REMARKS ON A RESULT OF FADDEEV

CIPRIAN SORIN ACATRINEI

Horia Hulubei National Institute for Nuclear Physics and Engineering
P.O. Box MG-6, 077125 Bucharest, Romania
acatrinei@theory.nipne.ro

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We give an alternative proof of a result obtained by Faddeev in 1995, which states that the generators of a pair of Heisenberg-Weyl groups span the whole space of bounded operators acting on $L_2(\mathcal{R})$. In the process, we provide a simple understanding of the result.

In 1995, L.D. Faddeev obtained the following interesting result [1]. Consider the Hilbert space of one-dimensional quantum mechanics, $\mathcal{H}$, with the operators Q and P satisfying the Heisenberg commutation rule

$$|Q,P| = i\hbar I \tag{1}$$

and their exponentials

$$u = e^{iqP/\hbar}, \quad v = e^{ipQ/\hbar}, \tag{2}$$

which satisfy the Weyl commutation relations

$$uv = e^{2\pi i\theta} vu, \quad \theta = \frac{pq}{2\pi\hbar} = \frac{pq}{h}. \tag{3}$$

Above, q and p are real numbers, having the dimensionality of coordinate and momentum, respectively; θ is taken to be irrational, which will be crucial in what follows. We used the notation of [1]. The operators P and Q generate the full algebra $\mathcal{B}$ of operators acting on $\mathcal{H}$, whereas u and v span $\mathcal{A}$, a proper subalgebra of $\mathcal{B}$. $\mathcal{A}$ is algebraically generated by u and v, which means that it is obtained as a closure of polynomials in u, v, u^{-1} and v^{-1}. Motivated by work on U(1) lattice current algebra [2, 3], Faddeev asked what generators should be added to u and v in order to generate the whole of $\mathcal{B}$. The answer provided in [1] is that, given

$$\hat{u} = v^{1/\theta}, \quad \hat{v} = u^{1/\theta}, \tag{4}$$

the set $\{u, v, \hat{u}, \hat{v}\}$ algebraically generates $\mathcal{B}$. It is to be noted that $\hat{u}$ and $\hat{v}$ satisfy

$$\hat{u}\hat{v} = e^{-2\pi i/\theta}\,\hat{v}\hat{u}, \tag{5}$$

and that they of course generate another subalgebra of $\mathcal{B}$, subsequently called $\hat{\mathcal{A}}$. Since u, v, commute with $\hat{u}, \hat{v}$, the elements of the two algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ commute with each other, and since they are shown to generate the whole of $\mathcal{B}$, this implies that a quantum mechanical degree of freedom is divided into two parts.

The proof given for this result in [1] is algebraic in nature, showing that any operator which commutes with the above four generators should be proportional to unity. We will give here a different argument, which will also provide a simple understanding of *why* the result holds true.

We want to find out what should be added to u and v in order to algebraically generate the whole of $\mathcal{B}$. To answer that, we note that the operators in $\mathcal{B}$ can perform two types of operations on a state vector belonging to $\mathcal{H}$. They can shift it, or leave it invariant, up to a factor. In the coordinate representation $|x\rangle$ one has

$$u|x\rangle = |x+q\rangle, \quad v|x\rangle = e^{ipx/\hbar}|x\rangle, \tag{6}$$

whereas in the momentum basis $|\pi\rangle$

$$v|\pi\rangle = |\pi + p\rangle, \quad u|\pi\rangle = e^{i\pi q\hbar}|\pi\rangle \tag{7}$$

u performs shifts by $\Delta_1 \equiv q$ on $|x\rangle$, while in the momentum representation v shifts by $\delta_1 \equiv p$. All we need is to add operators which provide other shifts, e.g. Δ_2 in coordinate space, δ_2 in momentum space, such that their linear combination (with integer coefficents) generates all the possible shifts. Now, the important fact is that given Δ_1, *any* shift Δ can be written to any degree of accuracy as

$$\Delta = n_1\Delta_1 + n_2\Delta_2, \tag{8}$$

with n_1 and n_2 integers, provided Δ_1 and Δ_2 are incommensurate, i.e. provided Δ_1/Δ_2 is irrational.

This can be understood beginning with commensurate shifts, which satisfy $\Delta_1/\Delta_2 = m_1/m_2, m_1, m_2$ integers. Then the minimum shift performed by an expression like $\Delta = n_1\Delta_1 + n_2\Delta_2$ is $\Delta_{\min} = \Delta_1/m_1 = \Delta_2/m_2$. Increasing m_1 or m_2 renders our resolution finer and finer. If we take now $\Delta_1/\Delta_2 = t$ to be irrational, that corresponds to ever increasing m_1 and/or m_2 , and hence renders an infinitesimal shift possible. Out of it, any real shift can be built up to accuracy Δ_{min}, which is arbitrarely small if t is irrational. It may happen that a given shift is exactly reproduced for finite n_1, n_2, for instance $2t-1=n_1\cdot t+n_1\cdot 1, t$ irrational. The basis $\{t,1\}$ will however generate a shift by 1/2 only approximately – although with any required accuracy – for finite n_1 and n_2.

The composition of commensurate shifts is ilustrated in Fig. 1a, with $\Delta_1/\Delta_2=2/7$ and a minimal shift $\Delta_{min}=\Delta_2-3\Delta_1=\Delta_1/2=\Delta_2/7$. Other examples are $\Delta_1/\Delta_2=2/3$, with $\Delta_{min}=\Delta_2-\Delta_1=\Delta_1/2=\Delta_2/3$, or $\Delta_1/\Delta_2=15/19$, for which $\Delta_{min}=4\Delta_2-5\Delta_1=\Delta_1/15=\Delta_2/19$.

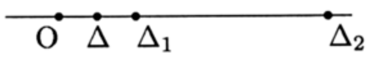

Figure 1a

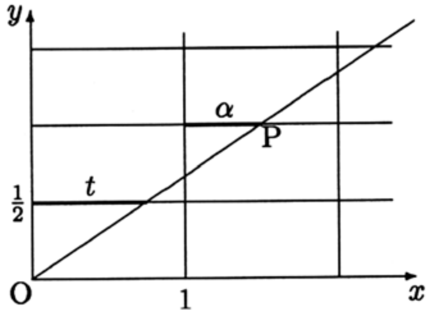

Figure 1b

The problem can also be mapped to a well known one, namely the foliation of a torus by a line with irrational slope. To see this divide Eq. (8) by Δ_2, to rewrite it as

$$n_1 t+n_2=\alpha;\quad \alpha=\frac{\Delta}{\Delta_2},\quad t=\frac{\Delta_1}{\Delta_2}. \tag{9}$$

As t is irrational, the same will hold for either $\{t,1\}$ or Δ/Δ_2, possibly for both. We suppose $\alpha=\Delta/\Delta_2$ is irrational (otherwise, we divide (8) by Δ_1, and proceed along the same lines).

Consider now the torus obtained by identifying points in $\mathcal{R}^2$ as follows (Fig. 1b):

$$x=x+\Delta_1,\quad y\equiv y+1/2. \tag{10}$$

The identification along y can be made in general as $y\equiv y+k, k$ integer, for our purposes. k=1/2 was chosen for convenience in drawing Fig. 1b. Draw now a line starting at the origin O, with (inverse) slope $2t$. Since t is irrational, the line will foliate the torus, filling it densely. Every time it intersects the horizontal cycle, it will do it at another point, coming thus arbitrely close to any point P, after an appropriate number of windings. In Fig. 1b the point P is touched after two windings along y, hence $\alpha = 2t - 1:\ n_1 = 2, n_2 = -1$.

In consequence, to generate the whole of $\mathcal{B}$ it is sufficient to add a shift in q incomensurate with the one performed by u, and a shift in p incommensurate with v. For any irrational numbers θ_1 and θ_2, such shifts are provided by the operators

$$\bar{u} = v^{1/\theta_1}, \quad \bar{v} = u^{1/\theta_2}, \tag{11}$$

respectively. The set $\{u, v, \bar{u}, \bar{v}\}$ thus algebraically generates the whole of $\mathcal{B}$. If we further require that the set $\{u, v\}$ commutes with the set $\{\bar{u}, \bar{v}\}$ (i.e. the algebras $\mathcal{A}$ and $\bar{\mathcal{A}}$ are commutant), we get the restrictions:

$$\theta = k_1\theta_1, \quad \theta = k_2\theta_2, \tag{12}$$

k_1, k_2 being integers. This is the general solution to the problem posed in [1]. Choosing the simplest possibility $k_1 = k_2 = 1$ one gets

$$\theta_1 = \theta_2 = \theta, \quad \bar{u} = \hat{v}, \quad \bar{v} = \hat{v}, \tag{13}$$

exactly the result of Faddeev. The slight generalization provided by (12) in comparison to (13) does not appear to be essential. What the above constructive proof rather achieved was to give an elementary "raison d'être" for Faddeev's result. It simply followed from the fact that any shift in $\mathcal{H}$ can be generated out of two given ones, provided they are incommensurate.

REFERENCES

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