bracket

$$\mathbf{Q} =: \langle\psi, \psi\rangle := \int d^3p \sum_\lambda \left[a^\dagger(\vec{p},\lambda)a(\vec{p},\lambda) - b^\dagger(\vec{p},\lambda)b(\vec{p},\lambda)\right] \tag{196}$$

gives just the charge operator corresponding to the internal $U(1)$ symmetry of the action (78) [19, 4]. It is obvious that all these operators are self-adjoint and represent the generators of the external or internal symmetry transformations of the quantum fields [36]. The conclusion is that, for fixed mass and spin, the helicity state vectors of the Fock space defined as common eigenvectors of the set $\{\mathbf{Q}, \mathbf{P}^i, \mathbf{W}\}$ form a complete system of orthonormalized vectors in generalized sense, *i.e.* the helicity basis.

The Hamiltonian operator $\mathbf{H} =: \langle \psi, H\psi \rangle :$ is conserved but is not diagonal in this basis since it does not commute with $\mathbf{P}^i$ and $\mathbf{W}$ as it follows from the commutation relations (165) and the properties (89). Its form in momentum representation can be calculated using the identity

$$H\, U_{\vec{p},\lambda}(t,\vec{x}) = -i\omega\left(p^i \partial_{p^i} + \frac{3}{2}\right) U_{\vec{p},\lambda}(t,\vec{x}), \tag{197}$$

and the similar one for $V_{\vec{p},\lambda}$, leading to

$$\mathbf{H} = \frac{i\omega}{2}\int d^3p\, p^i \sum_{\lambda}\left[a^{\dagger}(\vec{p},\lambda)\overleftrightarrow{\partial}_{p^i} a(\vec{p},\lambda) + b^{\dagger}(\vec{p},\lambda)\overleftrightarrow{\partial}_{p^i} b(\vec{p},\lambda)\right] \tag{198}$$

where the derivatives act as $f\overleftrightarrow{\partial} h = f\partial h - (\partial f)h$. Hereby it results the expected behavior of $\mathbf{H}$ under the space translations of $\tilde{E}(3)$ which transform the operators a and b according to (170). Moreover, it is worth pointing out that the change of the phase factors (184) associated with the transformations

$$a(\vec{p},\lambda) \to e^{-i\chi(\vec{p})} a(\vec{p},\lambda), \qquad b(\vec{p},\lambda) \to e^{-i\chi(\vec{p})} b(\vec{p},\lambda) \tag{199}$$

leave invariant the operators ψ, $\mathbf{Q}$, $\mathbf{P}^i$ and $\mathbf{W}$ as well as the equations (188), but transform the Hamiltonian operator,

$$\mathbf{H} \to \mathbf{H} + \omega\int d^3p [p^i \partial_{p^i}\chi(\vec{p})] \sum_{\lambda}\left[a^{\dagger}(\vec{p},\lambda) a(\vec{p},\lambda) + b^{\dagger}(\vec{p},\lambda) b(\vec{p},\lambda)\right]. \tag{200}$$

This remarkable property may be interpreted as a new type of gauge transformation depending on momentum instead of coordinates. Our preliminary calculations indicate that this gauge may be helpful for analyzing the behavior of the theory near $\omega \sim 0$.

In the simpler case of the left-handed massless field with the fundamental spinor solutions (185) and (186) we obtain similar results and we recover the standard rule of the neutrino polarization.

The quantum theory presented above opens many new interesting problems. One of them is if in the de Sitter geometry there exists an orbital analysis analogous to the Wigner theory of induced representations of the

Poincaré group [19]. This is necessary if we want to understand the meaning of the rest frames (of the massive particles) in the de Sitter spacetime and to find the "booster" mechanisms changing the value of p or even giving rise to waves of arbitrary momentum from those with $\vec{p}=0$. We believe that this theory may be done starting with the orbital analysis in M^5 since this helps us to find $SO(4,1)$-covariant definitions for our basic operators on M [2]. More precisely, for each momentum $q\in M_q^5$ we can write a five-dimensional momentum operator $P(q)$ of components $P^A(q)=\eta^{AC}q^B X_{(BC)}$ while a generalized five-dimensional Pauli-Lubanski operator in M has to be defined by $W_A=-\frac{1}{8}\varepsilon_{ABCDE}X^{(BC)}X^{(DE)}$. Then it is clear that for the representative momentum $\hat{q}=(\omega,0,0,0,-\omega)$ of the orbit $q^2=0$, associated to the little group $E(3)\subset SO(4,1)$, we recover our operators $P(\hat{q})=(H,\vec{P},-H)$ and $W=\hat{q}^A W_A$ as given by (160), (161) and (183), respectively. In this way one may construct generalized Wigner representations of the group $S(M)$ in spaces of spinors depending on momentum.

Finally, we would like to hope that our approach based on external symmetries could be an argument for a general tetrad gauge covariant theory of the quantum fields with spin in which the procedure of second quantization should be independent on the frames one uses.

APPENDIX

SOME PROPERTIES OF HANKEL FUNCTIONS

According to the general properties of the Hankel functions [33], we deduce that those used here, $H^{(1,2)}_{\nu_\pm}(z)$, with $\nu_\pm=\frac{1}{2}\pm ik$ and $z\in\mathbb{R}$, are related among themselves through

$$[H^{(1,2)}_{\nu_\pm}(z)]^*=H^{(2,1)}_{\nu_\mp}(z), \tag{201}$$

satisfy the equations

$$\left(\frac{d}{dz}+\frac{\nu_\pm}{z}\right)H^{(1)}_{\nu_\pm}(z)=ie^{\pm\pi k}H^{(1)}_{\nu_\mp}(z) \tag{202}$$

and the identities

$$e^{\pm\pi k}H^{(1)}_{\nu_\mp}(z)H^{(2)}_{\nu_\pm}(z)+e^{\mp\pi k}H^{(1)}_{\nu_\pm}(z)H^{(2)}_{\nu_\mp}(z)=\frac{4}{\pi z}. \tag{203}$$

[2] For plane wave spinors of a Dirac theory in M^5 see Ref. [37]

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