# MHD STEADY FLOW IN A CHANNEL WITH SLIP AT THE PERMEABLE BOUNDARIES

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The hydromagnetic steady flow of a viscous conducting fluid in a channel with slip at the permeable boundaries is investigated. Analytical solutions are constructed for the governing nonlinear boundary-value problem using perturbation method together with Padé approximation technique based on computer extended series solution and the important properties of the overall flow structure are discussed.

*Key words*: Channel flow, permeable walls, magnetic field, slip coefficient, Padé approximants.

## 1. INTRODUCTION

The study of flow of an electrically conducting fluid through a channel permeable walls not only possesses a theoretical appeal but also model many biological and engineering problems such as magnetohydrodynamics (MHD) generators, plasma studies, nuclear reactors, geothermal energy extraction, the boundary layer control in the field of aerodynamics, blood flow problems, etc. A survey of MHD studies in the technological fields can be found in Moreau ([15]). Furthermore, an extensive theoretical work has carried out on the hydromagnetic fluid flow in a channel under various situations e.g., Hartmann ([17]), Borkakati and Pop (1984), Makinde (2003), etc. Meanwhile, Beavers and Joseph ([5]) in their experimental work on boundary conditions at a naturally permeable wall confirmed the existence of slip at the interface separating the flow in the channel and the permeable boundaries. The importance of slip velocity on ultra-filtration performance has been well illustrated by Singh and Lawrence ([17]). Pal et al. ([16]) investigated the effect on slip on longitudinal dispersion of tracer particles in a channel bounded by porous media. The problem of laminar flow in channels of slowly varying width permeable boundaries was investigated in Makinde ([10]).

The main objective of the present paper is to study the combined effect of magnetic field and permeable walls slip velocity on the steady flow of an electrically conducting fluid in a channel of uniform width. Using Berman ([1]) similarity approach, the governing Navier-Stokes equations is reduced to a nonlinear boundary-value fourth order parameter dependent ordinary differential equation and semi-analytical and semi-numerical techniques of computer extended series solution coupled with Padé approximants is suggested and utilised for its solution, Makinde ([11]), Makinde ([14]). In the following sections, the problem is formulated, analysed and discussed.

## 2. MATHEMATICAL FORMULATION

Consider the steady laminar flow of an incompressible viscous conducting fluid in a channel with slip at the permeable boundaries under the influence of an externally applied homogeneous magnetic field. It is assumed that the fluid has small electrical conductivity and the electro-magnetic force produced is also very small. We choose a Cartesian coordinate system $(x,y)$ where $x$ lies along the center of the channel, y is the distance measured in the normal section such that $y=a$ is the channel's half width. Let $u$ and $v$ be the velocity components in the directions of $x$ and $y$ increasing respectively. Then, the continuity, Navier-Stokes equations governing the flow are:

$$\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1\partial P}{\rho\partial x}+\nu\nabla^2 u-\frac{\sigma_e B_0^2 u}{\rho}, \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1\partial P}{\rho\partial y}+\nu\nabla^2 v, \tag{3}$$

were $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, $p$ the fluid pressure, $\rho$ the fluid density, $\nu$ the kenematic viscosity coefficient, $\sigma_e$ the fluid conductivity, $B_0=(\mu_e H_0)$ the electromagnetic induction, $\mu_e$ the magnetic permeability and $H_0$ the intensity of magnetic field.

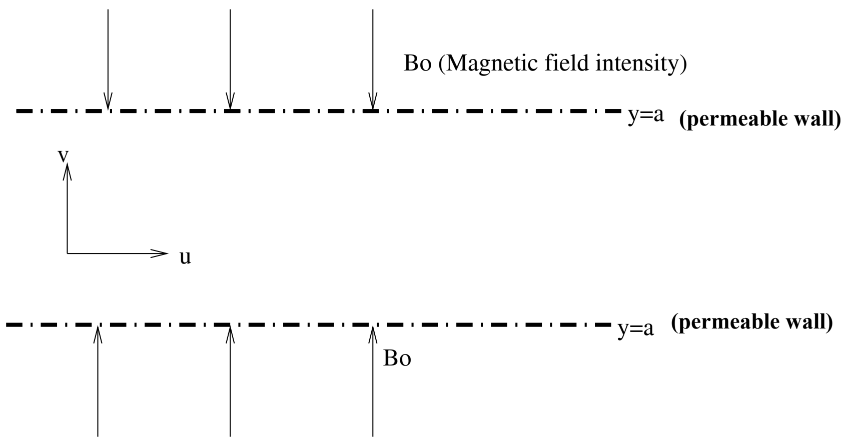


Fig. 1. Schematic diagram of the problem.

In order to complete the formulation of the problem, the boundary conditions have to be specified.
These can be written as follows:

$$\frac{\partial u}{\partial y}=0,\ v=0,\ on\ \ y=0\ \left(i.e.symmetry\right) \tag{4}$$

and

$$\mu\frac{\partial u}{\partial y}=-\beta u,\ v=V,\ on\ \ y=a. \tag{5}$$

The boundary condition (5) is the well known Beavers and Joseph (1967) slip condition, μ the dynamic viscosity coefficient, β the coefficient of sliding friction and $V$ characteristic wall suction velocity. The following dimensionless variables are introduced into Eqs. (1–5);

$$\begin{gathered} x=\frac{x'}{a},\ \ y=\frac{y'}{a},\ \ P=\frac{aP'}{\rho vV},\ u=\frac{u'}{V},\ \ v=\frac{v'}{V}, \\ R_e=\frac{V_a}{v},\ H=\sqrt{\frac{a\sigma B_0^2}{\rho V}},\ \ k=\frac{\mu}{a\beta}, \end{gathered} \tag{6}$$

and we obtain (neglecting the prime for clarity):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{7}$$

$$\mathrm{Re}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-H^2u, \tag{8}$$

$$\mathrm{Re}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial P}{\partial y}+\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{9}$$

$$\frac{\partial u}{\partial y}=0,\ v=0,\ on\ \ y=0, \tag{10}$$

$$u=-k\frac{\partial u}{\partial y},\ v=1,\ y=1, \tag{11}$$

where *Re* is the flow Reynolds number (with $Re > 0$ indicating suction and $Re < 0$ is for injection), *H* the Hartmann number and *k* is the slip parameter. We eliminate pressure *p* from the dimensionless governing Eqs. (8–9) and introduce stream-function Ψ and vorticity ω in the following manner:

$$u=\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\psi}{\partial x},\ \omega=\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}. \tag{12}$$

The governing equation (7)–(11) then become

$$\nabla^2\omega=\mathrm{Re}\left[\frac{\partial(\omega,\psi)}{\partial(x,y)}-H^2\frac{\partial^2\psi}{\partial y^2}\right],\ \omega=-\nabla^2\psi, \tag{13}$$

$$\frac{\partial\psi}{\partial y}=-k\frac{\partial^2\psi}{\partial y^2},\ \frac{\partial\psi}{\partial x}=-1,\ on\ \ y=1, \tag{14}$$

$$\frac{\partial^2\psi}{\partial y^2}=0,\ \ \frac{\partial\psi}{\partial x}=0,\ \ \ y=0, \tag{15}$$

Following Berman (1953), we introduce similarity variable:

$$\psi=xF(y),\ \ \omega=xG(y) \tag{16}$$

The dimensionless governing equations (13) together with the boundary conditions (14)–(15) in terms of similarity variables *F*, *G* can be written as

$$\frac{\partial^2 G}{\partial y^2}=\mathrm{Re}\left(G\frac{\mathrm{d}F}{\mathrm{d}y}-F\frac{\mathrm{d}G}{\mathrm{d}y}+H^2G\right),\ \ G=-\frac{\mathrm{d}^2F}{\mathrm{d}y^2}, \tag{17}$$

$$\frac{\mathrm{d}F}{\mathrm{d}y} = -k\frac{\mathrm{d}^2F}{\mathrm{d}y^2}, \quad F = -1, \quad on \quad y = 1, \tag{18}$$

$$\frac{\mathrm{d}^2F}{\mathrm{d}y^2} = 0, \quad F = 0, \quad on \quad y = 0. \tag{19}$$

Equations (17)–(19) above can be easily combined to form a nonlinear boundary-value fourth order parameter dependent ordinary differential equation.

## 3. METHOD OF SOLUTION

The non-linear nature of the equations (17)–(19) peclude its solution exactly, hence, we seek the solution in the form of power series in Re i.e.,

$$F = \sum_{i=0}^{\infty} \mathrm{Re}^i F_i, \quad G = \sum_{i=0}^{\infty} \mathrm{Re}^i G_i \tag{20}$$

We substitute the above expression (20) into (17)–(19) and collect the coefficients of like powers of Re, the resulting equations are:

**Zeroth Order**

$$\frac{d^2G_0}{dy^2} = 0, \quad G_0 = -\frac{d^2F_0}{dy^2}, \tag{21}$$

$$\frac{\mathrm{d}F_0}{\mathrm{d}y} = -k\frac{\mathrm{d}^2F_0}{\mathrm{d}y^2}, \quad F = -1, \quad on \quad y = 1, \tag{22}$$

$$\frac{\mathrm{d}^2F_0}{\mathrm{d}y^2} = 0, \quad F_0 = 0, \quad on \quad y = 0, \tag{23}$$

**Higher Order (n≥1)**

$$\frac{\partial^2 G_n}{\partial y^2} = \mathrm{Re}\left[\sum_{i=0}^{\infty}\left(G_i\frac{\mathrm{d}F_{n-i-1}}{\mathrm{d}y} - F_i\frac{\mathrm{d}G_{n-i-1}}{\mathrm{d}y}\right) + H^2G_{n-1}\right], \quad G_n = -\frac{\mathrm{d}^2F_n}{\mathrm{d}y^2}, \tag{24}$$

$$\frac{\mathrm{d}F_n}{\mathrm{d}y} = -k\frac{\mathrm{d}^2F_n}{\mathrm{d}y^2}, \quad F_n = 0, \quad on \quad y = 1, \tag{25}$$

$$\frac{\mathrm{d}^2F_n}{\mathrm{d}y^2} = 0, \quad F_n = 0, \quad on \quad y = 0, \tag{26}$$

We have written a MAPLE program that calculates successively the coefficients of the solution series. In outline, it consists of the following segments:

1. Declaration of arrays for the solution series coefficients e.g. *F=array*(0..20), *G=array*(0..20).

2. Input the leading order term and their derivatives i.e. $F_0$, $G_0$.

3.Using a MAPLE loop procedure, iterate to solve equations (24)–(26) for the higher order terms i.e $F_n$, $G_n$, $n = 1,2,3,...$ .

4. Compute the skin friction, axial pressure gradient and centerline axial velocity coefficients.

Some of the solution for stream-function and vorticity are then given as follows:

$$F = \frac{y^3 - 3(1-2k)y}{2(1+3k)} + \frac{\operatorname{Re} y\left(y^2-1\right)}{280(1+3k)^3}\left(y^4 + 3y^4k + 42H^2y^2k + 7H^2y^2 + \right. \\ \left. + 63y^2k^2H^2 + y^2 + 3y^2k - 18k - 7H^2 - 2 - 70H^2k - 147k^2H^2\right) + O\left(\operatorname{Re}^2\right), \tag{27}$$

$$G = \frac{-3y}{1+3k} - \frac{\operatorname{Re} y}{140(1+3k)^3}\left(70H^2y^2 + 21y^4 - 42H^2 - 9 - 336H^2k + \right. \\ \left. + 63y^4k + 420H^2y^2k - 63k + 630y^2k^2H^2 - 630k^2H^2\right) + O\left(\operatorname{Re}^2\right). \tag{28}$$

The non-dimensional form of the wall skin friction $t_w$ in terms of stream-function can be written as

$$t_w = -\frac{\mu x V}{a}\frac{d^2F}{dy^2}, \quad on \quad y = 1, \tag{29}$$

where μ is the dynamic viscosity. From the results in (27), we obtain the expression for $t_w$ explicitly as

$$t_\omega = \frac{\mu x V}{a}\left[\frac{3}{1+3k} + \frac{\operatorname{Re}\left(84H^2k + 28H^2 + 12\right)}{140(1+3k)^3} + \frac{\operatorname{Re}^2}{323400(1+3k)^5}\left(9240H^2 - \right.\right. \\ -4536k - 1848H^4 + 3152 - 27720k^2 + 11088H^2k - 133056k^2H^2 - \\ \left.\left. -249480H^2k^3 - 166320H^4k^3 - 127512H^4k^2 - 29568H^4k\right) + O\left(\operatorname{Re}^3\right)\right]. \tag{30}$$

From equation (8), we can determine the fluid pressure distribution. Let

$$\frac{\partial P}{\partial x} = xA, \tag{31}$$

using equations (12) and (16), we obtain

$$A = \frac{d^3F}{dy^3} - \mathrm{Re}\left[\left(\frac{dF}{dy}\right)^2 - F\frac{d^2F}{dy^2} + H^2\frac{dF}{dy}\right] \tag{32}$$

and explicity as

$$\begin{aligned} A = {} & \frac{3}{1+3k} - \frac{3\,\mathrm{Re}}{35(1+3k)^3}\big(315H^2k^3 + 315k^3 + 105k^2H^2 - 210k^2 - \\ & -42kH^2 - 21k - 14H^2 + 27\big) - \frac{\mathrm{Re}^2}{13475(1+3k)^5}\big(72765H^2k^4 + 6930H^4k^3 + \\ & +10395k^3 + 5313H^4k^2 - 2541k^2 - 23793k^2H^2 - 2506k + 1232H^4k - \\ & -8085H^2k - 693H^2 + 77H^4 - 234\big) + O\left(\mathrm{Re}^3\right). \end{aligned} \tag{33}$$

#### 4. SERIES SUMMATION AND IMPROVEMENT

We extend the solution series using a computer symbolic algebra package (MAPLE) in order to examine the effect of inertial forces, the Hartmann number and slip velocity on the flow structure. The first 15 coefficients for the above series representing the flow characteristics were obtained. We recast the series into several diagonal Padé approximants [*M*/*M*] in order to improve its usefulness. For instance, the series for the wall shear stress is transformed as follows,

$$A = \sum_{i=0}^{N} f_i\,\mathrm{Re}^i = \frac{\sum_{i=0}^{M} a_i\,\mathrm{Re}^i}{\sum_{i=0}^{M} c_i\,\mathrm{Re}^i} \tag{34}$$

where $N = M + M$ is the order of the series required for each approximant. The idea is to match the Taylor series expansion as far as possible, Makinde (1996). This method often evaluates accurately functions beyond the radius of convergence of the corresponding infinite series. It fails when we are evaluating near the zeros of the denominator of the fraction.

### 5. GRAPHICAL RESULTS AND DISCUSSION

Since the fluid is incompressible and viscous, the above mathematical analysis is very suitable for liquid. It is very important to note the an increase in the positive value of flow Reynolds number (*Re*) represents an increase in the fluid suction while an increase in the negative value of *Re* represents an increase in the fluid injection. In order to improve our results at moderately large suction and injection Reynolds number for various flow characteristics shown in the figures below, we have compared the numerical results obtained from several diagonal Padé approximants [*M*/*M*]. Figs. 2 and 3 show the fluid velocity profiles. A parabolic axial velocity profile is observed with maximum value at the channel centerline and minimum value at the walls. However, a general decrease in the magnitude of both axial and normal velocity profiles are noticed with an increase in both wall slip ($k$) and magnetic field intensity (H). The occurrence of negative axial velocity near the channel walls due to slip indicates the possibility of flow reversal near the walls. The wall skin friction with respect to flow Reynolds number are shown in the Figs. 4 and 5. The magnitude of the wall skin friction increases with suction and decreases with injection. Meanwhile, a general decrease in wall skin friction is observed with an increase in wall slip and a decrease in magnetic field intensity.

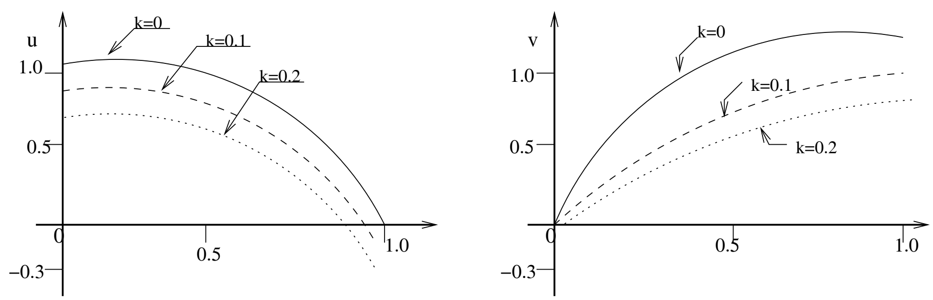


Fig. 2. Velocity profiles for different values of $k$; $Re$ = 1.0.

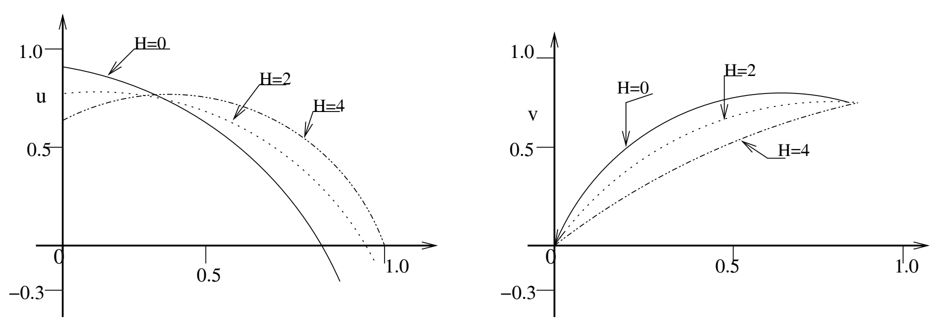


Fig. 3. Velocity profiles for different values of $H$; $Re$ = 1.0.

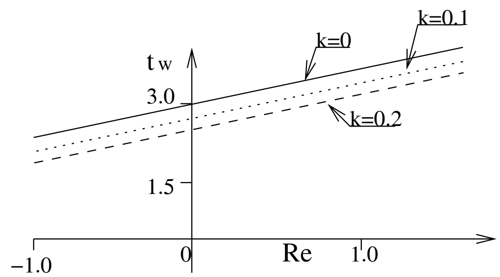


Fig. 4. Wall Skin Friction for different values of *k, H* = 0.5.

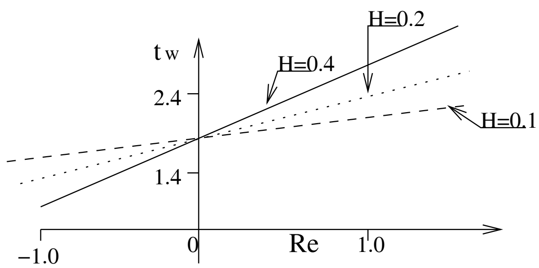


Fig. 5. Wall Skin Friction for different values of *H, k*= 0.1.

## CONCLUSION

We investigated the combined effects of wall slip and magnetic field on the steady flow of conducting viscous incompressible fluid in a channel with permeable boundaries. Our results revealed that the fluid velocity is reduced by both magnetic field and wall slip. We also noticed the presence of flow reversal near the wall due to wall slip. Generally, wall skin friction increases with suction and decreases with injection, however, both wall slip and magnetic field also have great influence on wall skin friction.

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