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γ-RIGID SOLUTION OF THE BOHR HAMILTONIAN FOR $\gamma = 30°$ COMPARED TO THE E(5) CRITICAL POINT SYMMETRY

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An exact, parameter free (up to overall scale factors) solution of the Bohr Hamiltonian is obtained by freezing the γ degree of freedom at $\gamma = 30°$. A structural similarity between the spectrum of the ground state band and the eigenvalue equation of the second order Casimir operator of the Euclidean algebra E(4), is identified, while, comparison of spectra and B(E2) transition rates reveals a good agreement with the E(5) critical point symmetry. Numerical results are also inreasonable agreement with experimental data in the Xe region around $A = 130$.

1. INTRODUCTION

The concept of shape phase transitions in atomic nuclei has recently been codified in the formalism of critical point symmetries, after the introduction of the E(5) [1] and X(5) [2] models by F. Iachello, followed by a still increasing amount of experimental evidence [3–6].

The formulation of these models is based on the original Bohr equation [7], by considering an infinite potential well in the β variable while making different assumptions about the γ degree of freedom, depending on the transition under consideration. Thus, in E(5) [1] the potential is γ-independent and the exact solution that is obtained describes the vibrational [U(5)] to γ-unstable [SO(6)] phase transition, whereas in X(5) [2] $\gamma \approx 0°$ and the approximate solution that is obtained describes the vibrational [U(5)] to prolate deformed [SU(3)] transition point. In the same context, another approximate solution with $\gamma \approx 30°$ called Z(5) [8] has been proposed, for the description of the prolate to oblate phase transition.

The work presented here is an exact solution of the Bohr Hamiltonian, called Z(4) [9], that is obtained by fixing γ at $\gamma = 30°$, following the Davydov

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and Chaban approach [10]. By fixing the γ degree of freedom, the dimensionality of the problem is reduced from five (the β and γ collective coordinates and the Euler angles) to four (β and the Euler angles), unlike all the other models where all five collective variables are taken into account.

A similarity in energies and B(E2) transition rates between Z(4) and E(5) is observed, while a coincidence of the spectrum of the ground state band of the Z(4) model with the spectrum of the second order Casimir operator of the Euclidean algebra E(4) gives a first clue on the relation of critical point symmetries to Lie symmetries. Numerical results lie close to experimental data from Xe around $A = 130$.

2. THE Z(4) MODEL

In the model of Davydov and Chaban [10] it is assumed that the nucleus is rigid with respect to γ-vibrations. Then the Hamiltonian depends on four variables (β, θ_i) and has the form [10]

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2\pi}{3}k\right)}\right] + U(\beta), \tag{1}$$

where β and γ are the usual collective coordinates [7], while Q_k ($k = 1, 2, 3$) are the components of angular momentum and B is the mass parameter. In this Hamiltonian γ is treated as a parameter and not as a variable. The kinetic energy term of Eq. (1) is different from the one appearing in the E(5) and X(5) models, because of the different number of degrees of freedom treated in each case (four in the former case, five in the latter). Introducing [1] reduced energies $\epsilon = (2B/\hbar^2)E$ and reduced potentials $u = (2B/\hbar^2)U$, and considering a wave function of the form $\Psi(\beta, \theta_i) = \phi(\beta)\psi(\theta_i)$, where θ_i ($i = 1, 2, 3$) are the Euler angles, separation of variables leads to two equations

$$\left[\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta} - \frac{\lambda}{\beta^2} + (\epsilon - u(\beta))\right]\phi(\beta) = 0, \tag{2}$$

$$\left[\frac{1}{4}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2\pi}{3}k\right)} - \lambda\right]\psi(\theta_i) = 0. \tag{3}$$

In the case of $\gamma = \pi/6$, the last equation takes the form

$$\left[\frac{1}{4}(Q_1^2 + 4Q_2^2 + 4Q_3^2) - \lambda\right]\psi(\theta_i) = 0. \tag{4}$$

This equation has been solved by Meyer-ter-Vehn [11], the eigenfunctions being

$$\psi(\theta_i)=\psi^L_{\mu,\alpha}(\theta_i)=\sqrt{\frac{2L+1}{16\pi^2(1+\delta_{\alpha,0})}}\left[\mathcal{D}^{(L)}_{\mu,\alpha}(\theta_i)+(-1)^L\mathcal{D}^{(L)}_{\mu,-\alpha}(\theta_i)\right] \tag{5}$$

with

$$\lambda=\lambda_{L,\alpha}=L(L+1)-\frac{3}{4}\alpha^2, \tag{6}$$

where $\mathcal{D}(\theta_i)$ denote Wigner functions of the Euler angles, L are the eigenvalues of angular momentum, while μ and α are the eigenvalues of the projections of angular momentum on the laboratory fixed $\hat{z}$-axis and the body-fixed $\hat{x}'$-axis respectively. α has to be an even integer [11].

Instead of the projection α of the angular momentum on the $\hat{x}'$-axis, it is customary to introduce the wobbling quantum number [11, 12] $n_w=L-\alpha$, which labels a series of bands with $L=n_w, n_w+2, n_w+4, \ldots$ (with $n_w>0$) next to the ground state band (with $n_w=0$) [11].

The "radial" Eq. (2) is exactly soluble in the case of an infinite square well potential ($u(\beta)=0$ for $\beta\le\beta_W$, $u(\beta)=\infty$ for $\beta>\beta_W$).

$$u(\beta)=\begin{cases}0 & \text{if } \beta\le\beta_W\\ \infty & \text{for } \beta>\beta_W\end{cases}, \tag{7}$$

Using the transformation $\phi(\beta)=\beta^{-1}f(\beta)$, Eq. (2) becomes a Bessel equation

$$\left[\frac{\partial^2}{\partial\beta^2}+\frac{1}{\beta}\frac{\partial}{\partial\beta}+(\epsilon-\frac{\nu^2}{\beta^2})\right]f(\beta)=0, \tag{8}$$

with

$$\nu=\sqrt{\lambda+1}=\sqrt{L(L+1)-\frac{3}{4}\alpha^2+1} \tag{9}$$

$$=\frac{\sqrt{L(L+4)+3n_w(2L-n_w)+4}}{2}. \tag{10}$$

Then the boundary condition $f(\beta_W)=0$ determines the spectrum,

$$\epsilon_{\beta;s,\nu}=\epsilon_{\beta;s,n_w,L}=(k_{s,\nu})^2, \qquad k_{s,\nu}=\frac{x_{s,\nu}}{\beta_W}, \tag{11}$$

where $x_{s,\nu}$ is the sth zero of the Bessel function $J_\nu(z)$. The eigenfunctions are

$$\phi(\beta)=\phi_{s,\nu}(\beta)=\phi_{s,n_w,L}(\beta)=\frac{1}{\sqrt{c}}\beta^{-1}J_\nu(k_{s,\nu}\beta), \quad c=\frac{\beta_W^2}{2}J^2_{\nu+1}(x_{s,\nu}) \tag{12}$$

where the normalization constant c is determined from the condition $\int_0^{\beta_w} \beta^3\phi^2(\beta)d\beta = 1$. The notation for the roots has been kept the same as in Ref. [2], while for the energies the notation $E_{s,n_w,L}$ will be used. The ground state band corresponds to $s = 1$, $n_w = 0$. This model will be called the Z(4) model.

The calculation of B(E2)s proceeds as in Ref. [8], the only difference being that the integrals over β have the form

$$I_\beta(s_i, L_i, \alpha_i; s_f, L_f, \alpha_f) = \int_0^{\beta_w} \beta\phi_{s_i,\nu_i}(\beta)\phi_{s_f,\nu_f}(\beta)\beta^3 d\beta, \tag{13}$$

since the volume element in the present case corresponds to four dimensions instead of five.

3. RELATION OF THE GROUND STATE BAND OF Z(4) TO E(4)

The ground state band of the Z(4) model is related to the second order Casimir operator of E(4), the Euclidean group in four dimensions. In order to see this, one can consider in general the Euclidean algebra in n dimensions, E(n), which is the semidirect sum [13] of the algebra T_n of translations in n dimensions, generated by the momenta

$$P_j = -i\frac{\partial}{\partial x_j}, \tag{14}$$

and the SO(n) algebra of rotations in n dimensions, generated by the angular momenta

$$L_{jk} = -i\left(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \right), \tag{15}$$

symbolically written as E(n) = $T_n \oplus_s$ SO(n) [14]. The generators of E(n) satisfy the commutation relations

$$[P_i, P_j] = 0, \qquad [P_i, L_{jk}] = i(\delta_{ik}P_j - \delta_{ij}P_k), \tag{16}$$

$$[L_{ij}, L_{kl}] = i(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il}). \tag{17}$$

From these commutation relations one can see that the square of the total momentum, P^2, is a second order Casimir operator of the algebra, while the eigenfunctions of this operator satisfy the equation

$$\left(-\frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r} + \frac{\omega(\omega+n-2)}{r^2} \right)F(r) = k^2F(r), \tag{18}$$

in the left hand side of which the eigenvalues of the Casimir operator of SO(n), $\omega(\omega+n-2)$ appear [15]. Putting

$$F(r)=r^{(2-n)/2}f(r), \tag{19}$$

and

$$\nu=\omega+\frac{n-2}{2}, \tag{20}$$

Eq. (18) is brought into the form

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+k^2-\frac{\nu^2}{r^2}\right)f(r)=0, \tag{21}$$

the eigenfunctions of which are the Bessel functions $f(r)=J_\nu(kr)$ [16]. The similarity between Eqs. (21) and (8) is clear.

The ground state band of Z(4) is characterized by $n_w=0$, which means that $\alpha=L$. Then Eq. (9) leads to $\nu=L/2+1$, while Eq. (20) in the case of E(4) gives $\nu=\omega+1$. Then the two results coincide for $L=2\omega$, *i.e.* for even values of *L*. One can easily see that this coincidence occurs only in four dimensions.

4. NUMERICAL RESULTS AND COMPARISONS TO E(5) AND EXPERIMENT

The lowest bands of the Z(4) model are given in Table 1. The notation L_{s,n_w} is used. All levels are measured from the ground state, $0_{1,0}$, and are

Table 1

Energy levels of the Z(4) model, measured from the $L_{s,n_w}=0_{1,0}$ ground state and normalized to the $2_{1,0}$ lowest excited state

s, n_w L	1,0	1,2	2,0	 L	1,1
0	0.000		2.954		
2	1.000	1.766	4.804	3	2.445
4	2.226	4.051	6.893	5	4.239
6	3.669	6.357	9.215	7	6.188
8	5.324	8.788	11.765	9	8.316
10	7.188	11.378	14.538	11	10.630
12	9.256	14.139	17.531	13	13.135
14	11.526	17.079	20.742	15	15.831
16	13.996	20.202	24.167	17	18.719
18	16.665	23.509	27.805	19	21.799
20	19.530	27.003	31.653	21	25.071

normalized to the first excited state, $2_{1,0}$. The ground state band is characterized by $s = 1$, $n_w = 0$, while the even and the odd levels of the γ_1-band are characterized by $s = 1$, $n_w = 2$, and $s = 1$, $n_w = 1$ respectively, and the β_1-band is characterized by $s = 2$, $n_w = 0$. These bands are also shown in Fig. 1, labelled by (s, n_w).

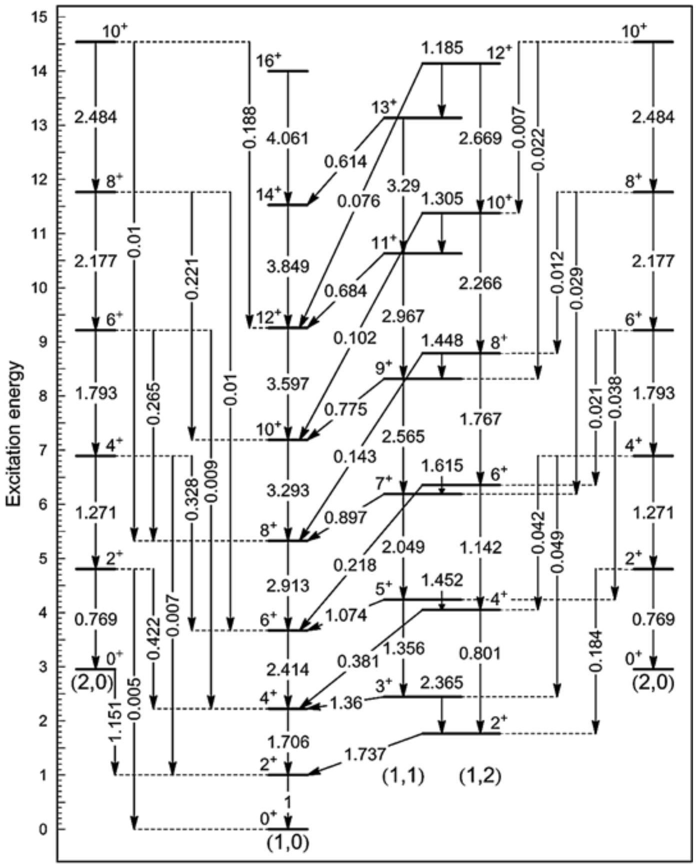

Fig. 1 – Intraband and interband B(E2) transition rates in the Z(4) model, normalized to the B(E2; $2_{1,0} \to 0_{1,0}$) rate. Bands are labelled by (s, n_w), their levels being normalized to $2_{1,0}$. The (2, 0) band is shown both at the left and at the right end of the figure for drawing purposes.

Both intraband and interband B(E2) transition rates, normalized to the one between the two lowest states, B(E2; $2_{1,0} \to 0_{1,0}$) are given in Fig. 1. The similarity between the spectra and B(E2) values of Z(4) and E(5), for which extensive numerical results can be found in Ref. [17], can be seen in Figs. 2(a) and 2(b), where the spectra of the ground state band and the β_1 band, as well as

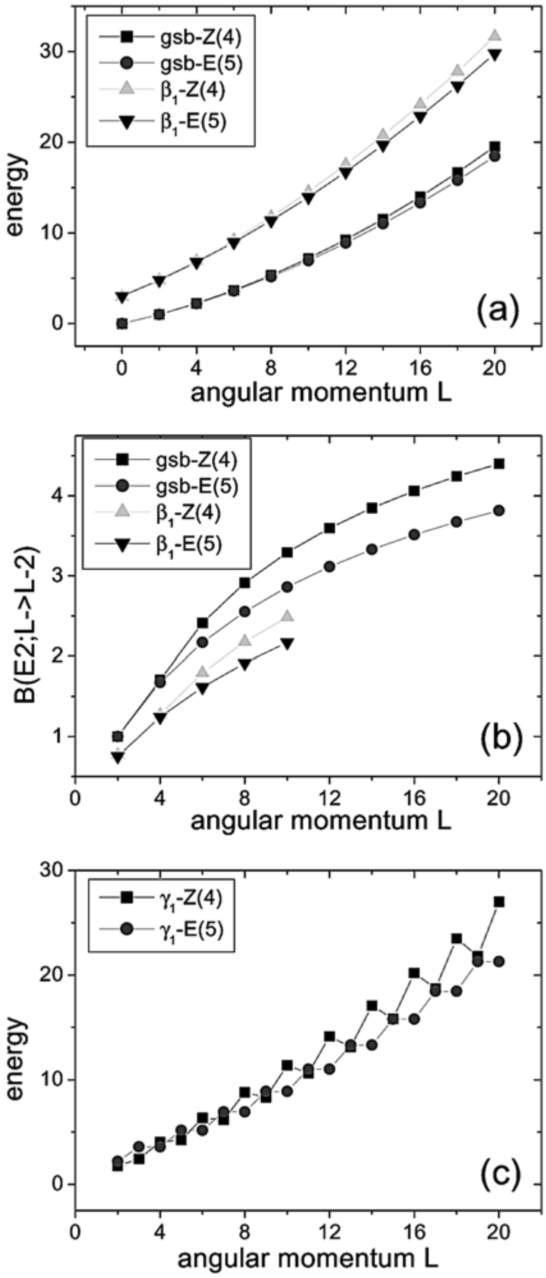

Fig. 2 – (a) Ground state band $[(s, n_w) = (1,0)]$ and first excited band $[(s, n_w) = (2,0)]$ of Z(4) (labeled as β_1-band) compared to the corresponding bands of E(5) [1, 17]. In each model all levels are normalized to the 2_1^+ state. (b) Intraband B(E2) transition rates within the same bands of Z(4) compared to the corresponding B(E2) rates of E(5). In each model all rates are normalized to the $2_1^+ \to 0_1^+$ rate. (c) The lowest "$K = 2$ band" of Z(4) [formed out of the (s, n_w) bands (1,2) and (1,1), labeled as γ_1], compared to the corresponding band of E(5).

their intraband B(E2)s are given. One can easily check that the similarity extends to interband transitions between these bands as well, for which the selection rules in the two models are the same. The main difference between Z(4) and E(5) appears, as expected, in the γ_1 band, the spectrum of which is shown in Fig. 2(c).

The predictions of the two models for the odd levels practically coincide, while the predictions for the even levels differ, since in the E(5) model the levels are exactly paired as (3,4), (5,6), (7,8), …, as imposed by the underlying $SO(5) \supset SO(3)$ symmetry [1, 17], while in the Z(4) model the levels are approximately paired as (4,5), (6,7), (8,9), …, which is a hallmark of rigid triaxial models [18]. The latter behaviour is never materialized fully, but situations intermediate between the two limits occur in the Ru-Pd, Xe-Ba (below $N = 82$), and Os-Pt regions.

Predictions of the Z(4) model are compared to existing experimental data for ^{128}Xe [19], ^{130}Xe [20], and ^{132}Xe [21] in Fig. 3. The reasonable agreement observed is in no contradiction with the characterization of these nuclei as O(6) nuclei [22], since it is known [22] that γ-unstable models (like O(6) [23]) and γ-rigid models (like Z(4)) yield similar predictions for most observables if γ_{rms} of the former equals γ_{rigid} of the latter.

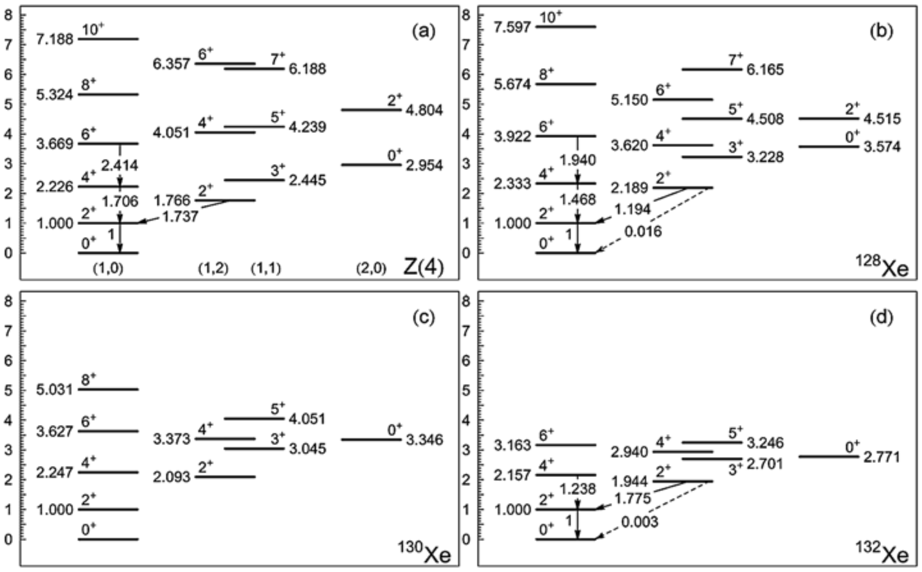

Fig. 3 – Comparison of the Z(4) predictions for (normalized) energy levels and (normalized) B(E2) transition rates (a) to experimental data for ^{128}Xe [19](b), ^{130}Xe [20](c), and ^{132}Xe [21](d). Bands in (a) are labelled by (s, n_w). See section 4 for further discussion.

5. DISCUSSION

In the present work an exact solution of the Bohr Hamiltonian with γ "frozen" to 30°, called Z(4), is obtained. Spectra and B(E2) transition rates of Z(4) resemble these of the critical point symmetry E(5), while the ground state band of Z(4) is related to the Euclidean algebra E(4), thus offering a first clue of connection between critical point symmetries and Lie algebras. Empirical evidence for Z(4) in the Xe region around $A = 130$ has been presented.

It should be emphasized, however, that neither the similarity of spectra and B(E2) values of Z(4) to these of the E(5) model, nor the coincidence of the ground state band of Z(4) to the spectrum of the Casimir operator of the Euclidean algebra E(4) clarify the algebraic structure of the Z(4) model, the symmetry algebra of which has to be constructed explicitly, starting from the fact that γ is fixed to 30°. The fact that the Bohr Hamiltonian for $\gamma = 30°$ possesses "accidentally" a symmetry axis (the body-fixed $\hat{x}'$-axis) has been early realized [24]. This "accidental" symmetry should also serve as the starting point for clarifying the symmetry underlying other solutions of the Bohr Hamiltonian obtained for $\gamma = 30°$ [8, 25, 26].

REFERENCES

1. F. Iachello, *Phys. Rev. Lett.*, **85**, 3580 (2000).
2. F. Iachello, *Phys. Rev. Lett.*, **87**, 052502 (2001).
3. R. F. Casten, N. V. Zamfir, *Phys. Rev. Lett.*, **85**, 3584 (2000).
4. R. M. Clark *et al.*, *Phys. Rev.*, C **69**, 064322 (2004).
5. R. F. Casten, N. V. Zamfir, *Phys. Rev. Lett.*, **87**, 052503 (2001).
6. R. M. Clark *et al.*, *Phys. Rev.*, C **68**, 037301 (2003).
7. A. Bohr, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. **26**, no. 14 (1952).
8. D. Bonatsos, D. Lenis, D. Petrellis, P. A. Terziev, *Phys. Lett.*, B **588**, 172 (2004).
9. D. Bonatsos, D. Lenis, D. Petrellis, P. A. Terziev, I. Ygitoglu, *Phys. Lett.*, B **621**, 102 (2005).
10. A. S. Davydov, A. A. Chaban, *Nucl. Phys.*, **20**, 499 (1960).
11. J. Meyer-ter-Vehn, *Nucl. Phys.*, A **249**, 111 (1975).
12. A. Bohr, B. R. Mottelson, *Nuclear Structure*, vol. II, Benjamin, New York, 1975.
13. B. G. Wybourne, *Classical Groups for Physicists*, Wiley, New York, 1974.
14. A. O. Barut, R. Raczka, *Theory of Group Representations and Applications*, World Scientific, Singapore, 1986.
15. M. Moshinsky, *J. Math. Phys.*, **25**, 1555 (1984).
16. M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.
17. D. Bonatsos, D. Lenis, N. Minkov, P. P. Raychev, P. A. Terziev, *Phys. Rev.*, C **69**, 044316 (2004).
18. A. S. Davydov, G. F. Filippov, *Nucl. Phys.*, **8**, 237 (1958).
19. M. Kanbe, K. Kitao, *Nucl. Data Sheets*, **94**, 227 (2001).
20. B. Singh, *Nucl. Data Sheets*, **93**, 33 (2001).
21. Yu. Khazov, A. A. Rodionov, S. Sakharov, B. Singh, *Nucl. Data Sheets*, **104**, 497 (2005).
22. R. F. Casten, *Nuclear Structure from a Simple Perspective*, Oxford University Press, Oxford, 1990.
23. F. Iachello, A. Arima, *The Interacting Boson Model*, Cambridge Univ. Press, Cambridge, 1987.
24. D. M. Brink, *Prog. Nucl. Phys.*, **8**, 99 (1960).
25. R. V. Jolos, *Yad. Fiz.*, **67**, 955 (2004) [*Phys. At. Nucl.*, **67**, 931 (2004)].
26. L. Fortunato, *Phys. Rev.*, C **70**, 011302(R) (2004).