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ALTERNATIVE DESCRIPTIONS OF BORROMEAN NUCLEI: HYPERSPHERICAL AND PSEUDOPOTENTIAL APPROACHES

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Three-body systems, *i.e.*, Borromean nuclei, are studied in the framework of the Boundary Condition Model, in which the relevant interactions are expressed in terms of the scattering lengths of the two-body subsystems. We present two alternative descriptions of such systems: a hyperspherical coordinate approach and a molecular approach. We demonstrate the role of virtual states in the two-body subsystems in producing a weekly bound three-body system, *i.e.*, a Borromean nucleus. We obtain relations between the spatial extension of the three-body bound state, its energy, and the scattering lengths in the subsystems. The results are compared with experimental data for the Borromean nucleus ^{11}Li and ^{14}Be.

1. INTRODUCTION

The study of the properties of nuclei near the neutron drip line is a highly interesting topic in low energy nuclear physics in recent years. The existence of a neutron halo, *i.e.*, a large spatial extension of the neutron density distribution, is an experimentally well established fact for some drip line nuclei [1–3]. Borromean nuclei are loosely bound systems of a core and two neutrons which, however, become unbound when one neutron or the core is removed [3]. Their structure calls for a three-body description, the conventional shell-model assumptions being insufficient.

In the weakly bound nuclei the continuum single particle states play an important role as they are close to the state of interest. The theoretical descriptions of such nuclei are based on extension of the conventional shell-model. One possibility is to use the Continuum Shell Model [4]. This model is based on Gamow states which are generalized eigenstates of the time-independent Schrödinger equation with complex energy eigenvalues. They are regular in origin and have an outgoing asymtotics. From such states one can form a basis, Berggren basis, [5]. It includes the resonances as distinct states, but there is also a continuum which should be truncated.

The Hartree-Fock–Bogoliubov or Hartree-Fock plus BCS approaches were extended to study the role of the pairing interaction into continuum, see *e.g.* [6]

and references therein. The main problem here is that the pairing field might change the single particle unbound wave functions to bound and therefore the correct asymptotic boundary conditions are very important.

An other possibility to describe weakly bound nuclei is based on Relativistic Mean Field Theory, see [7]. The starting point is to treat the nucleons as Dirac particles interacting through Lorentz scalar and vector mesons and isovector vector meson. In the mean field approximation the nucleons are moving in a large attractive scalar field and a large repulsive vector field. The scalar and vector fields explain naturally the strength of the spin-orbit interaction.

Appart from the general approaches which may be used for all weakly bound nuclei, for Borromean nuclei one can use particular approaches. Such systems can be described in the Faddeev picture [8, 9], revealing the existence of a large number of spatially extended bound states in the vicinity of the threshold of breakup into three particles (Efimov effect) [10, 11], provided the two-body forces between two constituents lead to a resonant state.

As was pointed out by Fedorov *et al.* [12], the Schrödinger picture is much simpler than the Faddeev one but is unable to describe all the three two-body subsystems in a symmetrical way. However, we are interested in problems for which one particle has a much larger mass than the other two (as *e.g.* two neutrons and a ^{9}Li core). In this special case we have two interacting light particles in the core potential. As the resonant character of the light-light interaction is essential for the Efimov effect in a system of two light and one heavy particle [10], the Schrödinger picture is thus very appropriate. For such loosely bound states it is reasonable to take into account the interaction with the core in terms of its scattering properties at zero energy, *e.g.* the particle-core *s*-wave scattering length. In the same way the particle-particle interaction will be described by its particle-particle scattering length. The neutron-neutron scattering length is $a = -18.5$ fm [13, 14] (compare with the neutron-proton singlet scattering length –23.7 fm which corresponds to a virtual state at ~ 0.07 MeV, [14]).

Therefore we will discuss in the following how one can describe the three-body bound states of two interacting weakly bound particles and a core in terms of their scattering lengths. We present two alternative descriptions of such systems: a hyperspherical coordinate approach [15] and a molecular approach [16].

In the present work we will use the Boundary Condition Method, which makes the relation between the interaction and the two-body scattering length particularly transparent, together with a simple ansatz for the wave function of the three-body system. The main aim of the work is to demonstrate explicitly, how a system of a particle in the field of two centers, with each of which it forms a resonant state, can be transformed into a bound 3-body state. We do not aim at quantitative results, but rather obtain qualitative relations between the parameters

of the Borromean bound state in the three-body system (energy and spatial extension) and the two-body scattering lengths.

2. FERMI PSEUDOPOTENTIALS – *s*-WAVE STATES

The basic idea of the Contact or Zero Range Potential Model is to replace the solution of the Schrödinger equation inside a potential well by boundary conditions on the wave function at $R = 0$, see *e.g.* [17]. In the Boundary Condition Method (BCM) the action of the internal field is thus replaced by imposing a boundary condition [17] for the logarithmic derivative at the origin

$$d/dr(r\Psi)|_{R=0} = -\alpha(r\Psi)|_{R=0} . \tag{1}$$

For a bound state for $r > R$ the wave function behaves as $\chi = r\Psi \sim exp(-\alpha r)$ and $\alpha = \sqrt{-2mE/\hbar^2}$. For a scattering state with energy $E = \hbar^2 k^2/2m$ the asymptotic wave function is given by the radiation condition

$$\Psi \to exp(i\vec{k}\vec{r}) + A(\vec{k})\exp(ikr)/r, \tag{2}$$

and the boundary condition relates the scattering amplitude, A, to the parameter α

$$A = -1/(\alpha + ik). \tag{3}$$

The negative of the zero-energy scattering amplitude is thus given by the inverse of the logarithmic derivative

$$a = -A(0) = 1/\alpha \tag{4}$$

The quantity $a = -1/(d\ln\chi/dr)|_{R=0}$ is the scattering length. For a bound state it is positive, $a > 0$, while for a virtual state it is negative, $a < 0$.

We may obtain an interpretation of the BCM by using the Born approximation in which the scattering amplitude is the Fourier transform of the interaction potential. For a contact potential, $V(r) = B\delta(\vec{r})$, the scattering amplitude becomes

$$A = -a = -mB/2\pi\hbar^2 \tag{5}$$

resulting in a connection between the potential strength and the scattering length, $B \sim a$.

The effect of the internal field is now represented by zero-range potential, $U(\vec{r}) = 2mV(\vec{r})/\hbar^2 = 4\pi a\delta^3(\vec{r})$, which is known as the Fermi Pseudopotential [14]. The Fermi Pseudopotential is repulsive for a interaction of a neutron with a nuclear medium described by a positive scattering length ($a > 0$). This result was used for describing the confinement of ultra-cold neutrons [18], since for

energies less than the Fermi Pseudopotential strength the neutrons are totally reflected by the material walls, provided the neutron-nucleus interaction of the material has a positive scattering length.

This idea can be extrapolated to the case of a neutron interacting with a nuclear medium, described by a negative scattering length. The Fermi Pseudopotential is then negative and a neutron is attracted by a nucleus if the neutron and nucleus (core) form a virtual state. However this (virtual) system state is unstable and manifests itself only in the scattering process.

3. HYPERSPHERICAL FORMALISM: TWO PARTICLES IN THE FIELD OF A HEAVY CORE

The Hamiltonian of the system of two identical particles in a short range potential, $V(|\vec{r}|)$, $V(|\vec{r}|) \neq 0$ iff $|\vec{r}| < R_0$ is given by

$$H = -\frac{\hbar^2}{2m}(\Delta_1 + \Delta_2) + V(|\vec{r}_1|) + V(|\vec{r}_2|) + v(|\vec{r}_1 - \vec{r}_2|) \tag{6}$$

where m is the common mass of the particles, $\vec{r}_1$ and $\vec{r}_2$ are their coordinates and ν is the particle-particle potential assumed to be of short range too, $v(|\vec{r}|) \neq 0$ iff $|\vec{r}| < r_0$.

The hyperspherical coordinates $(\rho, \Phi, \Omega_1, \Omega_2)$ are defined as [19]

$$\begin{aligned} r_1 &= \rho \cos\Phi \\ r_2 &= \rho \sin\Phi \end{aligned} \tag{7}$$

where $\rho \in [0, \infty)$, $\Phi \in [0, \pi/2]$ and $\Omega_i = (\theta_i, \varphi_i)$ are the angles describing the directions $\vec{r}_i$, where $\vec{r}_i$, $i = 1, 2$ are the coordinates of the two particles relative to the core.

If ρ is sufficiently large, ($\rho \gg R_0$, r_0), the Φ-motion is a free one except in three small regions

$$\begin{aligned} &\Phi \in [0, R_0/\rho] \\ &\Phi \in [\pi/4 - r_0/(\sqrt{2}\rho), \pi/4 + r_0/(\sqrt{2}\rho)] \\ &\Phi \in [\pi/2 - R_0/\rho, \pi/2] \end{aligned} \tag{8}$$

The kinetic part of the Hamiltonian can be written

$$\begin{aligned} T &= -\frac{\hbar^2}{2m}(\Delta_1 + \Delta_2) = \\ &= -\frac{\hbar^2}{2m}\left[\partial_\rho^2 + \frac{5}{\rho}\partial_\rho + \frac{1}{\rho^2}\left(\partial_\Phi^2 + \frac{4}{\tan 2\Phi}\partial_\Phi + \frac{\vec{L}_1^2}{\cos^2\Phi} + \frac{\vec{L}_2^2}{\sin^2\Phi}\right)\right] \end{aligned} \tag{9}$$

where $\vec{L}_i^2$ is the angular momentum operator related to the $\vec{r}_1$ degrees of freedom. Therefore the generalized angular momentum operator is

$$\Lambda^2 = \partial_\Phi^2 + \frac{4}{\tan 2\Phi}\partial_\Phi + \frac{\vec{L}_1^2}{\cos^2\Phi} + \frac{\vec{L}_2^2}{\sin^2\Phi} \tag{10}$$

and the eigenfunctions are given by

$$\mathcal{F}^{\lambda}_{l_1 m_1 l_2 m_2} = f^{\lambda}_{l_1 l_2}(\Phi) Y_{l_1 m_1}(\Omega_1) Y_{l_2 m_2}(\Omega_2) \tag{11}$$

where Y_{lm} are the familiar spherical harmonics and the functions $f^{\lambda}_{l_1 l_2}(\Phi)$ satisfy the equation

$$\left[\partial_\Phi^2 + \frac{4}{\tan 2\Phi}\partial_\Phi + \frac{l_1(l_1+1)}{\cos^2\Phi} + \frac{l_2(l_2+1)}{\sin^2\Phi}\right] f^{\lambda}_{l_1 l_2}(\Phi) = 4(1-\lambda^2) f^{\lambda}_{l_1 l_2}(\Phi) \tag{12}$$

In the following we restrict ourselves to the case $l_1 = l_2 = l$ as we expect that two neutrons outside of a core in a bound state close to zero energy have the same angular momentum given by the properties of the interaction with the core; for instance, if there is a particle-core resonance close to zero energy in some partial wave, that value of the angular momentum is expected to dominate in the structure of a possible bound state when the particle-particle interaction is properly taken into account. Moreover, the antisymetrization of the wave function is easier when $l_1 = l_2$, the hyperangular part having symmetry or antisymmetry for $\Phi \to \pi/2 - \Phi$ depending on the spin and orbital angular momentum part; for singlet spin state of spin one-half fermions and $l_1 = l_2 = 0$ the hyperangular part must be symmetric for $\Phi \to \pi/2 - \Phi$ and for $l_1 = l_2 = 1$ the hyperangular part could be symmetric or antisymmetric depending on the total orbital angular momentum. Due to the absence of the centrifugal barrier, the most favorable case for the existence of bound state is expected to be $l_1 = l_2 = 0$ and we will treat it in more detail.

We note that a change of variable, $z = \sin^2\Phi$, together with a change of function, $f^{\lambda}_{ll}(\Phi) = (\sin\Phi)^l (\cos\Phi)^l w(z)$, transforms the eq. (12) into the following one

$$z(1-z)w''(z) + \left[\left(\frac{3}{2} + l\right) - (3+2l)z\right] w'(z) - \left[(l+1)^2 - \lambda^2\right] w(z) = 0 \tag{13}$$

which is the Gauss equation for the hypergeometric function with the parameters

$$a = l + 1 + \lambda \qquad b = l + 1 - \lambda \qquad c = 3/2 + l$$

Therefore, the general solution of the equation (8) can be written

$$f_{ll}^{\lambda}(\Phi) = (\cos\Phi)^l \Big[C_1 (\sin\Phi)^l {}_2F^1(l+1+\lambda, l+1-\lambda, 3/2+l; (\sin\Phi)^2) \\ C_2 (\sin\Phi)^{-(l+1)} {}_2F^1(1/2+\lambda, 1/2-\lambda, 1/2-l; (\sin\Phi)^2) \Big] \tag{14}$$

Let us note that λ and $-\lambda$ correspond, due to the symmetry property of the hypergeometric function, to the same solution.

When there is no interaction, we have to impose regularity conditions in $\Phi = 0$ and $\Phi = \pi/2$ for the above wave function. The regularity condition in $\Phi = 0$ results into $C_2 = 0$ and, using the formula

$$ {}_2F^1(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F^1(a,b,a+b+1-c;1-z) + \\ + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F^1(c-a,c-b,c-a-b+1;1-z)$$

the regularity condition in $\Phi = \pi/2$ results into $\Gamma(l+1-\lambda) \to \infty$, or $\lambda = l+1+n$ where $n = 0, 1, 2, \ldots$ (for s-wave $\lambda = 1, 2, \ldots$). The general solution given by eq. (14) will reduce to Jacobi polynomials.

The radial equation for R^{λ} which gives the wave function

$$\Psi = f^{\lambda} \tfrac{1}{\rho^{5/2}} R^{\lambda}(\rho)$$

is as follows

$$-\frac{d^2R(\rho)}{d\rho^2} + \frac{4\lambda^2 - 1/4}{\rho^2} R(\rho) = \frac{2mE}{\hbar^2} R(\rho) \tag{15}$$

The above equation cannot have bound states if $\lambda^2 \geq 0$ and this is indeed the case where there is no interaction present.

When the interaction is added, the Schrödinger equation is not separable in hyperspherical coordinates. Our idea is to replace the interactions in the regions given by eqs. (8) by appropriate boundary conditions at $\Phi = 0$, $\Phi = \pi/4$ and also at $\Phi = \pi/2$.

These boundary conditions result into the dependence of eigenvalues of the angular equation and eigenfunctions on ρ, $\lambda(\rho)$ and $f^{\lambda}(\Phi;\rho)$. As our boundary conditions will be energy independent the hyperangular functions, $f^{\lambda}(\Phi;\rho)$, form a complet set for each value of ρ and the wave function can be written in general, up to the angular part,

$$\Psi = \sum\nolimits_{\lambda} f^{\lambda}(\Phi;\rho) \tfrac{1}{\rho^{5/2}} R^{\lambda}(\rho)$$

and the Schrödinger equation will result into a coupled set of equations for the radial functions $R^{\lambda}(\rho)$. We can truncate this infinite system of differential

equations and solve it numerically. The simplest truncation is realized by keeping only the component corresponding to the lowest eigenvalue of λ^2 for each ρ, and the result is

$$-\frac{d^2R(\rho)}{d\rho^2}-2\left(f(\Phi;\rho),\frac{d}{d\rho}f(\Phi;\rho)\right)_\Phi\frac{dR(\rho)}{d\rho}+ \\ +\left[\frac{4\lambda^2-1/4}{\rho^2}-\left(f(\Phi;\rho),\frac{d^2}{d\rho^2}f(\Phi;\rho)\right)_\Phi\right]R(\rho)=\frac{2mE}{\hbar^2}R(\rho) \tag{16}$$

where $(f_1,f_2)_\Phi$ is the scalar product for the angular functions

$$(f_1,f_2)_\Phi=\int_0^{\pi/2}f_1^*f_2\sin^2\Phi\cos^2\Phi\,d\Phi$$

and the hyperangular functions are supposed to be normalized. Eq. (16) reduces to eq. (15) when the angular overlap integrals are neglected; this can be done when the ρ-dependence of f_λ is weak. Therefore, the lowest eigenvalue of the angular equation (12) gives the effective potential in the (hyper)-radial equation (15) (of course, λ depends on ρ).

In what follows we concentrate on the solution of the angular equation for *s*-wave states.

3.1. *s*-WAVE STATES IN THE CORE POTENTIAL

As we expect that the case $l_1 = l_2 = 0$ (*s*-wave states), due to the absence of a centrifugal barrier, is the most favorable to produce bound states when the particle-particle interaction is added we treat this case in detail.

In this particular case, the hypergeometric functions appearing in the general solution given by eq. (14) reduce to combinations of trigonometric functions [20]:

$${}_2F^1(1+\lambda,1-\lambda,3/2;\sin^2\Phi)=\frac{\sin 2\lambda\Phi}{\lambda\sin 2\Phi}$$

$${}_2F^1(1/2+\lambda,1/2-\lambda,1/2;\sin^2\Phi)=\frac{\cos 2\lambda\Phi}{\cos\Phi}$$

Therefore, the general solution, eq. (14), can be written

$$f^\lambda(\Phi)=\frac{1}{\sin 2\Phi}\left(\frac{C_1}{\lambda}\sin 2\lambda\Phi+2C_2\cos 2\lambda\Phi\right) \tag{17}$$

For small Φ the above solution behaves as

$$f\sim C_1+C_2\frac{1}{\Phi}$$

Introducing the scattering length for the particle-core interaction, a_c, (we neglect the possible dependence on the core spin) the behaviour of the solution for small Φ, *i.e.* small r_2, must be

$$1-\frac{a_c}{r_2}=1-\frac{a_c}{\rho}\frac{1}{\Phi}$$

which results into the boundary condition

$$\frac{C_1}{C_2}=-\frac{\rho}{a_c} \tag{18}$$

For Φ close to $\pi/2$ the solution behaves as

$$f\sim(-C_1\cos\pi\lambda+2\lambda C_2\sin\pi\lambda)+\left(\frac{C_1}{2\lambda}\sin\pi\lambda+C_2\cos\pi\lambda\right)\frac{1}{\pi/2-\Phi}$$

and, similar to eq. (18), we must impose

$$\frac{-C_1\cos\pi\lambda+2\lambda C_2\sin\pi\lambda}{\frac{C_1}{2\lambda}\sin\pi\lambda+C_2\cos\pi\lambda}=-\frac{\rho}{a_c} \tag{19}$$

Therefore, if we neglect the particle-particle interaction, eqs. (18) and (19) result into an equation for the eigenvalues of the hyperangular operator which include the particle-core interaction

$$\tan\pi\lambda=-4\lambda\frac{\rho/a_c}{4\lambda^2-(\rho/a_c)^2} \tag{20}$$

The above equation allows us to obtain the eigenvalues λ, *i.e.* the effective potential in the hyperradial equation (15) when the particle-core interaction is treated in the scattering length approximation. Before we discuss the solution of this equation, let us remark that it is equivalent with the following two equations

$$\tan\frac{\pi\lambda}{2}=-\frac{\rho}{2a_c}\frac{1}{\lambda} \tag{21}$$

$$\tan\frac{\pi\lambda}{2}=\frac{2a_c}{\rho}\lambda \tag{22}$$

The equivalence can be proved using the identity

$$\tan x=\frac{2\tan x/2}{1-\tan^2 x/2}$$

The solutions (17), obtained from the equation (21) are symmetric for $\Phi\to\pi/2-\Phi$, so they are appropriate to describe two spin one-half particles in

the singlet spin state. These are the solutions that are useful to describe the wave function of the two halo neutrons in a Borromean nucleus. The other equation results into antisymmetric solutions and we do not study them; they vanish at $\Phi = \pi/4$ and a short-range particle-particle interaction (which in hyperspherical coordinates acts around $\Phi = \pi/4$) does not modify them.

Before we include the particle-particle interaction in the above formalism, let us study the solutions of eq. (21). When $a_c \gg 0$ (there is a loosely bound state of a particle in the core field) eq. (21) has two purely imaginary solutions and an infinite number of real solutions. For $\rho \ll a_c$ we can obtain in the leading order in ρ/a_c the lowest eigenvalue λ^2

$$\lambda_0^2 = -\frac{1}{\pi}\frac{\rho}{a_c} \tag{23}$$

which results into an attractive $1/\rho$ potential into the radial equation (15); this attractive potential reflects the existence of the particle-core bound state.

When $a_c \ll 0$ (there is a virtual state of a particle in the core field) eq. (21) has no imaginary solution. For $\rho \ll |a_c|$ we can obtain in the leading order in ρ/a_c the lowest eigenvalue λ^2

$$\lambda_0^2 = \frac{1}{\pi}\frac{\rho}{|a_c|} \tag{24}$$

which results into an repulsive $1/\rho$ potential into the radial equation (15).

To include the interaction between the two particles we have to calculate the behaviour of the solution in the interaction region $\Phi \in [\pi/4 - r_0/(\sqrt{2}\rho), \pi/4 + r_0/(\sqrt{2}\rho)]$. For the problem of two neutrons in the singlet spin state outside of a core it is reasonable to assume that the solution in this region is of the form $A(\rho)g_0^i(r_{12})$ (the relative state at zero energy is dominated by the *s*-wave) where $r_{12} = |\vec{r}_1 - \vec{r}_2| = \rho\sqrt{1 - \sin 2\Phi \cos\theta_{12}}$ and g_0^i is the zero-energy solution in the potential $v(r_{12})$; this solution around $\Phi = \pi/4$ behaves as

$$g_0^i \sim \frac{r_{12} - a}{r_{12}} = 1 - \frac{a}{\rho}\frac{1}{\sqrt{1 - \sin 2\Phi \cos\theta_{12}}}$$

(θ_{12} is the angle between $\vec{r}_1$ and $\vec{r}_2$ and a is the particle-particle scattering length). The above expression can be expanded in spherical harmonics, and we get for $\Phi < \pi/4$ $(r_1 > r_2)$

$$g_0^i \sim 1 - \frac{a}{\rho}\frac{4\pi}{\cos\Phi}\sum_l \frac{(\tan\Phi)^l}{2l+1}\sum_m Y_{lm}^*(\theta_2\varphi_2)Y_{lm}(\theta_1\varphi_1) \tag{25}$$

We can restrict ourselves for $\Phi \in [0, \pi/4]$, the other interval being obtained from symmetry considerations. The quantity of interest in our case is the component in the subspace of $l_1 = l_2 = 0$ states, *i.e.*

$$G(\Phi;\rho) \sim \int d\Omega_1 \int d\Omega_2 g_0^i \sim 1 - \frac{a}{\rho}\frac{1}{\cos\Phi} \tag{26}$$

(the ρ-dependent overall factors are not necessary). The solution given by eq. (17) must have the behaviour of $G(\Phi;\rho)$ for Φ approaching $\pi/4$ from below. This fact could be expressed as the identity of the logarithmic derivatives of f and G. We obtain

$$\left.\frac{\partial_\Phi f(\Phi;\rho)}{f(\Phi;\rho)}\right|_{\Phi=\pi/4} = 1\left(1 - \frac{1}{\sqrt{2}}\frac{\rho}{a}\right) \tag{27}$$

The logarithmic derivative in eq. (27) is calculated using the solution (17). From eqs. (18) and (27) we obtain

$$\tan\frac{\pi\lambda}{2} = -\frac{1}{2\lambda}\left[1 + \left(1 - \frac{1}{\sqrt{2}}\frac{\rho}{a}\right)\frac{\rho}{a_c}\right]\left[1 + \left(-\frac{1}{\sqrt{2}} - \frac{1}{4\lambda^2}\frac{a}{a_c}\right)\frac{\rho}{a}\right] \tag{28}$$

Eq. (28) is the main result of this section; it has an infinite number of real solutions, and, at least for sufficiently small ρ, two imaginary ones $\lambda = \pm i\bar{\lambda}$, $\bar{\lambda} \in R^+$. These imaginary solutions are of interest to our case as they result into the minimum λ^2 ($\lambda^2 < 0$, all the real solutions give $\lambda^2 > 0$ and the effective potential in eq. (15) will be repulsive). In the zeroth order in ρ/a and ρ/a_c, *i.e.* $\rho \ll |a|$ and $\rho \ll |a_c|$, we have the equation

$$\tanh\frac{\pi\bar{\lambda}_0}{2} = \frac{1}{2\bar{\lambda}_0} \tag{29}$$

which has a solution of order unity; it results into an attractive dipolar effective potential

$$V_{eff}(\rho) = -\frac{1/4 + 4\bar{\lambda}_0^2}{\rho^2} \tag{30}$$

in eq. (11) which has an infinite number of bound states (Efimov effect). Nevertheless, this behaviour is valid only for $\rho \ll |a|, |a_c|$. The correction to the solution of eq. (29) limits the number of Efimov states. Of course, the limit of eq. (28) for $a \to 0$ is exactly eq. (21) as it must be; $a \to 0$ corresponds to a vanishing particle-particle interaction.

In Fig. 1 we present an example of the effective potential obtained by solving numerically eq. (28). The quantities represented are the effective

hyperradial potential in MeV, $V(\rho) = \hbar^2/(2m)V_{eff}(\rho)$, where $V_{eff}(\rho)$ is given by eq. (30), and the hyperradius is in fm. We observe that the potential goes for small ρ to the first order approximation, eq. (29), and for large ρ to the asymptotic value $-1/(4\rho^2)$, which does not support bound states. From the numerical investigation of the solutions of eq. (28) we conclude that the most favourable case for the appearance of a spatially extended bound state is a large negative particle-core scattering length.

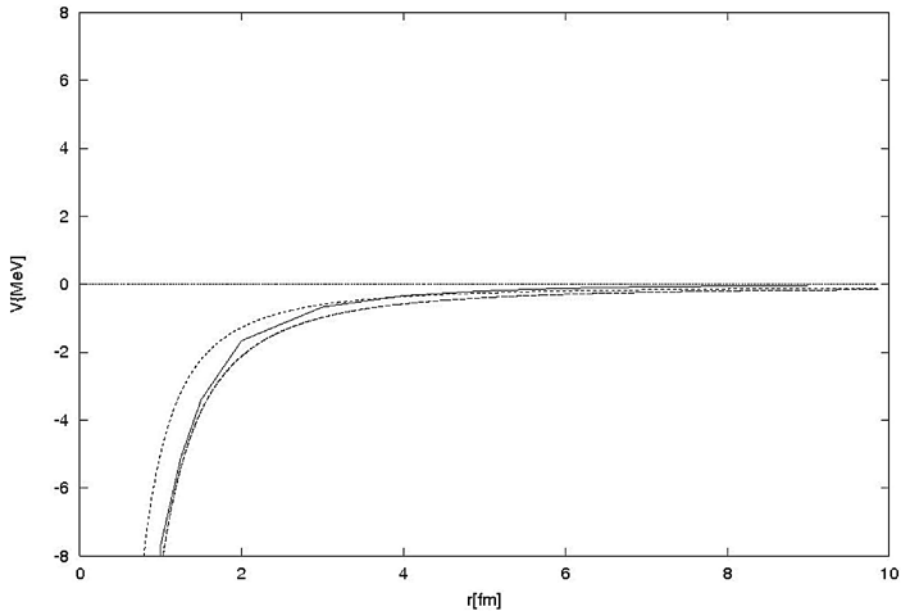

Fig. 1 – Effective hyperradial potential for a = –20 fm and a_c = 25 fm – solid line. Dashed line – the zeroth order approximation and dotted line – asymptotic value.

In the next section we will present an other possibility to describe a Borromean nucleus [16].

4. MOLECULAR DESCRIPTION

Let us consider a system formed by a core and a neutron in a virtual state. Now consider that another neutron approaches this (virtual-state) core-neutron system. The additional neutron and the core represent another virtual-state system. Also the two-neutron system forms a virtual state because the *n*-*n* scattering length is negative. It is then expected that the additional neutron will reinforce the attractive potential. Thus the core and the two neutrons attract each other more strongly, which could result in a bound system.

We will treat this problem as a particle "scattering" on two "centers" (either a neutron scattering on a system of core plus other neutron, or the core scattering on a two-neutron system). In particular, we consider the scattering of a very slow particle, the de Broglie wave length of which overlaps with both the scatterers. Secondary spherical waves are radiated by both scattering centers. If a bound three-body state arises, below threshold they become two evanescent waves.

The two-scatterers approach has been used initially for studying the interaction of pions [21] and kaons [22] with deuterium. Thereafter, the method was also used in studies of collisions of negative ions with atoms [23]. The same approach was used recently in the description of threshold resonances in double well scattering [24, 25]. We use it here for describing a Borromean nucleus in terms of the Boundary Condition Method. The coordinates for describing the system are represented in Fig. 2, where $R = |\vec{r}_1 - \vec{r}_2|$ is the distance between the two scattering centers (1, 2) and $\rho_i = |\vec{r} - \vec{r}_i|$, $i = 1, 2$ are the distances between the scattered particle and the centers.

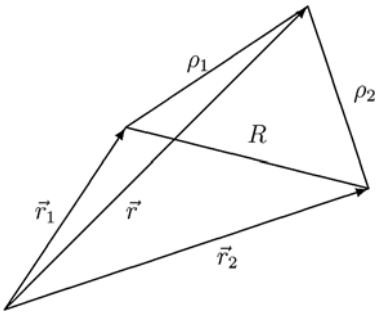

Fig. 2 – The three particle system and the coordinates.

We will consider first the problem of the third particle moving in the field of the other two particles considered fixed at a relative distance *R*. As an ansatz for the wavefunction we use a linear combination of wavefunctions of the scattered particle relative to the two scattering centers. A similar trial wave function is used in a method called Linear Combination of Atomic Orbitals (LCAO), an approximate method in Molecular Physics [26], where the molecular orbital (MO) is represented as sum of two atomic orbitals. This method has also been used in nuclear physics (called there LCNO), *e.g.*, in the scattering of two nearly identical nuclei $^{12}C + ^{13}C$. We thus use the ansatz

$$\Psi = C_1 \exp(-\alpha\rho_1)/\rho_1 + C_2 \exp(-\alpha\rho_2)/\rho_2 \tag{31}$$

It should be a good representation if the two wave functions do not overlap strongly.

One introduces boundary conditions corresponding to the non-overlapping scattering centers [27]. For the scattering of a particle on a system of two particles (1 and 2) they are

$$d/d\rho_1(\rho_1\Psi)|_{\rho_1=0} = -(1/a_1)(\rho_1\Psi)|_{\rho_1=0} \tag{32}$$

$$d/d\rho_2(\rho_2\Psi)|_{\rho_2=0} = -(1/a_2)(\rho_2\Psi)|_{\rho_2=0}, \tag{33}$$

where the a_i are the corresponding scattering lengths.

With eqs. (32, 33) one replaces the boundary conditions of the finite wells by those for zero-range potentials. This is only justified if the influence of the second well can be neglected at the boundary of the first, *i.e.*, if the ranges of the potential wells are small relative to the average distance between the scatterers. This should be well satisfied for the scattering of a neutron on the core-neutron system, but less well for the scattering of the core on the $n - n$ system.

By inserting the trial wave function Ψ, eq. (31), into the boundary conditions one obtains a set of equations for the constants C_1 and C_2,

$$(\alpha - 1/a_1)\cdot C_1 - \exp(-\alpha R)/R \cdot C_2 = 0 \tag{34}$$

$$-\exp(-\alpha R)/R \cdot C_1 + (\alpha - 1/a_2)\cdot C_2 = 0 \tag{35}$$

which depends parametrically on $R = |\vec{r}_1 - \vec{r}_2|$ the distance between the two scattering centers. The solution of this set of two equations implies a determinant condition for the decay constant α

$$(\alpha - 1/a_1)(\alpha - 1/a_2) - \exp(-2\alpha R)/R^2 = 0 \tag{36}$$

In the scattering case the corresponding trial wave function consists of the incoming plane wave and of two outgoing waves with amplitudes $A_1(\vec{k})$ and $A_2(\vec{k})$ radiated from the two scattering centers

$$\Psi = e^{i\vec{k}\vec{r}} + A_1(\vec{k})\exp(ik\rho_1)/\rho_1 + A_2(\vec{k})\exp(ik\rho_2)/\rho_2. \tag{37}$$

From the boundary conditions one similarly obtains equations for the scattering amplitudes A_1 and A_2,

$$A_1(ik + 1/a_1) + A_2\exp(ikR)/R = -\exp(i\vec{k}\vec{R}/2). \tag{38}$$

$$A_1\exp(ikR)/R + A_2(ik + 1/a_2) = -\exp(-i\vec{k}\vec{R}/2). \tag{39}$$

Observe that by setting $k \to i\kappa$ one obtains the bound state problem of eqs. (34, 35) except for the right-hand terms, originating from the incident wave on the two scatterers.

Using eq. (37) we can define an effective scattering amplitude on the system of the two interaction centers as:

$$A_{eff}(k) = A_1(k) + A_2(k) \tag{40}$$

The effective scattering length will be defined as $a_{eff} = -A_{eff}(0)$, see eq. (37); which results into

$$a_{eff}(k) = \frac{a_1\left(1-\frac{a_2}{R}\right)+a_2\left(1-\frac{a_1}{R}\right)}{1-\frac{a_1 a_2}{R^2}} = \frac{a_1+a_2-\frac{2a_1a_2}{R}}{1-\frac{a_1a_2}{R^2}} \tag{41}$$

the meaning of the reduced or effective scattering length.

In the following we discuss the various cases for the dependence of scattering length a_{eff} on the intercenter distance R and the scattering lengths of the two channels. The above equations are taken in the limit $\kappa = 0$, but the same qualitative results are obtained for non-zero $\kappa < 1/|a_2|$.

The effective scattering length a_{eff} is negative provided the scattering lengths of the two channels a_1 and a_2 are negative and also the condition $R^2 > |a_1| \cdot |a_2|$ is fulfilled. The virtual state is thus strengthened; the effective scattering length varies with R from $-|a_1|-|a_2|$ to $-\infty$. For the transition condition $R^2 = |a_1||a_2|$ the effective scattering length shifts to $+\infty$; the situation of a bound state is realized.

The effective scattering length a_{eff} remains attractive ($a_{eff} < 0$) even for repulsive scattering length of the second center, $a_2 = +|a_2|$, if $|a_1| > |a_2|$ and $R > 2|a_1||a_2|/(|a_1|-|a_2|)$. The virtual state is transformed into a bound one only for small separation of the two centers, $R < 2|a_1||a_2|/(|a_1|-|a_2|)$. When $|a_1| > |a_2|$ the effective scattering length is allways positive.

The above discussion may be given also in terms of Fermi pseudopotentials which are proportional to the scattering length. Thus a potential well which just has bound a neutron will repel another neutron. This is the physical meaning of Fermi Pseudopotential; it is attractive if no bound neutron exists in the potential well and it becomes repulsive if a neutron was just bound [17]. We can extend this reasoning to adding another neutron to a Borromean nucleus, *i.e.* to a four-body system. It follows that a Borromean nucleus cannot bind another neutron, *e.g.* ^{12}Li is not bound.

4.1. APPLICATION TO BORROMEAN NUCLEI

In general we can use directly the results obtained above only when the masses of the two centers are larger than the third particle mass. The Borromean Nucleus is a three-body nuclear system consisting of a core and two neutrons. In

this case the internuclear distance R is that between the two neutrons. Perhaps it is questionable if in this situation R can be treated as a parameter, because of the small neutron masses. However, in the framework of the Born-Oppenheimer Approximation, one can show that the terms neglected are important mainly at small intercenter separation where they are strongly repulsive such as to prevent the collapse at $R = 0$.

We may, however, permute the particle and scatterers. Another partition is core-neutron plus the additional neutron. The two scattering lengths are those of neutron-core and of neutron-neutron, and the internuclear distance R is that of neutron-core. The core and the neutron constitute a virtual state of the two-body system. The other neutron interacts attractively with both constituents of the two-body system. One obtains a bound state of the three-body system, provided the attractive interaction is strong enough. In this approach (partition) one scatterer (the ^{9}Li core) can be considered as fixed; the other scatterer (neutron) has a larger indeterminacy in position. However, both position uncertainties are smaller than in the other partition (^{9}Li scattered on the two neutrons).

We start from the established fact that a Borromean Nucleus exhibits weakly bound states, spatially extended. Consider a near zero energy state, *i.e.* $\alpha \to 0$. In this case one obtains $\exp(-2\alpha R)/R^2 \simeq 1/R^2 - 2\alpha/R + 2\alpha^2$, $(|\alpha R| < 1)$, and the determinant equation (36) becomes

$$\alpha^2 + (1/a_1 + 1/a_2 - 2/R)\alpha + (1/R^2 - 1/a_1 a_2) = 0, \tag{42}$$

resulting for α in

$$2\alpha = -(1/a_1 + 1/a_2 - 2/R) \pm \sqrt{(1/a_1 + 1/a_2)^2 + 4/a_1 a_2 - 4(1/a_1 + 1/a_2)/R}\,. \tag{43}$$

Alternatively one can express $1/R$ in terms of α as

$$1/R = \alpha \pm \sqrt{1/a_1 a_2 - (1/a_1 + 1/a_2)\alpha}\,. \tag{44}$$

For negative scattering lengths, $a_1 < 0$ and $a_2 < 0$, both parameters, α and R, are real.

Let us consider a model case with $|a_1| = |a_2| = a$. If $a_1 = a_2 = a > 0$ then α is complex for $a > R$. This pole, in the eighth octant of the k-plane, corresponds to a resonance. If $a_1 = -a_2 = a$ (or $-a_1 = a_2 = a$), (one bound- and one virtual-state), the solution is also complex $\alpha = 1/R \pm i/a$. One obtains a virtual state (seventh octant) provided $a < R$. The case of two repulsive potentials and also that of one repulsive – one attractive potential result into no bound state solutions (otherwise α would have to be real).

Consider now the case of two virtual states $a_1 = a_2 = -a$, $(a > 0)$. One obtains two real solutions

$$\alpha = 1/a + 1/R \pm \sqrt{2/a(1/a + 1/R)} \tag{45}$$

The first solution (+) should be rejected because it is not compatible with the physical constraint, $\alpha R < 1$, since we assumed that the scattering length should be much larger than the intercenter distance, such that the wave overlaps simultaneously with both scatterers. The second solution (–) has to be accepted because always $1 + 2a/R > 0$. It corresponds to a real negative energy $E = -\hbar^2\alpha^2/2m$. A positive α corresponds to a bound state. In the limit $R \to a$, (or $R^2 \to a_1 \cdot a_2$), a bound state of zero energy $\alpha \to 0$ appears; the spatial dimension becomes as large as the scattering length. Observe that α (energy) and R (dimension) move in opposite directions; a smaller energy implies a larger spatial extension.

The evaluation of the energy α of the Borromean nucleus needs the internuclear distance R and the two scattering lengths, a_1 and a_2. One may adopt also the inverse viewpoint, to calculate the internuclear distances R_{nn} (neutron-neutron) and R_{nC} (neutron-core) provided we know the energy and the two scattering lengths. The neutron n is bound by the core C with energy $E_{nC} = -\hbar^2\alpha^2/2m_{nC}$ as well as by the other neutron $\tilde{n}$ with energy $E_{n\tilde{n}} = -\hbar^2\alpha^2/2m_{n\tilde{n}}$.

The neutron-neutron scattering length is $a_{nn} = -18.5$ fm [13], and the n – ^{9}Li scattering length is $a_{nC} \leq -20$ fm [28]. The $2n$ separation energy in ^{11}Li is $E_0 = -300$ keV; the corresponding α parameter is $\alpha = 5 \cdot 10^{-2}$ fm^{-1}. According to other reported data [29, 30], the energy is $E_0 = -200$ keV which corresponds to $\alpha = 4 \cdot 10^{-2}$ fm^{-1}. One then obtains from the determinant equation (11) two values, $R_{nn} = 7.3$ fm and 8.2 fm, for the two separation energies, respectively. For the neutron and core as scatterers one obtains $R_{nC} = 7.5$ fm and 8.4 fm, respectively. One verifies, in both cases, that $\alpha R \simeq 0.4$, thus neglecting higher order terms is justified.

Because of the similar scattering lengths in all two-body subsystems, the distances between the three particles are seen to be of the same magnitude. The physical conclusion is that the two neutrons are located far-away from each other, *i.e.* they do not form of a di-neutron. Indeed, the deuteron size for both bound and virtual states is 2 to 2.5 fm, while from the above estimates the *n*-*n* distance in the halo is between ~ 7.3 fm and ~ 8.2 fm. It is interesting to compare the above results for the halo radius to those reported in the literature [31, 32]. The reported distance of the ^{9}Li core to the center of mass of the two neutrons is ~ 8 fm. According to the present calculations the distance between the core and the $2n$ center of mass is $\sqrt{R_{nC}^2 - R_{nn}^2/4}$, *i.e.* it is 6.5 fm or 7.3 fm. The last value is near the experimentally reported one.

The method could be used for other Borromean nuclei where the loosely bound neutrons are expected to be in an *s*-state, like ^{14}Be [33, 34]. The reported energy and neutron-core ^{12}Be scattering length are $E_0 = -1.34$ MeV and $a_{nC} \leq -10$ fm, which results in $\alpha = 1.0 \cdot 10^{-1}$ fm^{-1}, $R_{nn} = 4.6$ fm, and $R_{nC} = 4.1$ fm. The distance between the core and the $2n$ center of mass $\sqrt{R_{nC}^2 - R_{nn}^2/4}$ is 3.4 fm. The ^{14}Be Borromean Nucleus is more bound and less extended spatially than ^{11}Li. Comparing ^{11}Li and ^{14}Be, one can deduce that a Borromean Nucleus is more extended spatially the more it is weakly bound.

5. CONCLUSIONS

The results obtained allow to conclude that one can use the Boundary Condition Model (Fermi Pseudopotential Model) for describing the weakly bound states in a three-body system in a hyperspherical approach or in molecular approach. The mutual influence of the two-body subsystems results into the transition from a virtual state to a bound state one. The methods presented relate the properties of the three-body bound states (spatial extension and energy) to the two-body interactions (scattering lengths). To illustrate the method we have compared our results with the data available for the Borromean Nucleus ^{11}Li and obtain reasonable qualitative agreement.

The physical merit of the Boundary Condition Model is to highlight the role of virtual states in producing a Borromean nucleus. The present approaches to Borromean Nuclei, either in hyperspherical coordinates or using molecular ansatz, were restricted to s- waves. The generalization of the method to non-zero orbital angular momenta is a problem of current interest, see Ref. [35].

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