ON ANHARMONIC OSCILLATORS

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Received May 17, 2005

The exact solution is derived for the classical cubic anharmonic oscillator, and the first-order terms are computed in the perturbation series of the anharmonic correction. The self-consistent harmonic approximation is then employed for quartic oscillators, and its use for higher-order anharmonic oscillators is indicated.

There is a huge literature on anharmonic oscillators, both quantum and classical [1, 2]. Exact solutions are known for classical cubic and quartic anharmonic oscillators with and without dissipation, [3, 4] and detailed studies have been performed for forced classical oscillator with higher-order anharmonicities [5]. We present here a simple derivation of the exact solution for the classical cubic oscillator, and the first-order terms in the corresponding series expansion in powers of the anharmonicity.

Let $T = m\dot{u}^2/2$ be the kinetic energy and

$$U = \frac{1}{2}m\omega^2u^2 + \frac{1}{3}m\omega^2au^3 \tag{1}$$

the potential energy of a cubic anharmonic oscillator of mass m, frequency ω and anharmonicity parameter $a > 0$. The energy conservation gives

$$\dot{u}^2 = \frac{2}{m}(E-U) = \omega^2\left(x^2 - u^2 - \frac{2}{3}au^3\right), \tag{2}$$

for this oscillator, where $E = m\omega^2x^2/2 > 0$ is the energy. For $x^2 > 1/3a^2$ the velocity in (2) vanishes for $u_1 > 0$ and the motion is infinite for $u < u_1$. For $x^2 < 1/3a^2$ the velocity in (2) vanishes for $u_3 < u_2 < u_1$ and the motion is infinite for $u < u_3$ and finite for $u_2 < u < u_1$. For this finite motion (2) can also be written as $\dot{u}^2 = (2a\omega^2/3)(u_1-u)(u-u_2)(u-u_3)$, and the integral of motion reads

$$\int_{u_2}^{u}\frac{\mathrm{d}y}{\sqrt{(u_1-u)(u-u_2)(u-u_3)}} = \sqrt{2a/3}\,\omega t, \tag{3}$$

for $u_2 < u < u_1$ and the initial conditions $u = u_2$, $\dot{u} = 0$ for $t = 0$. The integral in (3) can be expressed by means of the elliptic function of the first kind $F(\varphi, k)$ by introducing $\sin\alpha = [(u_1 - u_3)(y - u_2)/(u_1 - u_2)(y - u_3)]^{1/2}$. [6] (p. 219, 3.131(5)). We obtain

$$F(\varphi, k) = \int_0^{\varphi} \frac{\mathrm{d}\alpha}{\sqrt{1 - k^2 \sin^2\alpha}} = \tau, \tag{4}$$

where

$$\sin\varphi = \sqrt{\frac{u_1 - u_3}{u_1 - u_2}} \sqrt{\frac{u - u_2}{u - u_3}}, \tag{5}$$

the modulus of the elliptic function is given by

$$k^2 = \frac{u_1 - u_2}{u_1 - u_3}, \tag{6}$$

and the dimensionless time τ is given by

$$\tau = \frac{1}{2}\sqrt{u_1 - u_3}\sqrt{2a/3}\,\omega t. \tag{7}$$

From (5) we obtain the solution

$$u = \frac{u_2 - k^2 u_3 \sin^2\varphi}{1 - k^2 \sin^2\varphi}, \tag{8}$$

or, making use of the Jacobi sine-amplitude $snF = sn\tau = sin\varphi$, [6] (p. 910), we get

$$u = \frac{u_2 - k^2 u_3 sn^2\tau}{1 - k^2 sn^2\tau}. \tag{9}$$

This is the exact solution of the cubic anharmonic oscillator. It oscillates between u_2 for $\varphi = n\pi$, and u_1 for $\varphi = (2n+1)\pi/2$, n being an integer. The period T of the motion is given by

$$K = \int_0^{\pi/2} \frac{\mathrm{d}\alpha}{\sqrt{1 - k^2 \sin^2\alpha}} = \frac{1}{4}\sqrt{2a(u_1 - u_3)/3}\,\omega T, \tag{10}$$

where K is the complete elliptic function. A similar exact solution can also be obtained for the quartic anharmonic oscillator [3].

It is worth noting that the infinite motion proceeds in a finite time. Indeed, let $u_1 > 0$ and $u_{2,3} = A \pm iB$ for $x^2 > 1/3a^2$. Then, the integral (3) becomes $F(\varphi, k) = \sqrt{2aD/3}\,\omega t$, where $k^2 = [1 + (u_1 - A)/D]/2$, $D = [(u_1 - A)^2 + B^2]^{1/2}$ and

$u = u_1 - D\tan^2(\varphi/2)$. One can see that $u \to -\infty$ for $\varphi \to \pi$, which means that motion goes to infinite in a finite time T_1 given by $2K = \sqrt{2aD/3}\omega T_1$.

It is often useful to have the solution of the cubic oscillator in the limit of the weak anharmonicity. In order to get this limit we need the approximate roots $u_{1,2,3}$ of the equation $x^2 - u^2 - \frac{2}{3}au^3 = 0$ in this limit. Introducing $z = 2au/3$ this equation becomes $z^3 + z^2 - \varepsilon^2 = 0$, where the perturbation parameter is $\varepsilon = 2ax/3$. It is easy now to solve perturbationally this equation; its solutions are given by $z_{1,2} = \pm\varepsilon(1 \mp \varepsilon/2 + \varepsilon^2/4)$ and $z_3 = -1 + \varepsilon^2$, or

$$u_1 = x(1 - \varepsilon/2 + \varepsilon^2/4), \quad u_2 = -x(1 + \varepsilon/2 + \varepsilon^2/4), \quad u_3 = -\frac{x}{\varepsilon}(1 - \varepsilon^2). \tag{11}$$

Making use of these expansions in powers of ε we obtain $k^2 = 2\varepsilon(1 - \varepsilon + 11\varepsilon^2/4)$ and $K = \pi(1 + \varepsilon/2 + \varepsilon^2/16)/2$. Using the same expansions in (10) we get the well-known second-order shift

$$\Omega = 2\pi/T = \omega(1 - 15\varepsilon^2/16) = \omega(1 - 5a^2x^2/12) \tag{12}$$

in frequency. Similarly, the angle φ is obtained from (4) as

$$\varphi = \frac{1}{2}\Omega t + \frac{\varepsilon}{4}\sin\Omega t + \frac{\varepsilon^2}{64}\sin 2\Omega t, \tag{13}$$

and the oscilaltor coordinate

$$u = -x\cos\Omega t - \frac{x\varepsilon}{4}(3 - \cos 2\Omega t) - \frac{x\varepsilon^2}{2}(2 - \frac{17}{8}\cos\Omega t + 2\cos 2\Omega t - \frac{11}{8}\cos 3\Omega t)\,. \tag{14}$$

It is worth noting that the renormalized frequency Ω appears in these expansions instead of the original frequency ω. All these expansions in powers of ε can also be obtained directly by solving perturbationally the equation of motion $\ddot{u} = -\omega^2(u + au^2)$, including the frequency renormalization.

Similar results can be obtained by the so-called self-consistent harmonic approximation. For instance, u^3 in (1) can be approximated by

$$u^3 = \frac{3}{2}(Au + Bu^2), \tag{15}$$

where $A = \overline{u^2}$, $B = \overline{u}$, the averages being taken over the motion and the coefficients 3/2 in (15) being chosen such as $\overline{u^3} = 3\overline{u}\,\overline{u^2}$. It is easy to see that the oscillator becomes then a displaced one, with the frequency $\Omega = \omega(1 + aB)^{1/2}$; the solution is $u = u_0\cos\Omega t - C$, where u_0 is an amplitude and $C = aA/2(1 + aB)$.

The condition $\overline{u^3} = 3\overline{u}\,\overline{u^2}$ is fulfilled only for small values of C, as expected ($\overline{u} = -C$, $u^2 = u_0^2/2 + C^2$, $u^3 = -3u_0^2 C/2$). It follows $C \cong au_0^2/4$ and $A \cong u_0^2/2$, $B = -C \cong -au_0^2/4$. The frequency shift is then given by

$$\Omega = \omega(1 + aB)^{1/2} \cong \omega(1 - a^2 u_0^2/8), \tag{16}$$

which compares rather satisfactorily with the exact result (12) $\Omega = \omega(1 - 5a^2 u_0^2/12)$, where $u_0 = x$.

A similar decomposition $u^4 = 3Au^2/2$ holds for the quartic anharmonicity in the potential energy $U = (m\omega^2/2)(u^2 + bu^4/2)$, where $A = \overline{u^2}$ and b is the anharmonic parameter. The condition $\overline{u^4} = 3(\overline{u^2})^2/2$ is then fulfilled exactly ($\overline{u^2} = A = u_0^2/2$, $\overline{u^4} = 3u_0^4/8$ for solution $u = u_0 \cos\Omega t$ and frequency $\Omega = \omega(1 + 3Ab/4)^{1/2}$). It follows the frequency shift given by

$$\Omega = \omega(1 + 3Ab/4)^{1/2} \cong \omega(1 + 3bu_0^2/16), \tag{17}$$

for small b, which again compares well with the exact result [1] $\Omega = \omega(1 + 3bu_0^2/8)$. Is is worth noting that the frequency shift is quadratic in amplitude for cubic anharmonicities, and linear for quartic anharmonicities.

Similar approximations can be used approximately for higher-order anharmonicities, without any loss of qualitative behaviour, and a satisfactory representation of the quantitative results.

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