A GENERALIZED MECHANISM FOR DIMENSIONAL REDUCTION

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We present a generalized version of dimensional reduction in particle mechanics. The collapse of four-dimensional phase-space to two dimensions is discussed in detail. An analysis of the same phenomenon in arbitrary dimensionalities is also provided.

Dimensional reduction in particle mechanics was apparently first put into evidence by Peierls [1], who argued that the dynamics of a particle living in (2+1)-dimensions becomes (1+1)-dimensional, once a strong magnetic field is applied, confining the particle to its lowest Landau level.

Additional insight into that phenomenon was provided in [2], and an extension of the set-up to the case of noncommutative mechanics [5] can be found, for instance, in [3]. In this last case, the dynamics is provided by a Hamiltonian H(q, p) and a non-canonical symplectic two-form ω. q_i and p_i, $i, j = 1,..., n$ are the phase-space variables of the system under consideration, and are denoted in the following by $x_a, a = 1,..., 2n.$ Dimensional reduction occurs when ω is degenerate. Up to now, only dynamical systems with constant symplectic forms were studied explicitely. The aim of this paper is to extend the analysis to the case in which the symplectic form is not constant, but a function of the phase-space variables, $\omega(q, p)$. The analysis is performed at the classical level. It extends to quantum mechanics whenever operator ordering issues related to $\omega(q, p)$ do not appear.

Let us first review the constant ω case. Consider

$$\omega = \sum_{a,b=1}^{2n} \omega^{ab} dx_a \Lambda dx_b. \tag{1}$$

Classically (1) generates extended Poisson brackets $\{x_a, x_b\} = \Theta_{ab}$, where Θ is the inverse of ω, $\Theta_{ab} = \left(\omega^{-1}\right)_{ab}$. Quantum mechanically, one replaces the

Poisson brackets with commutators. In (2+1)-dimensions (although the formalism extends straight forwardly to higher dimensionalities) the action

$$S=\int dt\left(\frac{1}{2}\omega_{ab}x_a\dot{x}_b-H\left(x\right)\right),\quad x_{1,2,3,4}=q_1,q_2,p_1,p_2, \tag{1}$$

engenders the equations of motion

$$\dot{x}_a=\left\{x_a,H\right\}=\Theta_{ab}\frac{\partial H}{\partial x_b},\quad \Theta_{ab}=\left(\omega^{-1}\right)_{ab},\quad a,b,=1,2,3,4. \tag{3}$$

Above, $\left\{A,B\right\}=\Theta_{a,b}\partial_a A\partial_b B;$ in particular, $\left\{x_a,x_b\right\}=\Theta_{ab}$. If we choose the symplectic form to be

$$\Theta=\begin{pmatrix}0&\theta&1&0\\-\theta&0&0&1\\-1&0&0&F\\0&-1&-F&0\end{pmatrix}\text{ or }\omega=\frac{1}{1-\theta F}\begin{pmatrix}0&-F&1&0\\F&0&0&1\\-1&0&0&-\theta\\0&-1&\theta&0\end{pmatrix}, \tag{4}$$

the nonzero Poisson brackets are $\left\{q_1,p_j\right\}=\delta_{ij},\left\{q_i,q_2\right\}=\theta,\left\{p_1,p_2\right\}=F,$ and the phase-space equations of motion become

$$\dot{q}_i=\frac{\partial H}{\partial p_i}+\theta\quad\varepsilon_{ij}\frac{\partial H}{\partial q_j},\quad \dot{p}_i=-\frac{\partial H}{\partial q_i}+F\varepsilon_{ij}\frac{\partial H}{\partial p_j},\quad \varepsilon_{12}=-\varepsilon_{21}=1. \tag{5}$$

Dimensional reduction occurs when θF=1. Then, the number of dynamical degrees of freedom is halved, since the equations of motion imply

$$\dot{q}_1=-\theta\dot{p}_2\quad and\quad \dot{q}_2=\theta\dot{p}_1. \tag{6}$$

The degeneracy is a consequence of Θ being singular. The four-dimensional phase space $\{q_1, q_2, p_1, p_2\}$ collapsed to a bidimensional one, spanned for instance by the canonically conjugate variables q_1 and q_2. As θ was taken to be constant, one has the identifications

$$q_1=-\theta p_2+c_1,\quad q2=\theta p_1+c_2. \tag{7}$$

Obviously, the equations of motion should now be solved by providing initial conditions for q_1, q_2, p_1 and p_2 at a given initial time, not by fixing boundary conditions for q_1, q_2 at two different times. Freedom in imposing the initial conditions is insured by the arbitrariness of c_1 and c_2.

An important remark is in order. If the original Hamiltonian is isotropic in the momenta, and separately in the coordinates, $H=H\left(p_1^2+p_2^2,q_1^2+q_2^2\right)$, the resulting system is not only integrable – as any one-dimensional system is – but also easily solvable, as the new Hamiltonian reads

$$H = H\left[\theta^2\left(q_1^2 + q_2^2\right), q_1^2 + q_2^2\right], \quad \{q_1, q_2\} = \theta. \tag{8}$$

The statement is obvious in the holomorphic coordinates

$$a = \frac{1}{\sqrt{2\theta}}\left(q_1 + iq_2\right), \qquad \bar{a} = \frac{1}{\sqrt{2\theta}}\left(q_1 - iq_2\right), \tag{9}$$

as those obey the equations $\left(H = h_\theta\left(\bar{a}a\right) = h\left(n\right) \text{ below}\right)$

$$\frac{\dot{a}}{a} = \frac{\dot{\bar{a}}}{\bar{a}} = i\frac{dh_\theta}{dn}, \tag{10}$$

solvable in terms of trigonometric functions. At the quantum level immediate solvability is also obvious, as the dimensionally reduced Hamiltonian is a function h_θ of the Hamiltonian of an harmonic oscillator, hence the spectrum will be discrete, with energy levels given by $E_n = h_\theta\left(\theta\left(n + 1/2\right)\right)$.

As already stated, we will consider in this paper the more general case in which $\Theta\left(x\right)$ in (3) is *x*–dependent.

We start with an arbitrary Hamiltonian *H*(*p*, *q*) in (2+1)-dimensions and the following nonzero fundamental Poisson brackets

$$\left\{q_1, q_2\right\} = \theta\left(q, p\right), \quad \left\{p_1, p_2\right\} = F\left(q, p\right), \quad \left\{q_i, q_j\right\} = \delta_{ij}. \tag{11}$$

The Jacobi identities read

$$\frac{\partial\theta}{\partial q_1} - F\frac{\partial\theta}{\partial p_2} = 0, \quad \frac{\partial\theta}{\partial q_2} + F\frac{\partial\theta}{\partial p_1} = 0, \tag{12}$$

$$\frac{\partial F}{\partial p_2} - \theta\frac{\partial F}{\partial q_2} = 0, \quad \frac{\partial F}{\partial p_1} + \theta\frac{\partial F}{\partial q_2} = 0. \tag{13}$$

The equations of motion are identical to the ones written in (3), except that now the Θ given in (4) is not constant. The equations of motion can be combined to give:

$$\dot{q}_1 + \theta\dot{p}_2 = \left(1 - F\theta\right)\frac{\partial H}{\partial p_1}, \quad \dot{q}_2 - \partial\dot{p}_1 = \left(1 - F\theta\right)\frac{\partial H}{\partial p_2}, \tag{14}$$

$$\dot{p}_1 - F\dot{q}_2 = -\left(1 - F\theta\right)\frac{\partial H}{\partial q_1}, \quad \dot{p}_2 - F\dot{q}_1 = -\left(1 - F\theta\right)\frac{\partial H}{\partial q_2}. \tag{15}$$

The above equations get halved in number if $F\theta = 1$, in which case one also has

$$\dot{q}_1 + \theta \dot{p}_2 = 0, \quad \dot{q}_2 - \theta \dot{p}_1 = 0. \tag{16}$$

The remarkable fact now is that one has

$$\dot{\theta} = 0, \tag{17}$$

which permits again the identification (7), although in this case θ was not anymore assumed to be constant, but a function of the coordinates and momenta. Let us prove Eq.(17). One has $\dot{\theta} = \frac{\partial \theta}{\partial q_i}\dot{q}_i + \frac{\partial \theta}{\partial p_i}\dot{p}_i$ which leads, using the Jacobi identities (12, 13), to

$$\dot{\theta} = \frac{\partial \theta}{\partial p_2}\left(\dot{p}_2 + F\dot{q}_1\right) + \frac{\partial \theta}{\partial p_1}\left(-\dot{p}_1 + F\dot{q}_2\right) = 0. \tag{18}$$

The last equality followed from $F = 1/\theta$ and from Eq. (16). If $F = 1/\theta$, the number of Jacobi identities also gets halved, as Eqs. (12, 13) give simply

$$\theta = \frac{\partial \theta}{\partial q_1} - \frac{\partial \theta}{\partial p_2} = 0, \quad \theta \frac{\partial \theta}{\partial q_2} + \frac{\partial \theta}{\partial p_1} = 0. \tag{19}$$

The general solution $\theta\left(q_1, p_2, p_1, p_2\right)$ of the system (19) is given implicitely by the algebric equation

$$\theta = f\left(q_1 + \theta p_2, q_2 - \theta p_1\right). \tag{20}$$

f is an arbitrary function of two independent variables. A particular solution for θ is found for each chosen f . Now, since (16) and (17) imply again, in spite of the nonconstancy of the symplectic form,

$$q_1 + \theta p_2 = c_1, \quad q_2 - \theta p_1 = c_2, \tag{21}$$

we observe that (20) can be rewritten as

$$\theta = f\left(c_1, c_2\right). \tag{22}$$

θ can thus be only a constant, in the dimensionaly reduced space! Consequently the solvability analysis performed above, for the case in which θ is constant from the beginning, remains valid.

We proceed to the general case of an arbitrary number of dimensions. Consider first a generalized electromagnetic background $F(q, p)$, living on a space with noncommutativity field $\theta\left(q, p\right)$, and flat metric $g_{ij} = \delta_{ij}$:

$$\left\{q^i, q^j\right\} = \theta^{ij}\left(q, p\right) \quad \left\{q^i, p^j\right\} = \delta^{ij} \quad \left\{p_i, p_j\right\} = F_{ij}\left(q, p\right). \tag{23}$$

The quantum mechanical version is obtained by replacing the Poisson brackets with commutators, $\{,\} \rightarrow -i[,]$. Although this is not necessary, we distinguish for clarity between up and down indices. The Jacobi identities read

$$\left\{q^k, F_{ij}\right\} = \frac{\partial F_{ij}}{\partial p_k} - \frac{\partial F_{ij}}{\theta q^m}\theta^{mk} = 0 \tag{24}$$

$$\left\{\theta^{ij}, p_k\right\} = \frac{\partial \theta^{ij}}{\partial q^k} + \frac{\partial \theta^{ij}}{\partial p_m}F_{mk} = 0 \tag{25}$$

$$\left\{F_{ij}, p_k\right\} + cyclic = \left(\frac{\partial F_{ij}}{\partial q^k} + \frac{\partial F_{ij}}{\partial p_m}F_{mk}\right) + cyclic = 0 \tag{26}$$

$$\left\{q^k, \theta^{ij}\right\} + cyclic = \left(\frac{\partial \theta^{ij}}{\partial p_k} - \frac{\partial \theta^{ij}}{\partial q^m}\theta^{mk}\right) + cyclic = 0. \tag{27}$$

They ensure the invariance of the commutation relations under time evolution, for a generic Hamiltonian $H(p, q)$. Explicitely:

$$\left\{\left\{q^m, p_n\right\} - \delta_n^m, H\right\} = -\partial_{p_s}H\left\{F_{ns}, q^m\right\} + \partial_s H\left\{\theta^{ms}, p_n\right\}, \tag{28}$$

$$\left\{\left\{q^m, q^n\right\} - \theta^{mn}, H\right\} = -\partial_{p_s}H\left\{\theta^{mn}, p_s\right\} - \partial_s H\left(\left\{\theta^{ns}, q^m\right\} + cyclic\right), \tag{29}$$

$$\left\{\left\{p_m, p_n\right\} - F_{mn}, H\right\} = -\partial_s H\left\{F_{mn}, q^s\right\} + \partial_{p_s}H\left(\left\{F_{mn}, p_s\right\} + cyclic\right), \tag{30}$$

and Eqs. (24–27) ensure the right hand side in (28–29) to be zero. The Jacobi identities above are seen to restrict F and θ to one of the following forms:

- F and θ both constant
- θ constant, and $F(q, p)$ constrained by (24, 26)
- F constant, and $\theta(q, p)$ constrained by (25, 27)
- $F(q, p)$ and $\theta(q, p)$, both constrained by (24–27)

In particular, (24) forbids an electromagnetic field strength to depend only on the coordinates, $F_{ij}(q)$, if $\theta^{ij} \neq 0$. We proceed to discuss all the above situations in turn; in most cases θ and F are related.

If both θ and F are constant, one has the simplest situation, intensively studied recently under the name of noncommutative quantum mechanics [5]. The dimensional reduction has been worked out previously [1, 2, 3].

In the case F constant and $\theta(q,p)$, Eq. (25) constrains the functional dependence of θ to be of the form $\theta(\overline{p}_m) = \theta(p_m - F_{mn}q^n)$. To satisfy (27) also, one can either block-diagonalize θ, or require $\partial_{\overline{p}_s}\theta^{ij}(\overline{p})\left[\delta_s^k + F_{sm}\theta^{sk}(\overline{p})\right] + cyclic = 0.$ The equations of motion do not change if *H*(*p*), they remain those of a classical particle in a constant electromagnetic field.

The Schrödinger equation in (2+1)-dimensions [4] confirms that: $\psi(q^1, p_2)$ satisfies the same equation as when $\theta \equiv 0.$ The only difference is that objects like $\psi(q^1, p_2)$ and $\psi(p_1, p_2)$ are not definable anymore. Two limiting cases exist:

1. $F = 0$ and $\theta(p),$ which implies a 'p-Bianchi identity' for θ,

$$\partial_{pk}\theta^{ij}(p) + \partial_{p_i}\theta^{jk}(p) + \partial_{pj}\theta^{ki}(p) = 0. \tag{31}$$

$\theta(p)$ plays the role of a magnetic field in momentum space [14]. If *H*(*p*), θ has no effect on the equations of motion, which remain those of a free particle.

2. $F^{-1} \to 0$ and $\theta \to \theta(q)$. If one allows a potential term also, $H(q,p) = p^2/2 + V(q)$, given that now $F \to \infty$, one gets $\ddot{q}_k \simeq F_{km}(\dot{q})(\dot{q}_m - \theta^{ms}\partial_s V)$. No dimensional reduction occurs here, as one cannot take $\theta F = 1$.

If *F*(*q*, *p*) is nonconstant and θ is constant, Eq. (24) requires that $F = F(\overline{q}^m) = F(q^m + \theta^{mn}p_n)$. To satisfy (26) one can either partially block-diagonalize F *and* θ such that they couple only pairs of directions, or just look for a solution of (26) for $F(\overline{q}^m)$. (26) is automatically satisfied in two dimensions (2D). For small θ one obtains, for a particle of mass *m*,

$$m\ddot{q}^m = F_{mn}(q^s + \theta^{st}p_t)\dot{q}^n \simeq F_{mn}(q^s)\dot{q}^n + m\partial_s F_{mn}(q)\theta^{st}\dot{q}^t\dot{q}^n. \tag{32}$$

We have thus the superposition of a usual electromagnetic background, linear in velocities, and of a force quadratic in the velocities, $\partial_s F(q)_{mn}\theta^{st}\dot{q}^{st}\dot{q}^t\dot{q}_n.\ \gamma_{tn}^s \equiv \partial_s F(q)_{mn}\theta^{st}$ simulates a gravitational connection. This interpretation is however valid only in one reference frame, since γ_{tn}^s does not behave like a Christoffel symbol under generalized coordinate transformations. No dimensional reduction occurs, as necessarely $\theta F \neq 1$. The limiting cases are:

1. $\theta = 0$ and $F(q)$, which is usual electromagnetism.

2. $\theta^{-1} \to 0$ and $F \to F(p)$, , which has the following dynamical content: If $H(p) = p^2/2$, get $\ddot{q}_k = F_{km}(\dot{q})\dot{q}_m$, i.e. first order differential equation in $\dot{q}$. If $H(q,p) = p^2/2 + V(q)$, due to $\theta \to \infty$, the potential term will dominate the dynamics; in a first approximation $\ddot{q}^k \approx \theta^{ks}\theta^{lm}\partial_l\partial_s V\partial_m V$.

The general case $F(q,p)$ and $\theta(q,p)$ permits an interesting study of dimensional reduction. It is our main interest here.

We will first show in general that relations of the type $F = \frac{1}{\theta}$ arise *if and only if* the q's and p's are not independent, i.e. if dimensional reduction takes place. We work in an arbitrary number of dimensions, with phase space given by $\{q^i, p_j\}$. First, assume there exists a relation between the q^i's and the p_j's say,

$$q^n = g^n(p_m) \qquad p_m = f_m(q^s) = (g^{-1})_m(q^s). \tag{33}$$

Asking consistency of the commutation relations, one obtains

$$\frac{\partial q^m}{\partial p_s} = -\theta^{ms} \qquad \frac{\partial p_k}{\partial q^n} = F_{kn}. \tag{34}$$

This follows, for instance, from $\{f(q(p)), q^s\} = \frac{\delta f}{\delta q^m}\theta^{ms} = -\frac{\delta f}{\delta q^m}\frac{\partial q^m}{\partial p_s}$.

Since $\frac{\partial q^m}{\partial p_k}\frac{\delta p_k}{\delta q^n} = \delta^m_n$, it follows that

$$\theta^{mk}F_{kn} = -\delta^m_n, \tag{35}$$

or $F_{12} \equiv F = \theta^{-1} \equiv \theta_{12}^{-1}$ in 2D parlance. (35) is thus enforced by (33) and (23). An alternative derivation uses the following chain of equalities, $\delta^m_n = \{q^m, p_n\} = \{g^m(p), p_n\} = \frac{\partial q^m}{\partial p_s}F_{sn} = -\theta^{mk}F_{kn}$. Eq. (35) is fully consistent with the commutation relations:

$$\{q^m, q^n\} = \{g^m(p), g^n(p)\} = \frac{\partial g^m}{\partial p_s}F_{st}\frac{\partial g^n}{\partial p_t} = \theta^{ms}F_{st}\theta^{nt} = \theta^{mn}. \tag{36}$$

Conversely, (35) implies (33), with the relationship between the q's and the p's actually taking the more precise from

$$q^m = -\theta^{mn}(q,p)\,p_n + c^m, \quad \forall H(p,q), \tag{37}$$

as already shown in 2D, cf. Eq.(21). To prove this in general, first notice that (35) and the equations of motion imply that

$$\dot{q}^m + \theta^{mn}\dot{p}_n = \left\{q^m + \theta^{mn} p_n, H\right\} - 0, \quad \forall H(p,q). \tag{38}$$

This already shows that dimensional reduction occurs, since the variations of the q's and the p's are related. Eq. (37) will follow from Eq. (38) if it is true that $\dot{\theta} = 0$. To prove that, first observe that Jacobi implies $\left\{F, q^i\right\} = 0$, hence

$$\dot{F} = \left\{F, p_1\right\}\partial_{p_i} H, \quad \dot{\theta} = \left\{\theta, q^j\right\}\partial_{q^j} H. \tag{39}$$

However, if $\theta F = 1$, it happens that both $\dot{\theta} = 0$ and $\dot{F} = 0$, $\forall H(p,q)$. This is so because if θ is a function of F or viceversa, then the nonzero terms in (39) become zero, due to $\left\{\theta, q^i\right\} = \frac{\partial\theta}{\partial F}\left\{F, q^i\right\} = 0,\ \left\{F, p_j\right\} = \frac{\partial F}{\partial\theta}\left\{\theta, p_j\right\} = 0.$

Thus $F(q,p)$ and $\theta(q,p)$ are constants of motion, for instance functions of the Hamiltonian. If $\dot{\theta} = 0$, then $\dot{q} = \theta\dot{p}$ *implies* $q = \theta p$, and viceversa, and (37) is proved. We note in passing that if $H(p), \dot{\theta} = 0$ immediately, due to Jacobi, cf. (39), and $\theta(q,p)$ is a constant of motion.

One may ask now: under what conditions are (37) and (34) compatible? It turns out (differentiating with respect to p) that those conditions are precisely the Jacobi identities, now meaning that the total variation of F or θ with respect to one set of coordinates (q xor p) is zero. That the Jacobi identity $\left\{\theta^{mn}, p_l\right\} = 0$ means zero total variation of θ in reduced space can be seen from the following sequence of equalities:

$$\left\{\theta^{mn}, p_l\right\} = \frac{\partial\theta^{mn}}{\partial q^l} + F_{sl}\frac{\partial\theta^{mn}}{\partial p_s} = F_{sl}\left(\frac{\partial\theta^{mn}}{\partial p_s} + \frac{\partial\theta^{mn}}{\partial q^r}\frac{\partial q^r}{\partial p_s}\right) = F_{sl}\frac{\delta\theta^{mn}}{\delta p_s} = \frac{\delta\theta^{mn}}{\delta q^l} \tag{40}$$

in which we used (34) repeatedly. Similarly, $\left\{F_{mn}, q^l\right\} = 0$ implies $\frac{\delta F_{mn}}{\delta q^t} = F_{st}\frac{\delta F_{mn}}{\delta p_s} = 0.$ This is satisfied either by constant F and θ, or by such a dependencies which cancels when $q = q(p)$. In 2D, such examples are $F_{12}(p_1, q_2) = \theta_{12}^{-1} = \frac{p_1}{q_2}; F_{12}(p_2, q_1) = \theta_{12}^{-1} = -\frac{p_2}{q_1}$, or a more general one,

$F_{12}\left(p_1, p_2, q_1, q_2\right) = \theta_{12}^{-1} = \dfrac{p_1 + p_2}{q_2 - q_1}$. It is easy to check that any of those F's satisfies $\left\{F, q^l\right\} = 0$, once $F = \theta^{-1}$.

It would be interesting to obtain solutions of the Jacobi identities in the general case. Then, one could see whether $\dot{\theta} = 0$ means that θ is trivially constant in the dimensionally reduced system, as we showed it happens in two dimensions, or whether it means that $\dot{\theta}_{ij} = 0$ gives nontrivial constants of motion.

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