FRW UNIVERSE WITH VARIABLE G AND Λ TERMS

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A new class of exact solutions of Einstein's field equations with a perfect fluid source, variable gravitational coupling G and cosmological term Λ for FRW spacetime is obtained by considering variable deceleration parameter models for the universe. The nature of the variables $G(t)$, $\Lambda(t)$ and the energy density $\rho(t)$ have been examined for three cases: (i) exponential, (ii) polynomial, and (iii) sinusoidal form. The special types of models for dust, Zel'dovich and radiating universe are also mentioned in all these cases. The behaviour of these models of the universe are also discussed in the light of recent supernovae Ia observations.

Key words: cosmology, deceleration parameter, variable gravitational and cosmological terms.

PACS: 98.80.Jk, 98.80.Es

1. INTRODUCTION

In Einstein's General Relativity (GR), there are two parameters, the gravitational constant G and the cosmological constant Λ which are considered to be time-independent [1]. A possible time variable G was suggested by Dirac [2] and this has been extensively discussed in literature [3, 4]. Since its introduction, its significance has been studied from time to time by various workers [5–7]. In modern cosmological theories, the cosmological constant remains a focal point of interest. A wide range of observations now compellingly suggest that the universe possesses a non-zero cosmological constant [8]. In the context of quantum field theory, a cosmological term corresponds to the energy density of vacuum. The birth of the universe has been attributed to an excited vacuum fluctuation triggering off an inflationary expansion followed by the super-cooling. The release of locked up vacuum energy results in subsequent reheating. The cosmological term, which is measure of the energy of empty

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space, provides a repulsive force opposing the gravitational pull between the galaxies. If the cosmological term exits, the energy it represents counts as mass because mass and energy are equivalent. If the cosmological term is large enough, its energy plus the matter in the universe could lead to inflation. Unlike standard inflation, a universe with a cosmological term would expand faster with time because of the push from the cosmological term [9]. Some of the recent discussions on the cosmological constant "problem" and on cosmology with a time-varying cosmological constant by Ratra and Peebles [10], Dolgov [11–13] and Sahni and Starobinsky [14] point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a "constant". However, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying Λ can be found. This entails that energy has to be conserved by a decrease in the energy density of the vacuum component followed by a corresponding increase in the energy density of matter or radiation [see also Weinberg [15] and Carroll, Press and Turner [16]].

Recent observations by Perlmutter *et al.* [17] and Riess *et al.* [18] strongly favour a significant and a positive value of Λ with magnitude $\Lambda(G\hbar/c^3) \approx 10^{-123}$. Their study is based on more than 50 type Ia supernovae with red-shifts in the range $0.10 \leq z \leq 0.83$ and these suggest Friedmann models with negative pressure matter such as a cosmological constant (Λ), domain walls or cosmic strings (Vilenkin [19], Garnavich *et al.* [20]). Recently, Carmeli and Kuzmenko [21] have shown that the cosmological relativistic theory predicts the value for cosmological constant $\Lambda = 1.934 \times 10^{-35}$ s^{-2}. This value of "Λ" is in excellent agreement with the recent estimates of the High-Z Supernova Team and Supernova Cosmological Project (Garnavich *et al.* [20]; Perlmutter *et al.* [17]; Riess *et al.* [18]; Schmidt *et al.* [22]). These observations suggest on accelerating expansion of the universe.

Since Dirac [2] first considered the possibility of a variable *G*, there have been numerous modifications of general relativity to make *G* vary with time [23]. Recently a modification linking the variation of *G* with that of Λ has been considered within the framework of general relativity by a number of workers [24–29].

The Einstein's field equations are a coupled system of non-linear differential equations. We seek physical solutions to the field equations for their applications of relevance to cosmology and astrophysics. In order to solve the field equations we normally assume a form for the matter content and killing vector symmetries by space time [30]. Solutions to the field equations may also be generated by applying a law of variation for Hubble's parameter proposed by Berman [31]. In simple cases the Hubble law yields a constant value of deceleration parameter. It is worth observing that most of the well-known models of Einstein's theory and

Brans-Deke theory with curvature parameter $k = 0$, including inflationary models have a constant deceleration parameter as borne out by the studies of several authors [31–36]. But redshift magnitude test has had a chequered history. During the 1960s and the 1970s, it was used to draw very categorical conclusions. The deceleration parameter q_0 was then claimed to lie between 0 and 1 leading to the suggestion that the universe is decelerating. But today, the view is entirely different. Observations (Knop *et al.* [37]; Riess *et al.*, [38]) of Type Ia Supernovae (SNe) allow us to probe the expansion history of the universe leading to the conclusion that the expansion of the universe is accelerating. So we can consider cosmological models with variable cosmological term and deceleration parameter. Vishwakarma and Narlikar [39] and Virey *et al.* [40] have reviewed the determination of the deceleration parameter from Supernovae data.

Recently Pradhan and Otarod [41] have studied a model of the universe with time dependent deceleration parameter and Λ-term in presence of perfect fluid. This study has prompted us to focus upon the problem of establishing a formalism for understanding the relativistic evolution for a time dependent deceleration parameter in an expanding universe with variable G and Λ in presence of perfect fluid. This paper is organized as follows. We present in Section 2 the metric and the field equations. In Section 3 we deal with a general solution. In Sections 4, 5 and 6 we deal with the solutions in exponential, polynomial and sinusoidal form. Finally in Section 7 we mention our main conclusions.

2. THE METRIC AND FIELD EQUATIONS

In standard coordinates $(x^a) = (t, r, \theta, \phi)$ the Robertson-Walker line element has the form

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],\tag{1}$$

where $S(t)$ is the cosmic scale factor. Without loss of generality the constant k takes only three values: 0,1 or –1. The constant k is related to spatial geometry of a 3-dimensional manifold generated by $t = $ constant. The Robertson-Walker spacetimes are the standard cosmological models supporting observations. For the case of variable cosmological term $\Lambda(t)$ and gravitational constant $G(t)$ the Einstein field equation

$$G_{ab} + \Lambda g_{ab} = 8\pi G(t)T_{ab}\tag{2}$$

yields

$$\frac{3}{S^2}(\dot{S}^2 + k) = 8\pi G\rho + \Lambda,\tag{3}$$

$$2\frac{\ddot{S}}{S} + \frac{(\dot{S}^2 + k)}{S^2} = -8\pi G p + \Lambda, \tag{4}$$

for the line element (1) where a dot indicate partial differentiation with respect to t. From Eqs. (3) and (4) we obtain the generalized continuity equation

$$\dot{\rho} + 3\frac{\dot{S}}{S}(\rho + p) + \frac{\dot{G}}{G}\mu + \frac{\dot{\Lambda}}{8\pi G} = 0. \tag{5}$$

This reduces to the conventional continuity equation when Λ and G are constants. In an attempt to obtain solutions to the field equations we assume, as is often done, that the classical conservation law, $T^{ab}_{;b} = 0$, also holds. Then Eq. (5) leads to

$$\dot{\rho} + 3\frac{\dot{S}}{S}(\rho + p), \tag{6}$$

$$8\pi\rho\dot{G} + \dot{\Lambda} = 0, \tag{7}$$

which facilitate the solution of the field equations. The result (6) is just the conventional continuity equation, and (7) simply relates G and Λ and does not explicitly contain the scale factor $S(t)$.

3. SOLUTION OF THE FIELD EQUATIONS

Equations (3)–(4) are two independent equations in five unknowns S, p, ρ, G, and Λ. In this paper we first assume an equation of state

$$p = \gamma\rho, \quad 0 \le \gamma \le 1. \tag{8}$$

Secondly, on the basis of supernovae searches, we consider the deceleration parameter to be variable and set

$$q = -\frac{S\ddot{S}}{\dot{S}^2} = -\left(\frac{\dot{H} + H^2}{H^2}\right) = b(\text{variable}), \tag{9}$$

where $H = \dot{S}/S$ is the Hubble parameter. The above equation may be rewritten as

$$\frac{\ddot{S}}{S} + b\frac{\dot{S}^2}{S^2} = 0. \tag{10}$$

The general solution of Eq. (10) is given by

$$\int e^{\int \frac{b}{S} dS} dS = t + m, \tag{11}$$

where m is an integrating constant.

In order to solve the problem completely, we have to choose $\int \frac{b}{S} dS$ in such a manner that Eq. (11) be integrable.

Without loss of generality, we consider

$$\int \frac{b}{S} dS = \ln L(S), \tag{12}$$

and obtain from Eqs. (11) and (12)

$$\int L(S) dS = t + m. \tag{13}$$

Of course the choice of $L(S)$, in Eq. (13), is quite arbitrary but, since we are looking for physically viable models of the universe consistent with observations, we consider the following three cases.

4. SOLUTION IN THE EXPONENTIAL FORM

Let us consider $L(S) = \frac{1}{k_1 S}$, where k_1 is arbitrary constant.

In this case on integration of Eq. (13) gives the exact solution

$$S(t) = k_2 e^{k_1 t}, \tag{14}$$

where k_2 is an arbitrary constant.

Using Eq. (7) in the differentiated form of (3), we obtain

$$8\pi G \dot{\rho} = -\frac{6kk_1}{S^2}. \tag{15}$$

From Eqs. (3), (4) and (8), we have

$$8\pi(1+\gamma)G\rho = \frac{2k}{S^2}. \tag{16}$$

From Eqs. (15) and (16), we obtain

$$\frac{\dot{\rho}}{\rho} = -3k_1(1+\gamma), \tag{17}$$

which on integration gives

$$\rho = Ae^{-3k_1(1+\gamma)t}, \tag{18}$$

where A is an integrating constant. Using Eqs. (14) and (18) in (16) we get the value of G

$$G = \frac{ke^{(1+3\gamma)k_1 t}}{4\pi(1+\gamma)Ak_2^2}. \tag{19}$$

From Eq. (3) or (4), we find Λ as

$$\Lambda = \frac{(1+3\gamma)ke^{-2k_1 t}}{(1+\gamma)k_2^2} + 3k_1^2. \tag{20}$$

From Eq. (14), since scale factor can not be negative, we find $S(t)$ is positive if $k_2 > 0$. So in the early stages of the universe, $i.e.$, near $t = 0$, the scale factor of the universe had been approximately constant and had increased very slowly. Some time later, the universe had exploded suddenly and expanded to a large scale. This picture is consistent with Big Bang scenario.

From Eq. (18), we note that $\rho(t)$ is a decreasing function of time and $\rho > 0$ for all times. From Eq. (19), we observe that G is an increasing function of time. When the universe is required to have expanded from a finite minimum volume, the critical density assumption and conservation of energy-momentum tensor dictate that G increases in a perpetually expanding universe [24]. In most variable G cosmologies [42, 43] G is a decreasing function of time. But the possibility of an increasing G has also been suggested by several authors [44].

The behaviour of the universe in this model will be determined by the cosmological term Λ; this term has the same effect as a uniform mass density $\rho_{eff} = -\Lambda/4\pi G$, which is constant in space and time. A positive value of Λ corresponds to a negative effective mass density (repulsion). Hence, we expect that in the universe with a positive value of Λ, the expansion will tend to accelerate; whereas in the universe with negative value of Λ, the expansion will slow down, stop and reverse. From Eq. [20], we see that the cosmological term Λ is a decreasing function of time and it approaches a small positive value as time increase more and more. Recent cosmological observations (Garnavich $et\ al.$ [19], Perlmutter $et\ al.$ [16], Riess $et\ al.$ [17], Schmidt $et\ al.$ [21]) suggest the existence of a positive cosmological constant Λ with the magnitude $\Lambda(G\hbar/c^3 \approx 10^{-123}$. These observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ-term. Thus our model is consistent with the results of recent observations. In the following, we consider the variation of $\rho(t)$, $\Lambda(t)$ and $G(t)$ with time for three different special cases.

The expressions for $\rho(t)$, $\Lambda(t)$ and $G(t)$ corresponding to $\gamma = 0, 1, 1/3$ are given in Table 1.

From Table 1, we observe that in all three models $\rho > 0$ for all times and is a decreasing function of time. The cosmological term Λ is also decreasing

Table 1

Values of $\rho(t)$, $\Lambda(t)$ and $G(t)$ for dust, Zel'dovice and radiation exponential models

γ	$\rho(t)$	$\Lambda(t)$	$G(t)$
0	Ae^{-3k_1t}	$\frac{ke^{-2k_1t}}{k_2^2} + 3k_1^2$	$\frac{ke^{k_1t}}{4\pi Ak_2^2}$
1	Ae^{-6k_1t}	$\frac{2ke^{-2k_1t}}{k_2^2} + 3k_1^2$	$\frac{ke^{4k_1t}}{8\pi Ak_2^2}$
$\frac{1}{3}$	Ae^{-4k_1t}	$\frac{3ke^{-2k_1t}}{2k_2^2} + 3k_1^2$	$\frac{3ke^{k_1t}}{16\pi Ak_2^2}$

function of time and it approaches to a small positive value with large value of time. Thus our models have good agreement with recent observations.

5. SOLUTION IN THE POLYNOMIAL FORM

Let $L(S) = \dfrac{1}{2k_3\sqrt{S+k_4}}$, where k_3 and k_4 are constants.

In this case, on integrating, Eq. (13) gives the exact solution

$$S(t) = \alpha_1 t^2 + \alpha_2 t + \alpha_3, \qquad (21)$$

where α_1, α_2 and α_3 are arbitrary constants.

By using the same techniques as described in previous section 4, we find the expressions for $\rho(t)$, $G(t)$ and $\Lambda(t)$:

$$\rho(t) = B(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^{-3(1+\gamma)}, \qquad (22)$$

$$G(t) = \frac{[2\alpha_1^2 t^2 + 2\alpha_1\alpha_2 t + \alpha_2^2 + k - 2\alpha_1\alpha_3](\alpha_1 t^2 + \alpha_2 t + \alpha_3)^{(1+3\gamma)}}{4\pi(1+\gamma)B}, \qquad (23)$$

$$\Lambda = \frac{(1+3\gamma)\{(2\alpha_1 t + \alpha_2)^2 + k\} + 4\alpha_1(\alpha_1 t^2 + \alpha_2 t + \alpha_3)}{(1+\gamma)(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^2}, \qquad (24)$$

where B is an integrating constant.

From Eq. (21), since scale factor can not be negative, we find $S(t)$ is positive if α_1, α_2 and α_3 are all greater the zero. From Fig. 1, it can be seen that in the early stages of the universe, *i.e.*, near $t = 0$, the scale factor of the universe had been approximately constant and had increases very slowly. At a specific time the universe has exploded suddenly and it has expanded to large scale. This fits with big bang scenario.

From Eq. (22), we observe that $\rho(t) > 0$ for $B > 0$. We also see that $\rho(t)$ is a decreasing function of time. Fig. 2 clearly shows this behaviour of $\rho(t)$.
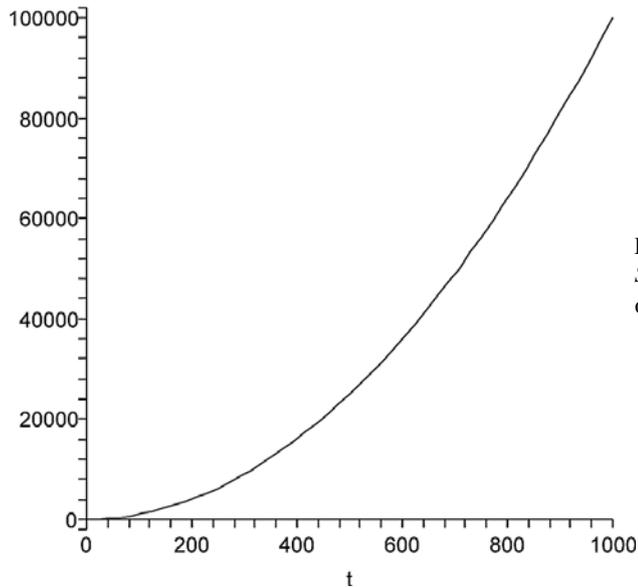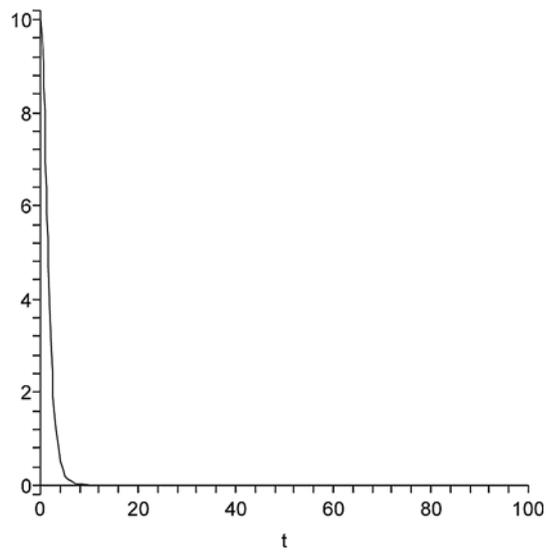

Fig. 1 – The plot of scale factor $S(t)$ *vs* time with parameters $\alpha_1 = 0.1$, $\alpha_2 = 0.01$ and $\alpha_3 = 1.0$.

Fig. 2 – The plot of energy density $\rho(t)$ *vs* time with parameters $\alpha_1 = 0.1$, $\alpha_2 = 0.01$, $\alpha_3 = 1.0$, $\gamma = 0.0$ and $B = 1.0$.

From Eq. (23), we note that $G(t)$ is an increasing function of time. Fig. 3 also shows this behaviour of G as increasing function of t. From Eq. (26), we observe that the cosmological term is a decreasing function of time and it approaches a small value as time progresses (*i.e.* the present epoch), which explains the small

and positive value of Λ at present (Perlmutter *et al.* [16]; Riess *et al.* [17, 38]; Knop *et al.* [21] Garnavich *et al.* [19]; Schmidt *et al.* [21]). Fig. 4 clearly shows this behaviour of Λ as decreasing function of time.

Fig. 3 – The plot of gravitational term $G(t)$ *vs* time with parameters $k = 1.0$, $\alpha_1 = 0.1$, $\alpha_2 = 0.01$, $\alpha_3 = 1.0$, $\gamma = 0.0$ and $B = 1.0$.
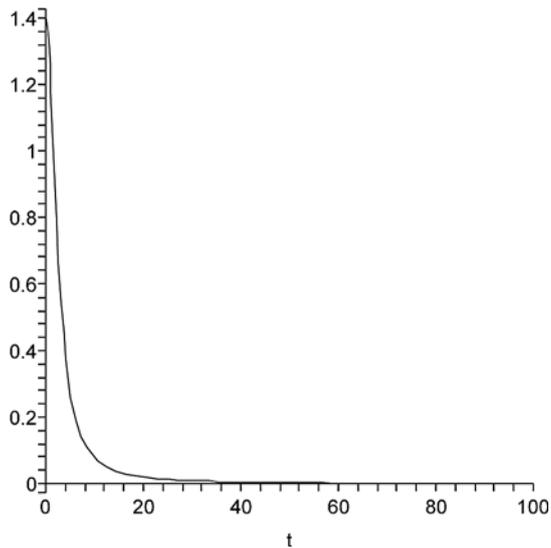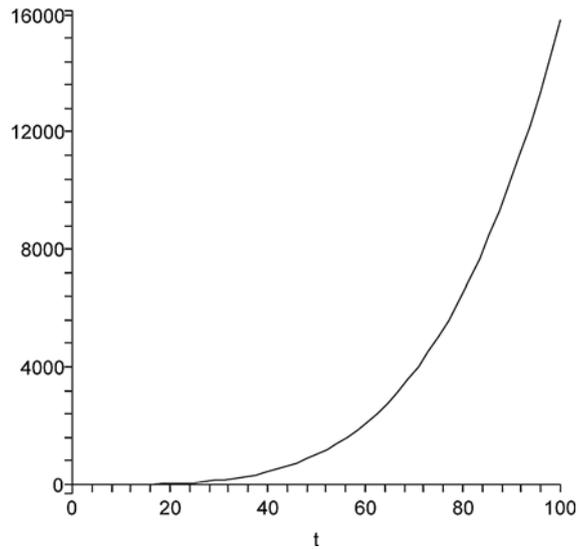

Fig. 4 – The plot of cosmological term $\Lambda(t)$ *vs* time with parameter $k = 1.0$, $\alpha_1 = 0.1$, $\alpha_2 = 0.01$, $\alpha_3 = 1.0$ and $\gamma = 0.0$.

The expressions for $\rho(t)$, $G(t)$ and $\Lambda(t)$ for dust universe ($p = 0$, $\rho > 0$), Zel'dovich universe ($p = \rho$), and radiative universe $\left(p = \dfrac{\rho}{3} \right)$ can be obtained by putting $\gamma = 0, 1, \dfrac{1}{3}$ in Eqs. (22), (23) and (24) respectively.

6. SOLUTION IN THE SINUSOIDAL FORM

Let $L(R) = \dfrac{1}{\beta\sqrt{1 - R^2}}$, where β is constant.

In this case, on integrating, Eq. (13) gives the exact solution

$$S = M\sin(\beta t) + N\cos(\beta t) + \beta_1, \tag{25}$$

where M, N and β_1 are constants.

By using the same techniques as described in previous section 4, we find the expressions for $\rho(t)$, $G(t)$ and $\Lambda(t)$:

$$\rho(t) = B[M\sin\beta t + N\cos\beta t + \beta_1]^{-3(1+\gamma)}, \tag{26}$$

$G(t) =$

$$= \frac{[\beta^2\{(M^2 + N^2) - \beta_1(M\sin\beta t + N\cos\beta t)\} + k][M\sin\beta t + N\cos\beta t + \beta_1]^{(1+3\gamma)}}{4\pi(1+\gamma)B}, \tag{27}$$

$$\Lambda = \frac{(1+3\gamma)[\beta^2(M\cos\beta t - N\sin\beta t)^2 + k]}{(1+\gamma)[M\sin\beta t + N\cos\beta t + \beta_1]^2} + \frac{2\beta^2(M\sin\beta t + N\cos\beta t)}{(1+\gamma)[M\sin\beta t + N\cos\beta t + \beta_1]}. \tag{28}$$

Since, in this case, we have many alternatives for choosing values of M, N, β, β_1, it is just enough to look for suitable values of these parameters, such that the physical initial and boundary conditions are satisfied. We are trying to find feasible interpretation and situations relevant to this case.

In Fig. 5 we plot scale factor versus time. It is observed here that at early stages of the universe, the scale of the universe increases gently and then decreases sharply, and afterwards it will oscillate for ever.

In Fig. 6 we plot energy density $\rho(t)$ versus time and in Fig. 8 we note the run of cosmological constant $\Lambda(t)$ versus time. From these figures we conclude that at the early stages of the universe the matter is created as a result of loss of vacuum energy and at a particular epoch it has started to oscillate for ever due to sinusoidal property. In Fig. 7 we plot the $G(t)$ versus time. From it we conclude that in the begining at the early stages of the universe $G(t)$ increases gently with time and at a particular epoch it decreases sharply, and afterwards it will oscillate for ever. It is worth to mention here that in this case oscillation takes place in positive quadrant which is physically relevant.

The expressions for $\rho(t)$, $G(t)$ and $\Lambda(t)$ for dust universe ($p = 0$, $\rho > 0$), Zel'dovich universe ($p = \rho$), and radiative universe $\left(p = \dfrac{\rho}{3}\right)$ can be obtained by putting $\gamma = 0$, 1, $\dfrac{1}{3}$ in Eqs. (26), (27) and (28) respectively.

7. DISCUSSION AND CONCLUSION

In this paper we have described a new class of FRW cosmological models of the universe with a perfect fluid as the source of matter by applying a variable deceleration parameter. The "constants" G and Λ are allowed to depend on the cosmic time. Generally, the models are expanding, non-shearing and isotropic in nature.

The cosmological terms in the models in Sections 4, 5 are a decreasing function of time and this approaches a small value as time increases (*i.e.* present epoch). The value of the cosmological "terms" for these models are found to be small and positive which is supported by the results from recent supernovae Ia observations (Perlmutter *et al.* [16]; Riess *et al.* [17,38]; Knop *et al.* [37]; Garnavich *et al.* [19]; Schmidt *et al.* [21]). The cosmological term in Section 6 also decreases while time increases to a specific instant. During this period as we can understand from Fig. 8, we will have enough matter creation to force the universe to oscillate for ever due to sinusoidal property of Λ. This means we always have annihilation and creation of matter permanently. At this point one more sentence may be added to our discussion and *i.e.* as the graphs for Λ and ρ in this case the explosion of the universe at the early stages of its creation has been only a consequence of matter creation. Thus, the implications of time varying Λ and G are important to study the early eavolution of the universe.

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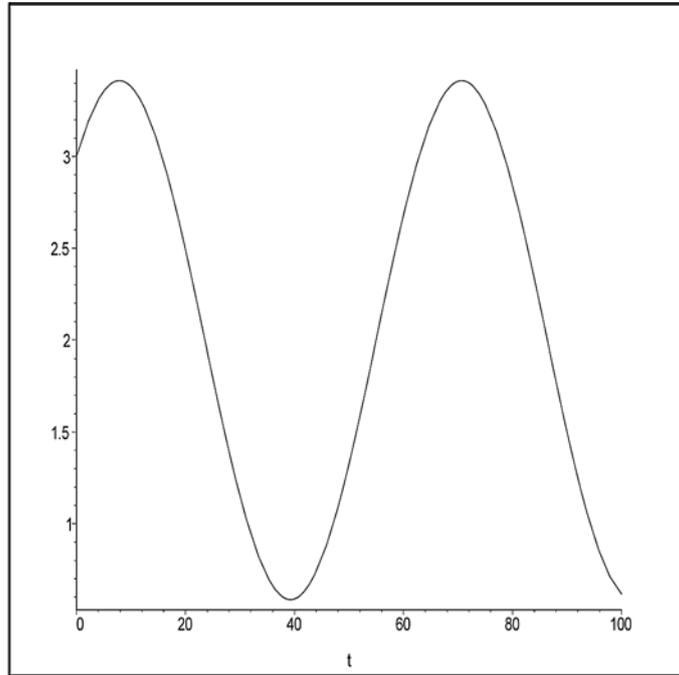
Fig. 5 – The plot of scale factor $S(t)$ *vs* time with parameter $M = 1.0$, $N = 1.0$, $\gamma = 0.0$, $\beta = 0.10$ and $\beta_1 = 2.0$.
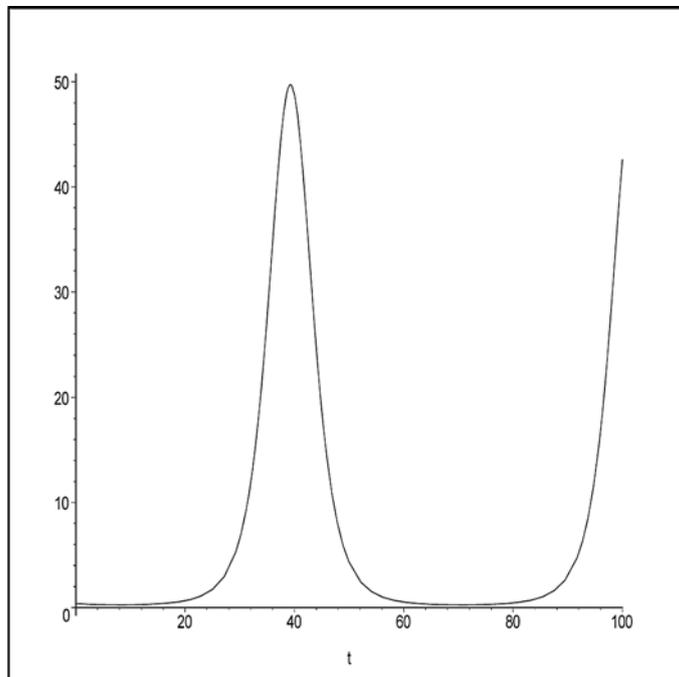

Fig. 6 – The plot of energy density $\rho(t)$ *vs* time with parameter $M = 1.0$, $N = 1.0$, $B = 10.0$, $\gamma = 0.0$, $\beta = 0.10$ and $\beta_1 = 2.0$.
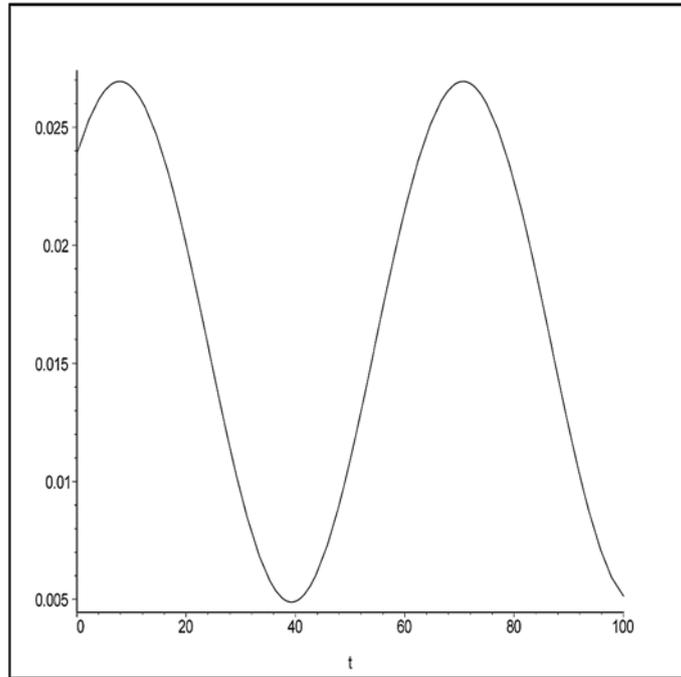
Fig. 7 – The plot of gravitational term $G(t)$ *vs* time with parameter $k = 1.0$, $M = 1.0$, $N = 1.0$, $B = 10.0$, $\gamma = 0.0$, $\beta = 0.10$ and $\beta_1 = 2.0$.
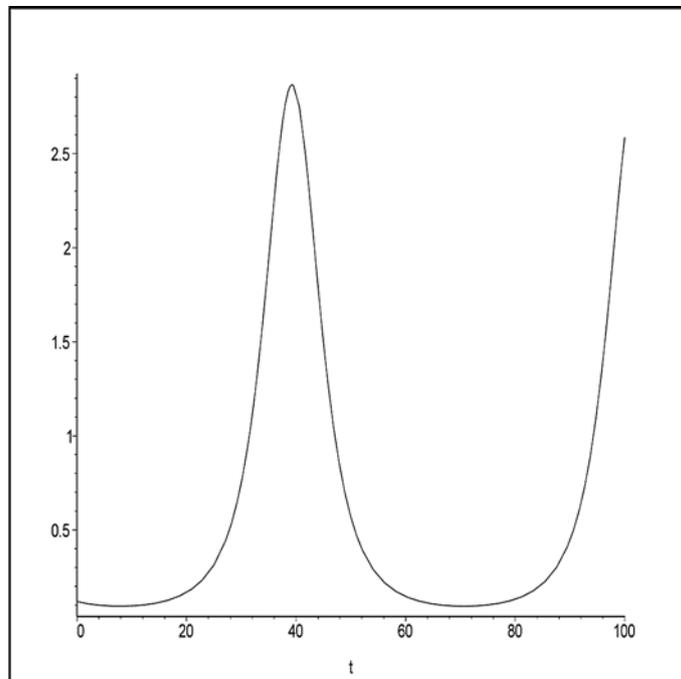

Fig. 8 – The plot of cosmological term $\Lambda(t)$ *vs* time with parameter $k = 1.0$, $M = 1.0$, $N = 1.0$, $B = 10.0$, $\gamma = 0.0$, $\beta = 0.10$ and $\beta_1 = 2.0$.