PARTICLE ACTIONS IN THE SUPERSPACE, SQUARE ROOT OPERATORS AND QUARTIONS

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In this work the physical and mathematical interpretation of the square root quantum operators is analyzed from the point of view of the theoretical group. To this end, we considered the relativistic geometrical action of a particle in the superspace in order to quantize it and to obtain the spectrum of physical states with the Hamiltonian remaining in the natural square root form. The generators of the group SO(3,1) are introduced and the quantization of this model is performed completely. The obtained spectrum of physical states, with the Hamiltonian operator in square root form, is compared with the spectrum obtained with the Hamiltonian in the standard form (*i.e*: quadratic in momenta). We show that the only states that the square root Hamiltonian can operate correspond to the representations with the lowest weights $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$.

Key words: superparticle, Hamiltonian formulation, relativistic theories.

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1. INTRODUCTION AND SUMMARY

The problem of the square root operator in theoretical physics, in particular in Quantum Mechanics and QFT, is well known [1]. Several attempts to avoid the problem of locality and quantum interpretation of Hamiltonian as square root operator were described in the literature: differential pseudoelliptic operators, several expansions of the fractional-exponential operator, etc. [2]. The main characteristic of all these attempts is to eliminate the square root of the Hamiltonian. In this manner, the set of operators into the square root operates freely on the physical states, paying the price to lose locality and quantum interpretation of the spectrum of a well formulated field theory.

Recently [3, 4, 5], several works have appeared where the problem of the quantization procedure and the square root operators was carefully analyzed. In these articles it was demonstrated for different simple problems (harmonic oscillator, massive particle on hyperboloid, etc.) that the spectrum changes drastically if the hamiltonian operator has the square root form or does not: the explicit computation of the Casimir operator of the symmetry group puts this difference in evidence.

In this work, strongly motivated for the several fundamental reasons described above, we considered the simple model of superparticle of Volkov and Pashnev [6], that is the type G4 in the description of Casalbuoni [7, 8], in order to quantize it and to obtain the spectrum of physical states with the Hamiltonian remaining in the natural square root form. To this end, we used the Hamiltonian formulation described by Lanczos in [9] and the inhomogeneous Lorentz group as a representation for the obtained physical states [10, 11, 12]. The quantization of this model is performed completely and the obtained spectrum of physical states, with the Hamiltonian operator in its square root form, is compared with the spectrum obtained with the hamiltonian in the standard form (*i.e.*: quadratic in momenta). We show that the only states that the square root Hamiltonian can operate correspond to the representations with the lowest weights $\lambda_{1,2} = \frac{1}{4}$ and $\lambda_{1,2} = \frac{3}{4}$. In this manner, we also show that the superparticle relativistic actions as of Ref. [6] are a good geometrical and natural candidate to describe quartionic states [13, 14, 15] (semions).

2. THE SUPERPARTICLE MODEL

In the superspace the coordinates are given not only by the spacetime x_μ coordinates, but also for anticommuting spinors θ^α and $\overline{\theta}^{\dot{\alpha}}$. The resulting metric [6, 16] must be invariant to the action of the Poincare group, and also invariant to the supersymmetry transformations

$$x'_\mu = x_\mu + i\left(\theta^\alpha (\alpha)_{\alpha\dot{\beta}} \overline{\xi}^{\dot{\beta}} - \xi^\alpha (\alpha)_{\alpha\dot{\beta}} \overline{\theta}^{\dot{\beta}}\right);\quad \theta'^\alpha = \theta^\alpha + \xi^\alpha;\quad \overline{\theta}'^{\dot{\alpha}} = \overline{\theta}^{\dot{\alpha}} + \overline{\xi}^{\dot{\alpha}}$$

The simplest super-interval that obeys the requirements of invariance given above is the following

$$\mathrm{d}s^2 = \omega^\mu \omega_\mu + a\omega^\alpha \omega_\alpha - a^* \omega^{\dot{\alpha}} \omega_{\dot{\alpha}} \tag{1}$$

where

$$\omega_\mu = \mathrm{d}x_\mu - i\left(\mathrm{d}\theta\sigma_\mu \overline{\theta} - \theta\sigma_\mu \mathrm{d}\overline{\theta}\right);\quad \omega^\alpha = \theta^\alpha;\quad \omega^{\dot{\alpha}} = \overline{\theta}^{\dot{\alpha}}$$

are the Cartan forms of the group of supersymmetry [16].

The spinorial indexes are related as follows

$$\theta^{\alpha} = \varepsilon^{\alpha\beta}\theta_{\beta}; \quad \theta_{\alpha} = \theta^{\beta}\varepsilon_{\beta\alpha}; \quad \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}; \quad \varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}; \quad \varepsilon_{12} = \varepsilon^{12} = 1$$

and of analog manner for the spinors with punctuated indexes. The complex constants a and a^* in the line element (1) are arbitrary. This arbitrarity for the choice of a and a^* is constrained by the invariance and reality of the interval (1).

As we have extended our manifold to include fermionic coordinates, it is natural to extend also the concept of trajectory of point particle to the superspace. To do this we take the coordinates $x(\tau)$, $\theta(\tau)$ and $\bar{\theta}^{\dot{\alpha}}(\tau)$ depending on the evolution parameter τ. Geometrically, the function action that will describe the world-line of the superparticle is

$$S = -m\int_{\tau 1}^{\tau 2} \mathrm{d}\tau \sqrt{\overset{\circ}{\omega}_{\mu}\overset{\circ}{\omega}^{\mu} + a\dot{\theta}^{\alpha}\dot{\theta}_{\alpha} - a^{*}\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}} = \int_{\tau 1}^{\tau 2} \mathrm{d}\tau L\left(x, \theta, \bar{\theta}\right) \tag{2}$$

where $\overset{\circ}{\omega}_{\mu} = \dot{x}_{\mu} - i\left(\dot{\theta}\sigma_{\mu}\bar{\theta} - \theta\sigma_{\mu}\dot{\bar{\theta}}\right)$ and the upper point means derivative with respect to the parameter τ, as usual.

The momenta, canonically conjugated to the coordinates of the superparticle, are

$$\mathcal{P}_{\mu} = \partial L/\partial x^{\mu} = \left(m^2/L\right)\overset{\circ}{\omega}_{\mu}$$

$$\mathcal{P}_{\alpha} = \partial L/\partial\dot{\theta}^{\alpha} = i\mathcal{P}_{\mu}\left(\sigma^{\mu}\right)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} + \left(m^2 a/L\right)\dot{\theta}_{\alpha}$$

$$\mathcal{P}_{\dot{\alpha}} = \partial L/\partial\dot{\bar{\theta}}^{\dot{\alpha}} = i\mathcal{P}_{\mu}\theta^{\alpha}\left(\sigma^{\mu}\right)_{\alpha\dot{\alpha}} - \left(m^2 a/L\right)\dot{\bar{\theta}}_{\dot{\alpha}} \tag{3}$$

It is difficult to study this system in the Hamiltonian formalism framework because of the constraints and the nullification of the Hamiltonian. As the action (2) is invariant under reparametrizations of the evolution parameter

$$\tau \to \tilde{\tau} = f\left(\tau\right)$$

one way to overcome this difficulty is to make the dynamic variable x_0 the time. For this, it is sufficient to use the chain rule of derivatives (with special care of the anticommuting variables)[1] and to write the action in the form

$$S = -m\int_{\tau 1}^{\tau 2} \dot{x}_0 \mathrm{d}\tau \sqrt{\left[1 - iW^{0}_{,0}\right]^2 - \left[x^i - W^{i}_{,0}\right]^2 + a\dot{\theta}_a\dot{\theta}^{\alpha} - a^{*}\dot{\bar{\theta}}_{\dot{\alpha}}\dot{\bar{\theta}}^{\dot{\alpha}}}$$

[1] We take the Berezin convention for the Grassmannian derivatives: $\delta F(\theta) = \frac{\partial F}{\partial\theta}\delta\theta$.

where the $W^{\mu}_{,0}$ was defined by

$$\overset{\circ}{\omega}{}^{0} = \dot{x}^{0}\left[1 - iW^{0}_{,0}\right]$$

$$\overset{\circ}{\omega}{}^{i} = \dot{x}^{0}\left[x^{i}_{,0} - iW^{i}_{,0}\right]$$

whence $x_0(\tau)$ turns out to be the evolution parameter

$$S = -m\int_{x_0(\tau_1)}^{x_0(\tau_2)} \mathrm{d}x_0 \sqrt{\left[1 - iW^{0}_{,0}\right]^2 - \left[x^{i} - W^{i}_{,0}\right]^2 + a\dot{\theta}^{\alpha}\dot{\theta}_{\alpha} - a^{*}\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}} \equiv \int \mathrm{d}x_0 L$$

Physically this parameter (we call it the dynamical parameter) is the time measured by an observer's clock in the rest frame.

Therefore, the invariance of a theory with respect to the invariance of the coordinate evolution parameter means that one of the dynamic variables of the theory ($x_0(\tau)$ in this case) becomes the observed time with the corresponding non-zero Hamiltonian

$$\begin{aligned} H &= \mathcal{P}_{\mu}\dot{x}^{\mu} + \Pi^{\alpha}\dot{\theta}_{\alpha} + \Pi^{\dot{\alpha}}\dot{\theta}_{\alpha} - L = \\ &= \sqrt{m^2 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^{\alpha}\Pi_{\alpha} - \frac{1}{a^{*}}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)} \end{aligned} \tag{4}$$

where

$$\Pi_{\alpha} = \mathcal{P}_{\alpha} + i\mathcal{P}_{\mu}\left(\sigma^{\mu}\right)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}$$

$$\Pi_{\dot{\alpha}} = \mathcal{P}_{\dot{\alpha}} - i\mathcal{P}_{\mu}\theta^{\alpha}\left(\sigma^{\mu}\right)_{\alpha\dot{\alpha}}$$

That gives the well-known mass shell condition and losing, from the quantum point of view, the operatibility of the Hamiltonian.

In the work [6], where this type of superparticle action was explicitly presented, the problem of nullification of Hamiltonian was avoided in the standard form. This means that the analog to a mass shell condition (4) in superspace was introduced by mean of a multiplier (einbein) to obtain a new Hamiltonian

$$H = \frac{\varkappa}{2}\left\{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^{\alpha}\Pi_{\alpha} - \frac{1}{a^{*}}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)\right\} \tag{5}$$

With this Hamiltonian it is clear that in order to perform the quantization of the superparticle the problems disappear: $\mathcal{P}_0$ is restored into the new Hamiltonian, and the square root is eliminated. The full spectrum from this Hamiltonian was obtained in [6] where the quantum Hamiltonian referred to the center of mass was

$$H_{cm} = m^2 - M^2 + \frac{2^{3/2} M}{|a|}\left[1 - (\sigma_0)_{\alpha\dot{\beta}}\, \bar{s}^{\dot{\beta}} s^{\alpha}\right] \tag{6}$$

with the mass distribution of the physical states being the following: two scalar supermultiplets $M_{1s} = \frac{2^{1/2}}{|a|} + \sqrt{\frac{2}{|a|} + m^2}$ and $M_{2s} = \sqrt{\frac{2}{|a|} + m^2} - \frac{2^{1/2}}{|a|}$; and one vector supermultiplet $M_v = m$.

We will show in this report that it is possible, in order to quantize the superparticle action, to remain the Hamiltonian in the square root form. As it is very obvious, in the form of square root the Hamiltonian operator is not linearly proportional to the operator $n_s = \bar{s}^{\dot{\beta}} s^{\alpha}$. The Fock construction for the Hamiltonian into the square root form agrees formally with the description given above for the reference [6], but the operability of this Hamiltonian is over basic states with lowest helicities $\lambda = 1/4, 3/4$. This means that the superparticle Hamiltonian preserving the square root form operates over physical states of particles with fractionary quantum statistics and fractional spin (quartions).

3. HAMILTONIAN TREATMENT IN LANCZO'S FORMULATION

In order to solve our problem from the dynamical and quantum mechanical point of view, we will use the formulation given in [9, 17]. This hamiltonian formulation for dynamical systems was proposed by C. Lanczos and allows us to preserve the square root form in the new Hamiltonian. We start from expression (4)

$$H = \sqrt{m^2 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a}\Pi^{\alpha}\Pi_{\alpha} - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}$$

if

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} \equiv \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = g\left(\mathcal{P}_0, \mathcal{P}_i, \Pi_{\alpha}, \Pi_{\dot{\alpha}}, x_0, x_i, \theta_{\alpha}, \bar{\theta}_{\dot{\alpha}}\right)$$

with the arbitrary function g given by

$$g = \frac{\sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a}\Pi^{\alpha}\Pi_{\alpha} - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}}{\sqrt{m^2 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a}\Pi^{\alpha}\Pi_{\alpha} - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)} + \mathcal{P}_0} \tag{7}$$

the new hamiltonian $\mathcal{H}$ takes the required "square root" form

$$\mathcal{H} \equiv g\left(H + \mathcal{P}_0\right) = \sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot\alpha}\Pi_{\dot\alpha}\right)} \tag{8}$$

and the variable $\mathcal{P}_0$ is clearly identified by the dynamical expression

$$\frac{\mathrm{d}P_0}{\mathrm{d}\tau} = -g\frac{\partial\mathcal{H}}{\partial x^0} \quad \text{or} \quad \frac{\mathrm{d}\mathcal{P}_0}{\mathrm{d}\tau} = -\frac{\partial\mathcal{H}}{\partial t} \tag{9}$$

This means that $\mathcal{P}_0 = -H + const.$

In order to make an analysys of the dynamics of our problem, we can compute the Poisson brackets between all the canonical variables and its conjugate momentum [6, 7, 8]

$$\dot{\mathcal{P}}_\mu = \left\{\mathcal{P}_\mu, \mathcal{H}\right\}_{pb} = 0 \tag{10}$$

$$\dot\theta^\alpha = \left\{\theta^\alpha, \mathcal{H}\right\}_{pb} = \frac{1}{a}\frac{\Pi^\alpha}{\mathcal{H}} \tag{11}$$

$$\dot{\bar\theta}^{\dot\alpha} = \left\{\bar\theta^{\dot\alpha}, \mathcal{H}\right\}_{pb} = -\frac{1}{a^*}\frac{\Pi^{\dot\alpha}}{\mathcal{H}} \tag{12}$$

$$\dot x_\mu = \left\{x_\mu, \mathcal{H}\right\}_{pb} = \frac{1}{\mathcal{H}}\left\{\mathcal{P}_\mu + \frac{i}{a}\Pi^\alpha(\sigma_\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta} + \frac{i}{a^*}\theta^\alpha(\sigma_\mu)_{\alpha\dot\beta}\Pi^{\dot\beta}\right\} \tag{13}$$

$$\dot\Pi_\alpha = \left\{\Pi_\alpha, \mathcal{H}\right\}_{pb} = \frac{2i}{a^*\mathcal{H}}P_{\alpha\dot\beta}\Pi^{\dot\beta} \tag{14}$$

$$\dot\Pi_{\dot\alpha} = \left\{\Pi_{\dot\alpha}, \mathcal{H}\right\}_{pb} = \frac{-2i}{a\mathcal{H}}\Pi^\beta\mathcal{P}_{\beta\dot\alpha} \tag{15}$$

where $\mathcal{P}_{\alpha\dot\beta} \equiv \mathcal{P}_\mu(\sigma^\mu)_{\alpha\dot\beta}$. From the above expressions the set of classical equations to solve is easily seen

$$\ddot\Pi_\alpha = -\left(\frac{4\mathcal{P}^2}{|a|^2\mathcal{H}^2}\right)\dot\Pi_{\dot\alpha} \tag{16}$$

$$\ddot\Pi_{\dot\alpha} = -\left(\frac{4\mathcal{P}^2}{|a|^2\mathcal{H}^2}\right)\dot\Pi_\alpha \tag{17}$$

Assigning $\frac{4\mathcal{P}^2}{|a|^2\,\mathcal{H}^2} \equiv \omega^2$, and having account for $\Pi^+_{\dot\alpha} = -\Pi_{\dot\alpha}$, the solution to the equations (16) and (17) takes the form

$$\begin{aligned}\Pi_\alpha &= \xi_\alpha e^{i\omega\tau} + \eta_\alpha e^{-i\omega\tau} \\ \Pi_{\dot\alpha} &= -\bar\eta_{\dot\alpha} e^{i\omega\tau} - \bar\xi_{\dot\alpha} e^{-i\omega\tau}\end{aligned} \tag{18}$$

By means of the substitution of above solutions into (14) and (15), we find the relation between ξ_α and η_α

$$\eta_\alpha = \left(\frac{2}{a^*\mathcal{H}\omega}\right)\mathcal{P}_{\alpha\dot\beta}\bar\xi^{\dot\beta}$$

From eqs. (18) and above we obtain

$$\Pi_\alpha = \xi_\alpha e^{i\omega\tau} + \left(\frac{2}{a^*\mathcal{H}\omega}\right)\mathcal{P}_{\alpha\dot\beta}\bar\xi^{\dot\beta} e^{-i\omega\tau} \tag{19}$$

$$\Pi_{\dot\alpha} = -\left(\frac{2}{a\mathcal{H}\omega}\right)\xi^\beta \mathcal{P}_{\beta\dot\alpha} e^{i\omega\tau} - \bar\xi_{\dot\alpha} e^{-i\omega\tau} \tag{20}$$

where we used the fact that the constant two-component spinors ξ_α verify $\bar\xi_{\dot\alpha} = \xi_\alpha^+$. Integrating expressions (11) and (12), we obtain explicitly the following

$$\theta_\alpha = \zeta_\alpha - \frac{i}{a\mathcal{H}\omega}\left[\xi_\alpha e^{i\omega\tau} - \frac{2}{a^*\mathcal{H}\omega}\mathcal{P}_{\alpha\dot\beta}\bar\xi^{\dot\beta} e^{-i\omega\tau}\right] \tag{21}$$

$$\bar\theta_{\dot\alpha} = \bar\zeta_{\dot\alpha} + \frac{i}{a^*\mathcal{H}\omega}\left[-\frac{2}{a\mathcal{H}\omega}\xi^\beta \mathcal{P}_{\beta\dot\alpha} e^{i\omega\tau} + \bar\xi_{\dot\alpha} e^{-i\omega\tau}\right] \tag{22}$$

where ζ_α and $\bar\zeta_{\dot\alpha} = \zeta_\alpha^+$ are two-component constant spinors.

Analogically, from expression (13), we obtain x_μ in explicit form

$$\begin{aligned} x_\mu = q_\mu - \frac{1}{\mathcal{H}}\left[\mathcal{P}_\mu - \frac{\omega\mathcal{H}}{\mathcal{P}^2}\left(\xi\sigma_\mu\bar\xi\right)\right]\tau + \frac{1}{\mathcal{H}\omega}\left[\frac{1}{a}e^{i\omega\tau}\left(\xi\sigma_\mu\bar\zeta\right) + \frac{1}{a^*}e^{-i\omega\tau}\left(\zeta\sigma_\mu\bar\xi\right)\right] + \\ + \frac{\mathcal{P}_\mu}{2\mathcal{P}^2}\left[\zeta^\alpha\xi_\alpha e^{i\omega\tau} - \bar\zeta^{\dot\alpha}\bar\xi_{\dot\alpha} e^{-i\omega\tau}\right] \end{aligned} \tag{23}$$

4. QUANTIZATION

Because of the correspondence between classical and quantum dynamics, the Poisson brackets between coordinates and canonical impulses are transformed into quantum commutators and anti-commutators

$$\left[x_\mu, \mathcal{P}_\mu\right] = i\left\{x_\mu, \mathcal{P}_\mu\right\}_{pb} = -ig_{\mu\nu}$$

$$\begin{aligned}\left\{\theta^{\alpha},\mathcal{P}_{\beta}\right\}&=i\left\{\theta^{\alpha},\mathcal{P}_{\beta}\right\}_{pb}=-i\delta^{\alpha}_{\beta}\\ \left\{\theta^{\dot\alpha},\mathcal{P}_{\dot\beta}\right\}&=i\left\{\theta^{\dot\alpha},\mathcal{P}_{\dot\beta}\right\}_{pb}=-i\delta^{\dot\alpha}_{\dot\beta}\end{aligned} \tag{24}$$

and the new Hamiltonian (8) operates quantically as follows

$$\sqrt{m^2-\mathcal{P}_0\mathcal{P}^0-\left(\mathcal{P}_i\mathcal{P}^i+\frac{1}{a}\Pi^{\alpha}\Pi_{\alpha}-\frac{1}{a^*}\Pi^{\dot\alpha}\Pi_{\dot\alpha}\right)}\left|\Psi\right\rangle=0 \tag{25}$$

where $\left|\Psi\right\rangle$ are the physical states. From the (anti)commutation relations (24) it is possible to obtain easily the commutators between the variables ξ_{α}, $\bar\xi_{\dot\alpha}$, ζ_{α}, $\bar\zeta_{\dot\alpha}$, q_{μ}, $\mathcal{P}_{\mu}$

$$\left\{\xi_{\alpha},\bar\xi_{\dot\alpha}\right\}=-\mathcal{P}_{\alpha\dot\alpha}\qquad\left\{\zeta_{\alpha},\bar\zeta_{\dot\alpha}\right\}=-\left(\frac{1}{2\mathcal{P}^2}\right)\mathcal{P}_{\alpha\dot\alpha}\qquad\left[q_{\mu},\mathcal{P}_{\mu}\right]=-ig_{\mu\nu} \tag{26}$$

To obtain the physical spectrum we use the relations given by (26) the hamiltonian taking the following form

$$\mathcal{H}=\sqrt{m^2-\mathcal{P}_0\mathcal{P}^0-\mathcal{P}_i\mathcal{P}^i-\frac{2^{3/2}\sqrt{(\mathcal{P}_{\mu})^2}}{|a|}-\frac{2^{3/2}}{|a|\sqrt{(\mathcal{P}_{\mu})^2}}\xi^{\alpha}\mathcal{P}_{\alpha\dot\beta}\bar\xi^{\dot\beta}} \tag{27}$$

Passing to the center of mass of the system, and defining new operators $s_{\alpha}=(1/\sqrt{M})\xi_{\alpha}$, $\bar s_{\dot\alpha}=(1/\sqrt{M})\bar\xi_{\dot\alpha}$, $d_{\alpha}=\sqrt{2M}\zeta_{\alpha}$, $\bar d_{\dot\alpha}=\sqrt{2M}\,\bar\zeta_{\dot\alpha}$, where $M=\mathcal{P}_0$, $\mathcal{H}_{cm}$ is

$$\mathcal{H}_{cm}=\sqrt{m^2-M^2+\frac{2^{3/2}M}{|a|}\left[1-\left(\sigma_0\right)_{\alpha\dot\beta}\bar s^{\dot\beta}s^{\alpha}\right]} \tag{28}$$

being

$$\left\{s_{\alpha},\bar s_{\dot\alpha}\right\}=-\left(\sigma_0\right)_{\alpha\dot\alpha}\qquad\left\{d_{\alpha},\bar d_{\dot\alpha}\right\}=-\left(\sigma_0\right)_{\alpha\dot\alpha} \tag{29}$$

the anti-commutation relations of the operators s_{α}, $\bar s_{\dot\alpha}$, d_{α}, $\bar d_{\dot\alpha}$. Now the question is: how the square-root $\mathcal{H}$ Hamiltonian given by expression (28) does operate on a given physical state? The problem of locality and interpretation of the operator like (25) is very well known. Several attemps to avoid these problems were given in the literature [1, 2]: differential pseudoelliptic operators, several expansions of the fractional-exponential operator, etc. The main characteristic of all these attempts is to eliminate the square root of the Hamiltonian. In this manner, the set of operators into the square root operates freely on the physical states, paying the price to lose locality and quantum interpretation of the spectrum of a well possessed field theory.

Our plan is: to take the square root to a bispinor in order to introduce the physical state into the square root Hamiltonian. In the next section we will perform the square root of a bispinor and obtain the mass spectrum given by the Hamiltonian $\mathcal{H}$.

5. MASS SPECTRUM AND SQUARE ROOT OF A BISPINOR

The square root from a spinor was extracted in Kharkov in 1965 by S. S. Sannikov [11]. Taking the square root from a spinor was also performed by P. A. M. Dirac [12] in 1971.

We know that the group $SL(2,\mathbb{C})$ is locally isomorph to $SO(3, 1)$, and $SL(2, R)$ is locally isomorph to $SO(2, 1)$. For instance, the generators of the group $SO(3, 1)$ for our case can be constructed from the usual operators a, a^+ (or q and p) in the following manner. We start from an irreducible unitary infinite dimensional representation of the Heisenberg-Weyl group, which is realized in the Fock of spaces of states of one-dimensional quantum oscillator [14, 15, 10]. Creation operators and annihilation operators of these states obey the conventional commutation relations $\left[a^+, a\right] = 1 \quad \left[a, a\right] = \left[a^+, a^+\right] = 0.$ To describe this representation to the Lorentz group one may also use the coordinate-momentum realization (q, $p = -i\dfrac{\partial}{\partial q}$) of the Heisenberg algebra, which relates to the a, a^+ realization by the formulas

$$a^+ = \frac{q - ip}{\sqrt{2}} \qquad a^+ = \frac{q + ip}{\sqrt{2}} \tag{30}$$

as usual. Let us introduce the spinors

$$L_\alpha = \begin{pmatrix} a_1 \\ a_1^+ \end{pmatrix} \quad L_{\dot\alpha} = \begin{pmatrix} a_2 \\ a_2^+ \end{pmatrix} \tag{31}$$

The commutation relations take the form

$$\left[L_\alpha, L_\beta\right] = i\varepsilon_{\alpha\beta} \qquad \left[L_{\dot\alpha}, L_{\dot\beta}\right] = i\varepsilon_{\dot\alpha\dot\beta} \tag{32}$$

The generators of $SL(2,\mathbb{C})$ are easily constructed [15] from L_α and $L_{\dot\alpha}$

$$\begin{aligned} S_{\alpha\beta} &\equiv iS_\mu(\sigma^\mu)_{\alpha\beta} = \frac{1}{4}\left\{L_\alpha, L_\beta\right\} \\ S_{\dot\alpha\dot\beta} &\equiv iS_\mu(\sigma^\mu)_{\dot\alpha\dot\beta} = \frac{1}{4}\left\{L_{\dot\alpha}, L_{\dot\beta}\right\} \end{aligned} \tag{33}$$

and satisfy the commutation relation

$$\left[S_\mu, S_\nu\right] = -i\varepsilon_{\mu\nu\rho}S^\rho \tag{34}$$

Then the quantities

$$\Phi_\alpha \equiv \langle\Psi|L_\alpha|\Psi\rangle \quad \Phi_{\dot\alpha} \equiv \langle\Psi|L_{\dot\alpha}|\Psi\rangle \tag{35}$$

are the two-components of a bispinor, and $|\Psi\rangle$ is the square root of this bispinor, that is very easy to verify. Notice that the four components of the bispinor operate on the same function $|\Psi\rangle$. In terms of q the basic vectors of the representation can be written as [14, 10, 11]

$$\langle q|n\rangle = \varphi_n(q) = \pi^{-1/4}\left(2^n n!\right)^{-1/2} H_n(q) e^{-q^2/2} \tag{36}$$

$$\int \mathrm{d}q \varphi_m^*(q)\varphi_n(q) = \delta_{mn} \tag{37}$$

(where are the Hermite polynomials) and form a unitary representation of $SO(3, 1)$, and

$$|n\rangle = (n!)^{-1/2}\left(a^+\right)^n|0\rangle \tag{38}$$

the normalized basic states where the vacuum vector is annihilated by a. The Casimir operator, that is $S_\mu S^\mu$, has the eigenvalue $\lambda(\lambda-1) = -\frac{3}{16}$ (for each subgroup $ISO(2, 1)$ given by eq. (33)) and indeed corresponds to the representations with the lowest weights $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$. The wave functions which transform as linear irreducible representation of $ISO(2, 1)$, subgroup of $ISO(3, 1)$ generated by operators (33) are

$$\Psi_{1/4}(x, \theta, q) = \sum_{k=0}^{+\infty} f_{2k}(x, \theta)\varphi_{2k}(q) \tag{39}$$

$$\Psi_{3/4}(x, \theta, q) = \sum_{k=0}^{+\infty} f_{2k+1}(x, \theta)\varphi_{2k+1}(q) \tag{40}$$

We can easily seen that the hamiltonian $\mathcal{H}$ (28) operates over the states $|\Psi\rangle$, which become into $\mathcal{H}$ as its square Φ_α and $\Phi_{\dot\alpha}$. It is natural to associate, up to a proportional factor, the spinors d_α and $\bar{d}_{\dot\alpha}$ with

$$d_\alpha \to \left(\Phi_{1/4}\right)_\alpha \equiv \langle\Psi_{1/4}|L_\alpha|\Psi_{1/4}\rangle \quad \bar{d}_{\dot\alpha} \to \left(\Phi_{1/4}\right)_{\dot\alpha} \equiv \langle\Psi_{1/4}|L_{\dot\alpha}|\Psi_{1/4}\rangle \tag{41}$$

and of analog manner the spinors s_α and $\bar{s}_{\dot\alpha}$ with

$$s_\alpha \to (\Phi_{3/4})_\alpha \equiv \langle \Psi_{3/4} | L_\alpha | \Psi_{3/4} \rangle \quad \bar{s}_{\dot\alpha} \to (\Phi_{3/4})_{\dot\alpha} \equiv \langle \Psi_{3/4} | L_{\dot\alpha} | \Psi_{3/4} \rangle \tag{42}$$

The relations (41) and (42) give a natural link between the spinors $\xi_\alpha \left(\bar{\xi}_{\dot\alpha}\right)$ and $\zeta_\alpha \left(\bar{\zeta}_{\dot\alpha}\right)$, solutions of the dynamical problem, with the only physical states that can operate freely with the Hamiltonian $\mathcal{H}$: the "square root" states $|\Psi\rangle$ from the bispinors Φ_α.

Commutation relations (29) obey the Clifford's algebra for spinorial creation-annihilation operators. In this manner, operators s_α and d_α in the representation given by the associations (41) and (42) acting on the vacuum give zero: $s_\alpha |0\rangle = d_\alpha |0\rangle = 0$. The Fock's construction in the center of mass consists in the following vectors:

$$\begin{array}{lll} S_1 = |0\rangle e^{iMt} & \Xi_{1\alpha} = \bar{d}_{\dot\alpha}|0\rangle e^{iMt} & P_1 = \bar{d}^{\beta}\bar{d}_{\dot\beta}|0\rangle e^{iMt} \\ \Xi_{2\alpha} = \bar{s}_{\dot\alpha}|0\rangle e^{iMt} & V_{\alpha\beta} = \bar{s}_{\dot\alpha}\bar{d}_{\dot\beta}|0\rangle e^{iMt} & \Xi_{3\alpha} = \bar{s}_{\dot\alpha}\bar{d}^{\beta}\bar{d}_{\dot\beta}|0\rangle e^{iMt} \\ P_2 = \bar{s}^{\alpha}\bar{s}_{\dot\alpha}|0\rangle e^{iMt} & \Xi_{4\alpha} = \bar{d}_{\dot\alpha}\bar{s}^{\dot\beta}\bar{s}_{\dot\beta}|0\rangle e^{iMt} & \\ S_2 = \bar{d}^{\beta}\bar{d}_{\dot\beta}\bar{s}^{\alpha}\bar{s}_{\dot\alpha}|0\rangle e^{iMt} & & \end{array} \tag{43}$$

From expression (38) and taking into account that the number operator is $\bar{s}^{\dot\beta} s^{\alpha} \equiv n_s$, because $\bar{s}^{\dot\beta}$ and s^{α} work as creation-annihilation operators, we can easily obtain the mass for the different supermultiplets:

i) $n_s = 0 \to M_{1s} = -\frac{2^{1/2}}{|a|} + \sqrt{\frac{2}{|a|^2} + m^2}$; Scalar supermultiplet $(S_1, \Xi_{1\alpha}, P_1)$

ii) $n_s = 1 \to M_v = m$; Vector supermultiplet.

iii) $n_s = 2 \to M_{2s} = \sqrt{\frac{2}{|a|^2} + m^2} + \frac{2^{1/2}}{|a|}$; Scalar supermultiplet.

We emphasize now that the computations and algebraic manipulations given above were with $\bar{d}_{\dot\alpha} \to (\Phi_{1/4})_{\dot\alpha}$ and $\bar{s}_{\dot\alpha} \to (\Phi_{3/4})_{\dot\alpha}$ (the square of the true states) into the square root Hamiltonian. Notice from expressions (35), (41) and (42) that the physical states for the Hamiltonian in the square root form are one half the number of physical states for the Hamiltonian quadratic in momenta.

Looking at the commutation relations (29) one can see that the square norm of the vector supermultiplet and the states $\Xi_{1\alpha}$, $\Xi_{4\alpha}$ become negative (*i.e.*: the spectrum has "ghosts").

On the other hand, there is the possibility in the discussed problem to eliminate these "ghost states". This possibility is connected with the fact that the product of the masses of scalars supermultiplets equals to the square of the mass of the vector supermultiplet. There is a way: if we fix the mass of the first scalar supermultiplet

$$M_{1s} = \frac{2^{1/2}}{|a|} + \sqrt{\frac{2}{|a|^2} + m^2} \equiv \mu \tag{44}$$

sending the parameter $a \to 0$ and $m \to \infty$ keeping the condition (44), we get infinite values for the mass of the second scalar and vector supermultiplet.

It means, as it was pointed in [6], that effective contributions of these multiplets to the processes of scattering is equal to zero, and only one scalar supermultiplet with a fixed mass μ is left in the model. The discussed procedure is analogous to the transition from linear to non-linear realization σ-models through taking the limit when the mass of σ-particles goes to infinity.

It is interesting to note that the arbitrary c-parameters a and a^* generate a deformation of the usual line element for a superparticle in proper time, and this deformation is responsible, in any meaning, for the multiplets given above. This is not a casuality: one can easily see how the quantum Hamiltonian (28) is modified in the center of mass of the system by the c-parameters a and a^*. The implications of this type of superparticle actions with deformations of the quantization will be analyzed in a future work.

6. CONCLUSIONS

In this work the problem of the square root quantum operators was analyzed considering the simple model of superparticle of Volkov and Pashnev [6]. The quantization of this model was performed completely and the obtained spectrum of physical states, with the Hamiltonian operator in its square root form, was compared with the spectrum obtained with the Hamiltonian in the standard form (i.e: quadratic in momenta). To this end we used the Hamiltonian formulation described by Lanczos in [9] and the inhomogeneous Lorentz group as a representation for the obtained physical states [10, 11, 12] without any other manipulation like the usual quantum equations from the mathematical or operatorial point of view . We have shown that, in contrast to [1], the only states that the square root Hamiltonian can operate correspond to the representations with the lowest weights $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$. For instance, we conclude that quantically it is not the same to operate with the square root Hamiltonian as that with its square; the main problem is not the square root operator itself but the group theoretical description for the states under which such type of Hamiltonians operate. It is interesting to see that the results presented here for the superparticle are in complete agreement with the results, symmetry group and discussions for non-supersymmetric examples given in references [3, 4, 5]; and seeing that the lowest weights of the states under the square root Hamiltonian can operate, and because no concrete action is known to describe particles with fractionary statistics, superparticle relativistic actions as of [6] can be good geometrical and natural candidates to describe quartionic states [13, 14, 15, 10] (semions).

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